An analysis of risk measures

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Several recent articles on risk management indicate that a quantile measure of losses such as value-at-risk may not contain enough, or the right, information for risk managers. This paper presents a comprehensive empirical analysis of a set of left-tail measures (LTMs): the mean and standard deviation of a loss larger than the VAR (MLL and SDLL) and the VAR. We investigate the empirical dynamics of the LTMs. We present a robust and unified framework, the Arch quantile regression approach, in estimating the LTMs. Our Monte Carlo simulation shows that the VAR is appropriate for risk management when returns follow Gaussian processes, but the MLL strategy and strategies accounting for the SDLL are useful in reducing the risk of large losses under non-normal distributions and when there are jumps in asset prices.

1 Introduction

In recent years value-at-risk (VAR) has become a popular tool in the measurement and management of financial risk. VAR is defined as the loss in market value over a given time horizon that is exceeded with probability τ, which is often set at 0.01 or 0.05. VAR is an easily interpretable measure of risk that summarizes information regarding the distribution of potential losses. The popularity of the VAR method results from both the need of various institutions to manage risk and the specifications of government regulations (see Jorion, 2000, and Saunders, 1999).

Although VAR is a relatively simple concept, its measurement is in fact a challenging task. This difficulty explains why most of the research on VAR has been concerned with estimation and forecasting.[1] Recently, several papers have shed light on the use of VAR as a risk management tool. Ahn *et al.* (1999) analyze the problem of optimally managing risk by minimizing VAR using options.

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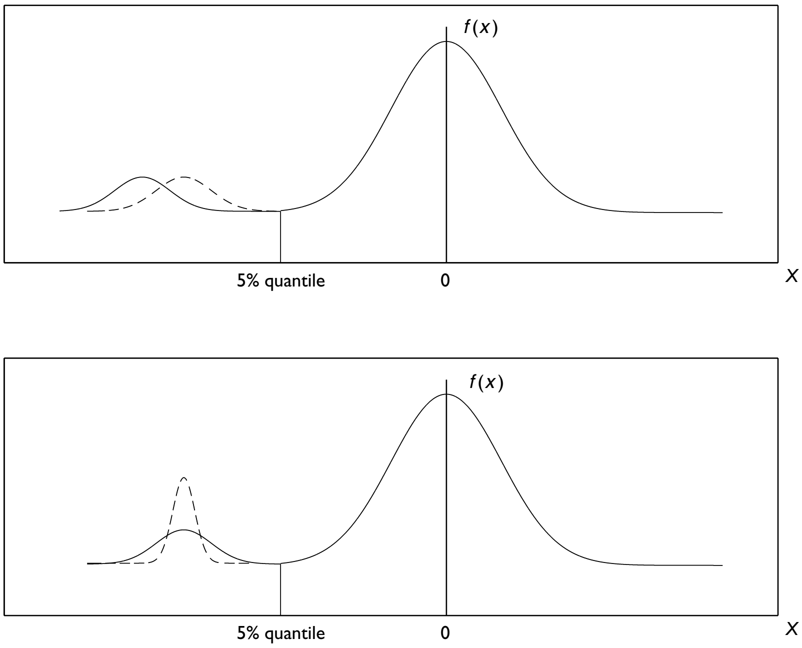
Klüppelberg and Korn (1998) and Alexander and Baptista (1999) conduct a mean–variance analysis using VAR to manage the risk of portfolios. It is well known that some flaws in the VAR approach exist, notably the (often implicit) assumption that market returns are distributed normally over a short horizon. Yet this assumption may be inappropriate in periods of market stress such as the Long Term Capital Management debacle in 1998.

At a more fundamental level, Basak and Shapiro (2000) demonstrate that risk management practices under the VAR approach may yield an unintended result. They find that the VAR risk manager who focuses myopically on VAR often optimally chooses a larger exposure to risky assets than non-risk managers and consequently incurs larger losses when losses occur. In compliance with the VAR constraint, the VAR risk manager is willing to incur losses, and it is optimal for him to incur losses in those states against which it is most expensive to insure. Although the probability of a loss is fixed, when a large loss occurs it is larger than when the agent is not engaging in VAR risk management. The authors suggest an alternative risk management model based on the expectation of a loss, which they call "limited expected loss" (LEL).

In this paper we conduct an empirical analysis of the various left-tail measures (LTMs): the mean and standard deviation of a loss larger than the VAR (MLL and SDLL, respectively), as well as the VAR itself. A key objective of this study is to examine if the estimated MLL and SDLL provide any new information. Our analysis shows that these measures do not give new information in normal market conditions, but when large losses are more likely the information in the MLL and SDLL cannot be captured by the VAR. This is illustrated in the upper panel of Figure 1, which shows a situation where the 5% VAR is the same for both distributions while the expected losses on the tails are different. The deficiency in information about the mean of the tail can be corrected by the use of measures such as MLL. The lower panel of Figure 1 illustrates a situation where the mean of the tail is the same but the standard deviation is different, so here the SDLL provides useful information. However, in practice there are so few observations in the tail that one would be hard pressed to determine if peaks occurred in the tails. Furthermore, it is likely that any peaks in the tails would indicate the presence of regime switches.

We estimate the LTMs under general conditions using the quantile regression method. The basic idea of quantile regression is to treat the simple ordinary quantile calculation as an optimization problem and extend this optimization problem into general regression models. The quantile regression method was introduced by Koenker and Bassett (1978) and has now become a popular, robust approach for statistical analysis.[2] These authors showed how a simple minimization problem yielding the ordinary sample quantiles in the location model can be naturally generalized to the linear model, generating a new class of statistics called regression quantiles. Given the estimated quantiles, generating the LTMs is a relatively straightforward task.

FIGURE 1 The inability of the VAR measure to capture tail shapes



This figure shows situations where the VAR measure is unable to capture the various shapes of the left tail. The upper panel shows the situation where the conditional means of the left tails (large losses) are different while the 5% VARs are the same. The lower panel shows the situation where the conditional variances of the left tails (large losses) are different while the 5% VARs are the same.

Since the pioneering work of Koenker and Bassett (1978, 1982), the asymptotic theory of the ordinary sample quantiles has been extended to the joint asymptotic behavior of a finite number of regression quantiles. Koenker and Zhao (1996) study conditional quantile estimation and inference under Arch models and derive limiting distributions of the regression quantiles. The Arch-quantile regression is an excellent way to estimate the conditional quantiles and LTMs under conditional heteroskedasticity. It would be desirable if the above quantile regression Arch model could be extended to the case of quantile regression Garch (Engle and Manganelli, 1999). In this case, just like the maximum-likelihood estimation of conventional Garch models, a non-linear quantile regression estimation is needed. Although candidate algorithms are available, the convergence of these estimation procedures and the limiting distributions have not yet been fully developed. For this reason, we use in this paper the quantile regression model with Arch effect. Notice that an ARMA process can be asymptotically represented by an AR process; with an appropriately chosen number of lags, the Arch quantile regression model can also practically provide a good approximation for Garch models.

We conduct Monte Carlo simulation of risk-managed portfolios using different LTMs as risk measures. The analysis indicates that the VAR measure reduces risks as well as other LTMs do when the return process is conditionally Gaussian – even when it exhibits conditional heteroskedasticity. However, the VAR measure no longer works well when the return process makes discrete jumps over time. The MLL measure works better in reducing large losses, and the SDLL appears to be useful in further reducing risk. Our empirical findings support the theoretical arguments of Artzner *et al.* (1999), Basak and Shapiro (2000) and Wang (2000) in that the VAR measure ignores information when the return process is not normal.

The remainder of the article is organized as follows. Section 2 introduces and defines the left-tail measures. Section 3 presents the quantile regression methodology for estimating the quantile function and the LTMs. Section 4 provides data descriptions and conducts the quantile regression estimation. We also report and discuss empirical results from the estimated LTMs. In Section 5 we present Monte Carlo simulation analyses on the performance of the various LTMs in a simple risk management framework for an index fund. Section 6 presents concluding remarks.

2 Left-tail measures

For ease of exposition, we define value-at-risk as the *percentage* loss in market value over a given time horizon that is exceeded with probability τ. That is, for a time series of returns on an asset, $\{r_t\}_{t=1}^{n}$, find VAR_t such that

$$\Pr\left(r_t < -\mathrm{VAR}_t \mid I_{t-1}\right) = \tau \tag{1}$$

where I_{t-1} denotes the information set at time $t-1$. From this definition it is clear that finding a VAR is essentially the same as finding a $100\tau\%$ quantile of the conditional distribution of returns.

We consider, as complements of VAR, the conditional moments of losses that exceed the VAR at a given level. Define the conditional mean of losses larger than the $100\tau\%$ level VAR (denoted by MLL) as

$$-\mathrm{E}\left[r_t \mid r_t \leq -\mathrm{VAR}_t(\tau);\, I_{t-1}\right] \tag{2}$$

and the standard deviation of losses larger than the VAR (denoted by SDLL) as

$$\sqrt{\mathrm{E}\left\{\left[r_t - \mathrm{E}\left(r_t \middle| r_t \leq -\mathrm{VAR}_t(\tau);\, I_{t-1}\right)\right]^2 \middle| r_t \leq -\mathrm{VAR}_t(\tau);\, I_{t-1}\right\}}$$

Similarly, we may consider higher-order conditional moments, such as skewness and kurtosis of losses larger than the $100\tau\%$ VAR. In general, we call these measures left-tail measures (LTMs) because they measure the quantile, the mean, standard deviation and higher moments of the $100\tau\%$ tail.

3 Estimating left-tail measures by quantile regression

Most existing estimation methods for risk management must assume a particular distribution for innovations – typically, a normal distribution conditional on past information. However, empirical studies of economic time series indicate that many financial variables display non-normal characteristics. Some series tend to be skewed and leptokurtic and, in particular, many financial returns have fat tails. Risk measures based on the assumption of normality generally underestimate the risk because the conditional distribution of asset returns has tails that are fatter than those of a conditional normal distribution.

To avoid underestimating risk, Hull and White (1998) use a mixture of normal distributions to model the return of financial assets. The accuracy of their estimates still, however, depends on the correctness of the distributional specifications. Generally the estimates are not robust to deviations from the assumed distributions, and their estimation may be poor if extreme realizations are not accurately modeled. Estimation may be particularly poor for models with conditional heteroskedasticity since the estimation of time-varying variance is very sensitive to large innovations. Thus one can see why the estimation of risk measures under general distributional assumptions is an important issue.

In this paper we suggest using a quantile regression method to calculate risk measures. This method does not depend on distributional assumptions for innovations and can be used to estimate the VAR, MLL and SDLL.

3.1 Quantile regression

Consider a random variable Y which is characterized by its distribution function $F(y)$. The τth quantile of Y is defined by

$$Q_Y(\tau) = \inf\{\, y \mid F(y) \geq \tau \,\}$$

Similarly, if we have a random sample $\{y_1, \cdots, y_n\}$ from the distribution F, the τth sample quantile can be defined as

$$\widehat{Q}_Y(\tau) = \inf\{\, y \mid \widehat{F}(y) \geq \tau \,\}$$

where $\widehat{F}$ is the sample distribution function. Koenker and Bassett (1978) determined the sample quantile by solving the following minimization problem:

$$\min_{b \in \Re} \left[\sum_{t \in \{t: y_t \geq b\}} \tau \left| y_t - b \right| + \sum_{t \in \{t: y_t < b\}} (1-\tau) \left| y_t - b \right| \right] \tag{3}$$

and generalized the concept of quantiles to regression models.

If we consider a regression model

$$y_t = b' x_t + u_t \tag{4}$$

where x_t is a $k \times 1$ vector of regressors, with the first element normalized to 1. The regressors could just be lagged returns. Conditional on the regressor x_t, the τth quantile of Y,

$$Q_Y(\tau \,|\, x_t) = \inf\{\, y \,|\, F(y \,|\, x_t) \geq \tau \,\}$$

is a linear function of x_t:

$$Q_Y(\tau \,|\, x_t) = b(\tau)' x_t$$

where

$$b(\tau) = \left(b_1 + F_u^{-1}(\tau),\ b_2,\ \cdots,\ b_k\right)'$$

and $F_u(\cdot)$ is the cumulative distribution function of the residual. Koenker and Bassett (1978) consider an analogue of (3):

$$\min_{b \in \Re^k} \left[\sum_{t \in \{t: y_t \geq x_t b\}} \tau \left| y_t - x_t' b \right| + \sum_{t \in \{t: y_t < x_t b\}} (1 - \tau) \left| y_t - x_t' b \right| \right] \tag{5}$$

and show that the τth conditional quantile of y can be estimated by

$$\hat{Q}_Y(\tau \,|\, x_t) = \hat{b}(\tau)' x_t \tag{6}$$

where $\hat{b}(\tau)$ is the solution to (5) and is called the regression quantile. As a special case, the least absolute error estimator is the regression median, ie, the regression quantile for $\tau = 0.5$.

Now consider the following quantile regression model with conditional heteroskedasticity

$$u_t = \left(\gamma_0 + \gamma_1 |u_{t-1}| + \cdots + \gamma_q |u_{t-q}|\right) \varepsilon_t$$

where ε_t is independent identically distributed (iid) $(0, 1)$, and $\gamma_0 > 0$, $(\gamma_1, \cdots, \gamma_q)' \in \Re_+^q$. The series $\{u_t\}$ is then an Arch model. The τth quantile of u_t, conditional on u_{t-j}, $j \geq 1$, is given by $(\gamma_0 + \gamma_1 |y_{t-1}| + \cdots + \gamma_q |y_{t-q}|) F_\varepsilon^{-1}(\tau)$, where $F_\varepsilon^{-1}(\tau)$ is the quantile function of ε. The conditional quantile of u_t can be estimated from a quantile regression of u_t on its lagged values.

3.2 Estimating VAR by quantile regression

A plausible way of modeling a return process is to use an autoregressive specification. If we model for the returns process $\{r_t\}$ as

$$r_t = \alpha_0 + \sum_{i=1}^{k} \alpha_i r_{t-i} + u_t \tag{7}$$

the $100\tau\%$ value-at-risk of r_t is then determined by

$$-\alpha_0 - \sum_{i=1}^{k} \alpha_i r_{t-i} - Q_u(\tau \,|\, I_{t-1}) \tag{8}$$

where $Q_u(\tau \,|\, I_{t-1})$ is the τth conditional quantile of the residual process u_t. More generally, we may consider the following regression:

$$r_t = \alpha' x_t + u_t \tag{9}$$

where $x_t \in I_{t-1}$ is the vector of regressors, which usually includes lagged values of the dependent variable. For example, a general model might be $x_t = (1, r_{t-1}, \cdots, r_{t-k}, z_{1t}, \cdots, z_{vt})'$, where $\{z_{it}\}$ are useful covariates that explain r_t.

We denote the vector $(1, |u_{t-1}|, \cdots, |u_{t-q}|)'$ as Z_t and calculate the conditional quantile as

$$Q_u(\tau \,|\, I_{t-1}) = \gamma(\tau)' Z_t = \left(\gamma_0 Q_\varepsilon(\tau), \gamma_1 Q_\varepsilon(\tau), \cdots, \gamma_q Q_\varepsilon(\tau)\right)' Z_t$$

and $Q_\varepsilon(\tau) = F_\varepsilon^{-1}(\tau)$ is the quantile function of ε.[3]

Therefore, the VAR_t is the quantile of r_t in model (9) conditional on information to time $t-1$, which is a linear function of the conditional quantile of u_t:

$$\mathrm{VAR}_t(\tau) = -\alpha' x_t - \gamma(\tau)' Z_t \tag{10}$$

Thus, we can use a quantile regression method to estimate $\gamma(\tau)$ and $\mathrm{VAR}_t(\tau)$. In particular, $\gamma(\tau)$ can be estimated from the following problem:

$$\hat{\gamma}(\tau) = \arg\min_{\gamma \in \Re^k} \left[\sum_{t \in \{t: u_t \geq Z_t'\gamma\}} \tau \left| u_t - Z_t'\gamma \right| + \sum_{t \in \{t: u_t < Z_t'\gamma\}} (1-\tau) \left| u_t - Z_t'\gamma \right| \right] \tag{11}$$

Following an argument similar to that proposed by Koenker and Zhao (1996), it can be shown that $\hat{\gamma}(\tau)$ is a root-n consistent estimator[4] of $\gamma(\tau)$. In practice, we can replace u_t and Z_t by consistent estimators: $\hat{u}_t = r_t - \hat{\alpha}_0 - \sum_{i=1}^{k} \hat{\alpha}_i r_{t-i} - \sum_{i=1}^{v} \hat{\alpha}_{k+i} z_{it}$, and $\hat{Z}_t = (1, |\hat{u}_{t-1}|, \cdots, |\hat{u}_{t-q}\}|)'$. Under mild regularity conditions, it can be shown that the $\hat{\gamma}(\tau)$ based on $\hat{u}_t$ is still a root-n consistent estimator of $\gamma(\tau)$.

3.3 Estimating LTMs by quantile regression

Now we consider the estimation of MLL and SDLL based on quantile regressions. Suppose we want to calculate the expected return conditional on past information that is lower than a given level, say $\bar{r}$:

$$\mathrm{E}\left[r_t \mid r_t \le \bar{r}; I_{t-1}\right]$$

We rewrite this as

$$\mathrm{E}\left[r_t \mathbf{1}(r_t \le \bar{r}) \mid I_{t-1}\right] \tag{12}$$

where $\mathbf{1}(r_t \le \bar{r}) = 1$ if $r_t \le \bar{r}$ and 0 otherwise. If we denote the conditional distribution function of r_t as $F_t(\cdot)$, ie, $F_t(\cdot) = \Pr(r_t < \cdot \mid I_{t-1})$, then

$$E\left[r_t \mathbf{1}(r_t \le \bar{r}) \middle| I_{t-1}\right] = \int_{-\infty}^{\bar{r}} \xi \, \mathrm{d}F_t(\xi)$$

which can be approximated by a discrete Riemann function:

$$\sum_{\xi_i \le \bar{r}} \xi_i \left[F_t(\xi_i) - F_t(\xi_{i-1})\right]$$

We consider a partition of $\{\tau_i\}$ on $[0,1]$ and estimate the τ_i-th conditional quantile of r_t by quantile regression and denote the estimated τ_i-th conditional quantile of r_t as $\hat{Q}_r(\tau_i \mid I_{t-1})$. As $\max(\tau_i - \tau_{i-1}) \to 0$, the conditional mean of r_t smaller than $\bar{r}$ can be consistently estimated by the following discrete Riemann approximation:

$$\sum_{\tau_i \le \bar{\tau}} \hat{Q}_r\left(\tau_i \mid I_{t-1}\right)\left[\tau_i - \tau_{i-1}\right] \tag{13}$$

where

$$\bar{\tau} = \sup\left\{\tau_j : \hat{Q}_r(\tau_j \mid I_{t-1}) \le \bar{r}\right\}$$

Consequently, an estimation of the limited expected loss (LEL) is obtained.

Notice that

$$\mathrm{E}\left[r_t \middle| r_t \le \bar{r}; I_{t-1}\right] = \frac{\int_{-\infty}^{\bar{r}} \xi \, \mathrm{d}F_t(\xi)}{\int_{-\infty}^{\bar{r}} \mathrm{d}F_t(\xi)}$$

and the MLL at the 5% level is estimated immediately.

The SDLL and higher moments can also be estimated by similar discrete Riemann approximations based on quantile regression. Note that the above estimation procedures do not depend on distributional assumptions about the return process, a condition which provides a robust method for risk measurement.

Quantile regression method has the important property that it is robust to distributional assumptions, a property that is inherited from the robustness of the ordinary sample quantiles. In quantile regression, it is not the magnitude of the dependent variable that matters but its position relative to the estimated hyperplane. Quantile estimation is influenced only by the values of the dependent variable near the specified quantile of the conditional distribution. As a result, the estimated coefficient vector $\hat{\gamma}(\tau)$ is not sensitive to outlier observations. Such a property is especially attractive in financial applications since many financial data like (log) portfolio returns are usually fat-tailed.

The quantile regression model has a linear programming representation that makes the estimation easy. Notice that the optimization problem (11) may be reformulated as a linear program by introducing "slack" variables to represent the positive and negative parts of the vector of residuals (see Koenker and Bassett (1978) for a more detailed discussion). Estimation of regression quantiles can then be obtained in a finite number of simplex iterations.[5]

4 Empirical estimation and results

4.1 Data and estimation procedure

The data we use for estimating the regression quantiles and the LTMs are the daily returns of the S&P500 Index from August 1963 to January 1998. In addition, we use two series of interest rate data: the daily yields on the one-year Treasury bills and 10-year Treasury notes from January 1962 to December 2001. All data were obtained from the Center for Research in Security Prices. In Table 1, we see that the S&P500 data display negative skewness and excess kurtosis, and that the interest rates exhibit positive skewness, weak excess kurtosis and strong autocorrelations.

TABLE 1 Summary statistics of the data

		One-year Treasury bill		10-year Treasury note	
	Daily S&P500	**Yield**	**Change**	**Yield**	**Change**
Mean	0.0003	6.4034	−0.0001	7.1086	0.0001
Standard deviation	0.0088	3.0088	0.0949	2.8551	0.0689
Maximum	0.0910	17.3100	1.1000	15.8400	0.6500
Minimum	−0.2047	1.9200	−1.0800	3.8500	−0.7500
Skewness	−1.4707	0.7108	−0.1940	0.1741	−0.3145
Excess kurtosis	39.8046	1.3768	20.4705	0.9785	10.6144
AC(1)	0.1131	0.7477	0.1180	0.6738	0.0932
AC(2)	−0.0284	0.7463	0.0474	0.6719	0.0351
AC(3)	−0.0169	0.7452	−0.0010	0.6715	−0.0049
AC(4)	−0.0246	0.7498	0.0321	0.6763	−0.0202
AC(5)	0.0144	0.7673	0.0208	0.6990	0.0344
AC(10)	−0.0041	0.7434	0.0419	0.6730	0.0194

This table gives summary statistics for the daily returns of the S&P500 Index, along with the daily yield and the change in yield (annualized in per cent) of one-year Treasury bills and 10-year Treasury notes. AC(k) denotes autocorrelation of order k. The sample period for the daily S&P500 Index data is August 1963 to January 1998. The yield data are for January 1962 to December 2001. The source of the data is the Center for Research in Security Prices and the Federal Reserve.

TABLE 2 Estimated Arch equation parameters for the daily S&P500 data

	Quantile (τ)				
Parameter	**1%**	**2%**	**3%**	**4%**	**5%**
$\gamma_0(\tau)$	−0.0094	−0.0085	−0.0078	−0.0069	−0.0064
	(0.0011)	(0.0008)	(0.0007)	(0.0005)	(0.0004)
$\gamma_1(\tau)$	−0.3699	−0.1376	−0.0949	−0.0863	−0.1071
	(0.1162)	(0.0583)	(0.0643)	(0.0504)	(0.0433)
$\gamma_2(\tau)$	−0.1660	−0.0924	−0.1315	−0.1077	−0.1000
	(0.0759)	(0.0442)	(0.0388)	(0.0401)	(0.0384)
$\gamma_3(\tau)$	−0.4123	−0.2725	−0.2078	−0.2077	−0.1690
	(0.1108)	(0.0600)	(0.0591)	(0.0475)	(0.0506)
$\gamma_4(\tau)$	−0.1755	−0.1692	−0.1853	−0.1584	−0.1196
	(0.0928)	(0.0577)	(0.0595)	(0.0501)	(0.0476)
$\gamma_5(\tau)$	−0.1414	−0.1853	−0.2061	−0.2461	−0.2388
	(0.0774)	(0.0430)	(0.0399)	(0.0337)	(0.0308)
$\gamma_6(\tau)$	−0.2603	−0.1540	−0.1102	−0.1159	−0.1055
	(0.0986)	(0.0528)	(0.0473)	(0.0296)	(0.0358)
$\gamma_7(\tau)$	−0.0367	−0.1116	−0.0802	−0.0906	−0.0930
	(0.1140)	(0.0594)	(0.0482)	(0.0272)	(0.0391)
$\gamma_8(\tau)$	−0.2566	−0.2112	−0.0754	−0.1287	−0.0408
	(0.1128)	(0.0687)	(0.0469)	(0.0349)	(0.0367)

This table reports the estimated parameters of the daily Arch equations for quantiles, τ, 1% to 5%. The standard errors are in parentheses. We specify $u_t = (\gamma_0 + \gamma_1|u_{t-1}| + \cdots + \gamma_q|u_{t-q}|)\varepsilon_t$, where ε_t are iid (0, 1). We denote the vector $(1, |u_{t-1}|, \cdots, |u_{t-q}|)'$ as Z_t and calculate the conditional quantile as $Q_u(\tau | I_{t-1}) = \gamma(\tau)' Z_t$, where $\gamma(\tau)' = (\gamma_0 Q_\varepsilon(\tau), \gamma_1 Q_\varepsilon(\tau), \cdots, \gamma_q Q_\varepsilon(\tau))$, and $Q_\varepsilon(\tau) = F_\varepsilon^{-1}(\tau)$ is the quantile function of ε.

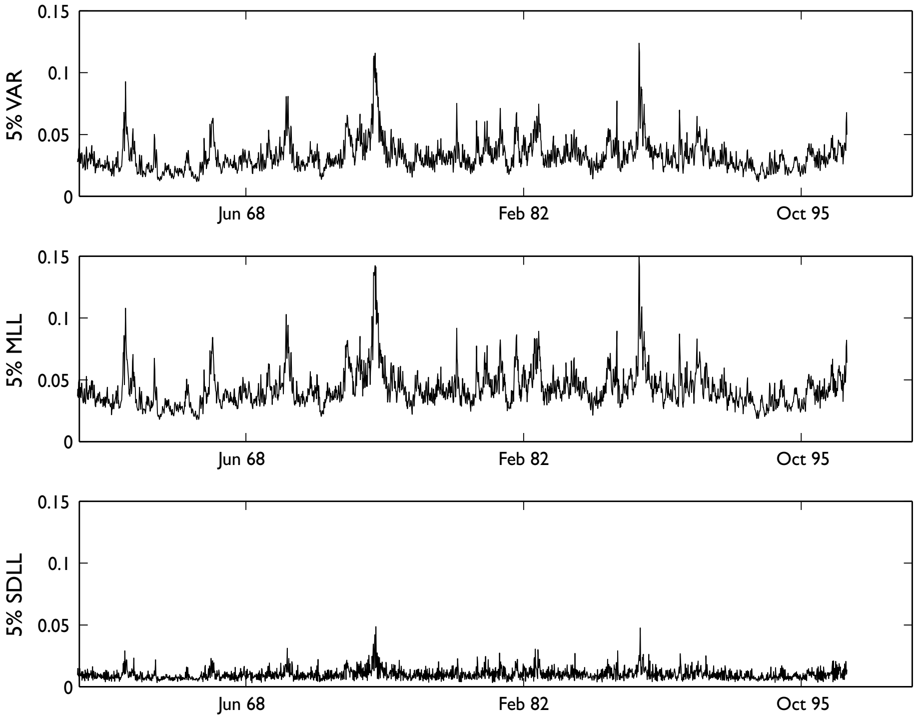
4.2 Estimated LTMs

To estimate the 5% VAR we use Equation (10) at the 5% quantile of the returns. To estimate the MLL and SDLL left-tail measures, we estimate the quantile function of returns using 8,668 observations for the daily data, a number that should be sufficient to yield unbiased estimates.

To estimate the quantile functions we estimate the mean equation (7) for the S&P500 returns. Since the data in Table 1 show weak autocorrelation, we use an AR(1) model and estimate the intercept to be 0.00032, with a t ratio of 3.5597, and the AR(1) coefficient to be 0.11739 with a t ratio of 2.6026.

Given the residuals from this estimation, we estimate the regression quantiles. Based on the size of the data set and the need to have a sufficient number of points to compute the MLL and SDLL, we decided to compute 50 regression quantiles between the 0% and 5% quantiles. Table 2 reports the Arch estimates for an AR(8) residual process. To conserve space, we only report results for five of the 50 quantile regressions. We see that most of the estimates of $\gamma_i(\tau)$ are neg-

FIGURE 2 Estimated left-tail measures for the daily S&P500 Index



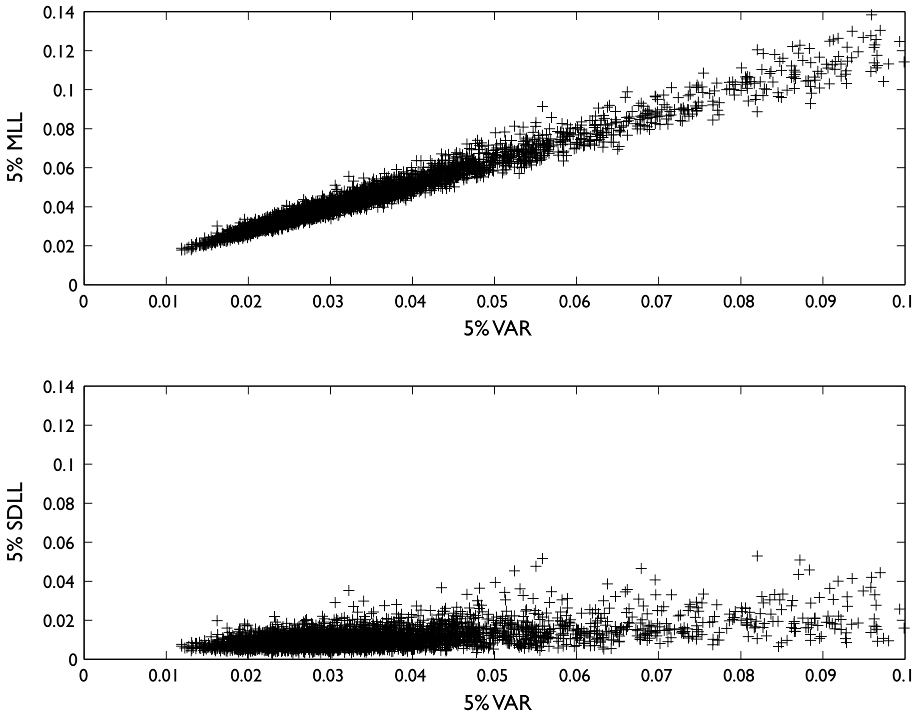
This figure shows estimated left-tail measures for the daily S&P500 Index. The top panel plots the 5% VAR, the middle panel plots the 5% MLL, and the bottom panel plots the 5% SDLL.

ative and that they are statistically significant for all listed τ and for lags smaller than eight, which indicates a strong Arch effect. We limit the lags to eight to balance estimation efficiency with the estimation of the Arch effect.

Given the estimated quantile functions in the 5% tail, we compute the MLLs and SDLLs and plot the results in Figure 2. The LTMs exhibit high volatility over the entire sample period – in particular during the 1987 market crash. In general, the 5% VAR and 5% MLL track each other very well. The SDLL seems to follow very similar dynamics, though it has smaller values. This close comparison indicates that the VAR drives the dynamics of the first and second tail moments.

Figure 3 plots the MLL and SDLL against the VAR. It is apparent that the MLLs are linearly related to the VAR, and a simple linear regression should be able to capture their relationship very well. The SDLL can also be modeled by a linear regression to the VAR, although such a regression would have a much smaller slope coefficient and a smaller R^2. We find that the MLL plot in Figure 3 is similar to that from the normal distribution. The SDLL plot in Figure 3 more

FIGURE 3 Estimated MLL and SDLL vs. VAR for the daily S&P500 Index



This figure shows estimated left-tail moments versus the VAR for the daily S&P500 Index. The top panel plots the 5% MLL versus the VAR. The bottom panel plots the 5% SDLL versus the VAR.

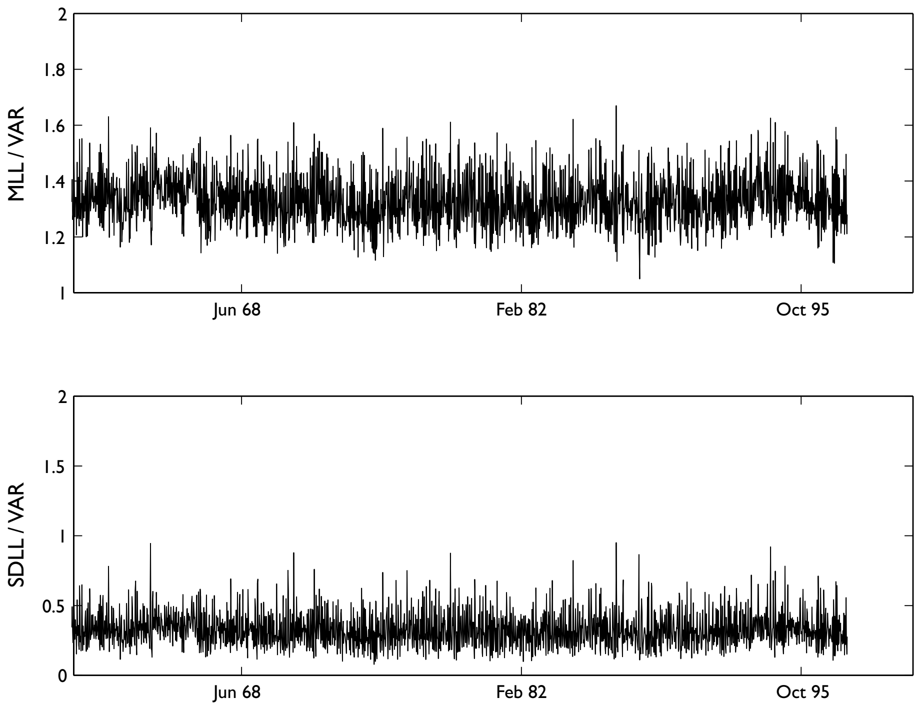
closely resembles that from the *t* distribution. Thus, the normal distribution may not be able to completely describe the left-tail measures at daily frequencies.

The dynamics of the MLL and the SDLL follow that of the VAR closely, but not exactly. We plot the ratios of the MLL to the VAR and the SDLL to the VAR in Figure 4. The estimated MLL/VAR ratio varies between 1.08 to 1.68, with an average of 1.38, and the SDLL/VAR ratio varies between 0.09 and 1.17, with an average of 0.45. The ratios change over time somewhat randomly, and their variation does not seem to be related to the volatility of returns.

5 Risk management performance of the LTMs

In this section we present Monte Carlo simulation analyses of risk management using the left-tail measures. We hope to determine if the LTMs are able to provide better protection against losses. Moreover, we would like to know if there is a trade-off between better risk management and lower returns or poorer portfolio performance as measured by the Sharpe ratio.

FIGURE 4 Ratios of MLL and SDLL to VAR for the daily S&P500 Index



This figure shows estimated ratios of the 5% MLL and SDLL to the 5% VAR for the daily S&P500 Index. The top panel plots the MLL/VAR ratios, and the bottom panel plots the SDLL/VAR ratios.

5.1 A risk management analysis

We conduct this analysis in the setting of a naive risk manager of a risk-managed index fund that tracks the S&P500 Index. We assume that there are limits on the risk exposure so that the manager invests all his funds in the "Index" as long as the estimated VAR or other LTMs are below the limits. If the limits are exceeded, the risk manager will invest less in the risky asset and shift the funds into a zero-return, risk-free asset until the VAR or other LTMs are just below their limits.[6] For example, if the risk manager sets the risk tolerance level at a loss of 3% of the value of the portfolio as measured by the 5% VAR, and if the estimated 5% VAR for the "Index" is 4%, the risk manager will invest only 75% of the value of the managed fund in the "Index" and put the remaining balance in a risk-free asset. This will reduce the estimated 5% VAR for the managed fund to an acceptable 3% loss from a 4% potential loss if the manager invests all funds in the "Index". To make the different risk management strategies comparable, we set the limits of these 5% LTMs at their sample averages of the S&P500 Index returns.

We first simulate several time series of returns, each with 5,000 observations, and describe the dynamics below. We then estimate 50 quantile regressions from 0% to 5% quantile to yield 50 sets of parameters for quantile forecasts conditional on available information. We then simulate 10 additional periods of returns and compute the LTMs using available information at the end of each of the 10 additional periods. We conduct a total of 400 simulations. The "Index" portfolio invests fully in this simulated series. We examine portfolio performance for various left-tail measures when returns follow various dynamics, such as a normal distribution, a Garch process, a jump process, and a regime-switching process.

We define the following three risk management strategies with different left-tail measures: (1) 5% VAR set at the sample average of the daily S&P500 Index (3.722%); (2) 5% MLL set at the sample average (4.869%); and (3) 5% MLL + 5% SDLL set at the sample average (5.992%). The first portfolio risk management strategy uses the popular value-at-risk measure, which serves as our benchmark for risk-managed portfolios. The second risk management strategy takes into account the conditional mean of losses in the 5% tail. The third strategy goes further by taking into account changes in the second moment of the left tail. There are more general ways of combining the first two moments of the tail – for example, $5\%\,\mathrm{MLL} + \pi_0 5\%\,\mathrm{SDLL}^{\pi_1}$, where π_0 is a weighting constant and π_1 is a magnifying constant to take risk-aversion into account. But we will leave further analysis for future research.

Given the above simple risk management and trading strategy, we measure the performance of the portfolios. The exercise is repeated 1,000 times to give a collection of asset returns for each left-tail measure. To evaluate the various LTMs as risk management strategies, we are interested in the frequency of large losses, as well as the means and standard deviations of returns. We also compute the Sharpe ratio of these portfolios.

5.2 Return dynamics

If the return process is conditionally normal, the VAR should provide a good measure of risk. But when larger losses are more likely than some smaller losses within the 5% tail, the VAR no longer provides a good measure of risk. Large losses may be more likely, for example, when the "Index" process contains a negative jump component. We simulate the following four return processes, each calibrated to the original time series data of the daily S&P500 Index.

Normally distributed returns

The first data-generating process we consider is a simple AR(1) process with Gaussian innovations. Let $\{r_t\}$ be the simulated returns; then

$$r_t = a_0 + a_1 r_{t-1} + \epsilon_t \tag{14}$$

TABLE 3 Performance of risk-managed portfolios with Gaussian returns

	Strategy			
	Index	**VAR**	**MLL**	**MLL + SDLL**
Mean ($\times 10^{-3}$)	0.3565	0.3507	0.2762	0.2869
Standard deviation	0.0113	0.0076	0.0086	0.0099
Sharpe ratio	0.0316	0.0463	0.0320	0.0291
Maximum return	0.0330	0.0246	0.0270	0.0283
Minimum return	−0.0378	−0.0203	−0.0227	−0.0219
Loss				
>3%	0.51	0	0	0
>2%	3.23	0.58	0.57	0.65
>1%	9.80	8.30	8.56	8.23

This table reports the performance of risk-managed portfolios using various left-tail measures under the Gaussian return process in Equation (14). The estimated parameters of the process (a_0, a_1, and σ) are 0.0002, 0.1491 and 0.0111. "Index" indicates that the portfolio is fully invested in the simulated index. The other three portfolios' limits, are respectively: 5% VAR at 1.2415%; 5% MLL at 1.7186%; and 5%MLL + 5%SDLL at 2.2857%.

where $\epsilon_t \sim N(0, \sigma^2)$, and the parameters, a_0, a_1, and σ are calibrated to our daily sample data of the S&P500 Index. These values are estimated to be 0.0002 (0.0001), 0.1491 (0.0112) and 0.0111 (0.0003), where standard errors of the estimates are listed in parentheses.

Table 3 reports the performance of risk-managed portfolios with the Gaussian returns. We see that the risk management strategies reduce the means of portfolio returns somewhat, although the standard deviations are also a bit smaller relative to the "Index" portfolio. This reduction in standard deviations translates into marginally larger Sharpe ratios for the risk management portfolios using VAR and MLL. However, there are costs to the reduction in risk: for some of the risk management portfolios, the maximum returns are smaller than that of the "Index" portfolio. For the Gaussian process, there are no returns lower than the -3% mark. For losses larger than 2%, the simulation indicates that the VAR measure and the MLL can reduce their occurrence from 3.32% of all returns to 0.57–0.58%. The second-moment measures do not seem to reduce risk any further. For losses larger than 1%, the risk-managed portfolios do marginally better than the "Index" portfolio, reducing losses from 9.8% to 8.23–8.56%.

In summary, for the Gaussian returns, the VAR measure works as well as the MLL measure and, due to a lack of variability in the tail, the second-moment measures do not seem to do better. The results are consistent with the work of Artzner *et al.* (1999), Basak and Shapiro (2000) and Wang (2000), and indicate that for long-horizon risk management (a month or longer), where normality in returns has been extensively documented, VAR is an adequate tool for risk measurement.

AR(1)–Agarch(1,1) model

Conditional heteroskedasticity has been widely documented for weekly, daily and intra-day returns (see, for example, Bollerslev, Chou and Kroner, 1992). We generate return series based on the AR(1)–asymmetric Garch(1, 1) model of Glosten, Jagannathan and Runkle (1993), which captures well the conditional heteroskedasticity and asymmetric volatility (see Engle and Ng, 1993; Bekaert and Wu, 2000; Wu, 2001; and Wu and Xiao, 2002):

$$r_t = a_0 + a_1 r_{t-1} + \epsilon_t, \qquad \epsilon_t \mid \mathrm{I}_t \sim N(0, \sigma_t^2)$$
$$\sigma_t^2 = \omega_0 + \omega_1 \sigma_{t-1}^2 + \omega_2 \epsilon_{t-1}^2 + \omega_3 \mathbf{1}_{\{\epsilon_{t-1} < 0\}} \epsilon_{t-1}^2 \tag{15}$$

where $\mathbf{1}_{\{\epsilon_{t-1} < 0\}}$ is an indicator function that equals 1 if $\epsilon_{t-1} < 0$ and 0 otherwise. This term captures the additional power that negative return shocks have in predicting future volatility. The model parameters for the mean equation, a_0 and a_1, are estimated to be 0.0002 (0.0001) and 0.1491 (0.0112), and the Garch equation parameters ω_i, $i = 0, 1, 2, 3$, are estimated to be 1.0231E−6 (5.3451E−11), 0.9239 (0.0028), 0.0800 (0.0049) and 0.0335 (0.0039).

Table 4 reports the performance of risk-managed portfolios with the asymmetric Garch returns. The risk management strategies marginally increase the means of portfolio returns and result in smaller standard deviations of returns relative to the "Index" portfolio. All the risk management strategies therefore yield higher Sharpe ratios. For losses larger than 2%, all risk management strategies reduce their occurrence from 2.04% of all returns to 0.55–0.57%. For losses larger than 1%, all risk management strategies can reduce the occurrence from 10.26% to 3.26–3.28%.

TABLE 4 Performance of risk-managed portfolios with Garch returns

	Strategy			
	Index	**VAR**	**MLL**	**MLL + SDLL**
Mean ($\times 10^{-3}$)	0.3443	0.4338	0.3836	0.4195
Standard deviation	0.0105	0.0055	0.0060	0.0065
Sharpe ratio	0.0327	0.0786	0.0638	0.0641
Maximum return	0.0418	0.0263	0.0285	0.0298
Minimum return	−0.0730	−0.0170	−0.0206	−0.0244
Loss				
> 3%	0.09	0	0	0
> 2%	2.04	0.57	0.56	0.55
> 1%	10.26	3.28	3.26	3.22

This table reports the performance of risk-managed portfolios using various left-tail measures under the asymmetric Garch return process in Equation (15). The estimated parameters for the mean equation (a_0 and a_1) are 0.0002 and 0.1491. The Garch equation parameters ω_i, $i = 0, 1, 2, 3$, are 1.0231E–6, 0.9239, 0.0800 and 0.0335. "Index" indicates that the portfolio is fully invested in the simulated index. The other three portfolios' limits are, respectively: 5% VAR at 1.2415%; 5% MLL at 1.7186%; and 5% MLL + 5% SDLL at 2.2857%.

In summary, the results for the Garch returns are similar to the Gaussian returns: the VAR measure works as well as the other measures due to a lack of large losses. Again, the results are consistent with the theoretical work on risk measures. Conditional heteroskedasticity does not seem to significantly reduce the effectiveness of the VAR as a risk management tool as long as the VAR estimator can capture the Arch effect.

AR(1)–Agarch(1, 1)–jump model

The main weakness of the value-at-risk is that, when there are large losses, it does not distinguish between two losses of different sizes. To demonstrate this, we add a negative jump component to the above AR(1)–asymmetric Garch(1, 1) model (see, for example, Bates (1991, 1996) for illustrations of jump models). The frequency of a jump in returns follows a Poisson process with parameter λ, and the size of the jump is the absolute value of a draw from a normal distribution with mean μ_J and standard deviation δ_J. To check their impact on the performance of risk measures, we set λ to 0.01 – which corresponds to one jump in 100 days – the mean jump size, μ_J, to 0 and the standard deviation of the jump, δ_J, to be four times the standard deviation of the normal distribution in the AR(1) model (4.98%). Since the negative jump is likely to reduce the mean of the return process, we add the mean of the jump component ($\sqrt{\pi}\,\lambda\delta_J/2$) to the data-generating process.

Table 5 reports the performance of risk-managed portfolios with the Garch–jump returns. The strategy using the VAR measure yields a mean smaller than those from the MLL and the MLL + SDLL strategies. The standard deviations

TABLE 5 Performance of risk-managed portfolios with Garch–jump returns

	Strategy			
	Index	**VAR**	**MLL**	**MLL + SDLL**
Mean ($\times 10^{-3}$)	0.3650	0.1934	0.2257	0.2368
Standard deviation	0.0226	0.0099	0.0072	0.0071
Sharpe ratio	0.0161	0.0196	0.0315	0.0334
Maximum return	0.0478	0.0213	0.0317	0.0317
Minimum return	−0.1510	−0.846	−0.0647	−0.0544
Loss				
> 3%	6.57	1.23	1.02	0.56
> 2%	8.23	2.56	1.26	1.20
> 1%	15.77	6.65	4.23	3.22

This table reports the performance of risk-managed portfolios using various left-tail measures under the Garch–jump return process. The jump intensity, λ, is 0.01 and standard deviation of the jump size, σ_J, is 4.98%. The Garch model is specified by Equation (15). The estimated parameters for the mean equation (a_0 and a_1) are 0.0002 and 0.1491. The Garch equation parameters ω_i, $i = 0, 1, 2, 3$, are 1.0231E–6, 0.9239, 0.0800 and 0.0335. "Index" indicates that the portfolio is fully invested in the simulated index. The other three portfolios' limits are, respectively: 5% VAR at 1.2415%; 5% MLL at 1.7186%; and 5% MLL + 5% SDLL at 2.2857%.

from the MLL and MLL + SDLL strategies are also smaller. The risk-managed portfolios all have higher Sharpe ratios than that of the "Index", the MLL strategy having the highest at 0.0315. The MLL strategy reduces the maximum loss from 15.10% to 6.47%, while MLL + SDLL reduces it further to 5.44%. All strategies are able to reduce the occurrence of losses larger than 3%, and the MLL + SDLL strategy reduces it the most, from 6.57% of all returns to 0.56%. For losses larger than 2%, all risk management strategies again reduce the occurrence of losses, the MLL + SDLL strategy reducing it from 8.23% to 1.20%. For losses larger than 1%, we see the same pattern: all risk management strategies reduce the occurrence of losses and MLL + SDLL reduces it the most, from 15.77% to 3.22%.

Thus, when there are jumps in the return process, the VAR strategy no longer appears to work as well as the MLL and MLL + SDLL strategies. Furthermore, the strategy that takes into account the second moment is the most useful in reducing the risk of large losses. Our empirical analysis supports the theoretical arguments of Artzner *et al.* (1999), Basak and Shapiro (2000) and Wang (2000) in finding that the VAR measure misses risk management information when the return process is not normal. One may argue that risk management is most needed when markets are subject to volatility and the risk of jumps. Our results indicate that VAR may not be able to deliver when it is most needed.

Regime-switching model

In this subsection we study the risk management of the left-tail measures under changes of regime. We calibrate the following model to our sample of daily S&P500 Index returns:

$$\begin{aligned} r_t &= a_{0,1} + a_{11} r_{t-1} + \sigma_1 \epsilon_t, \qquad \epsilon_t \,\big|\, I_t \sim N(0,1) \\ r_t &= a_{0,2} + a_{12} r_{t-1} + \sigma_2 \epsilon_t \end{aligned} \tag{16}$$

where σ_1 and σ_2 are the volatilities under state 1 and state 2, respectively. The transition matrix is constant and given by

$$\begin{pmatrix} P & 1-P \\ 1-Q & Q \end{pmatrix}$$

The estimated coefficients for $a_{0,1}$, $a_{0,2}$, a_{11}, and a_{12} (with standard errors in parentheses) are 0.0004 (8.081E−5), 2.368E−5 (0.0003), 0.1409 (0.0134), and 0.0941 (0.0224). So the return process is weakly autoregressive under both states. σ_1 and σ_2 are estimated to be 0.6087% (0.1088%) and 1.4004% (0.3216%). Returns in the second state have more than twice the volatility than returns in the first state, while the probability of staying in state 1 (P) is 0.9862 (0.0022) and the probability of staying in state 2 (Q) is lower at 0.9572 (0.0073). Hence, volatility is persistent in both states but slightly less so in the high-volatility state.

TABLE 6 Performance of risk-managed portfolios with regime-switching returns

	Strategy			
	Index	**VAR**	**MLL**	**MLL + SDLL**
Mean ($\times 10^{-3}$)	0.4038	0.3979	0.4071	0.4112
Standard deviation	0.0076	0.0072	0.0073	0.0071
Sharpe ratio	0.0531	0.0555	0.0558	0.0561
Maximum return	0.0271	0.0224	0.0226	0.0226
Minimum return	−0.1510	−0.846	−0.0647	−0.0544
Loss				
> 3%	0	0	0	0
> 2%	1.46	1.43	1.40	1.40
> 1%	6.49	5.69	4.61	5.60

This table reports the performance of risk-managed portfolios using various left-tail measures under the regime-switch return process. "Index" indicates that the portfolio is fully invested in the simulated index. The other three portfolios' limits are, respectively: 5% VAR at 1.2415%; 5% MLL at 1.7186%; and 5% MLL + 5% SDLL at 2.2857%.

Table 6 reports the performance of risk-managed portfolios with regime-switching returns. The risk management strategies marginally reduce the means of portfolio returns, but they also result in smaller standard deviations of returns than that of the "Index" portfolio. The net effect on the Sharpe ratio is slightly positive. There are no returns lower than the -3% mark. For losses larger than 2%, it seems that all risk management strategies can reduce their occurrence from 1.46% of all returns to 1.40–1.43%. For losses larger than 1%, all risk management strategies reduce the occurrence from 6.49% to 4.61–5.69%. The overall results appear similar to those from the Gaussian return process. The VAR measure is adequate and the MLL and MLL + SDLL measures perform marginally better in reducing the occurrence of large losses.

Robustness check with interest rate processes

As a robustness check, we apply our risk management analysis to two interest rate series. We first calibrate the yields on one-year Treasury bills and 10-year Treasury notes to the AR(1)–asymmetric Garch(1, 1) model of Glosten, Jagannathan and Runkle (1993). Risk management limits exposure to interest rate risks according to the left-tail measures of the change in yield. Let B_t be the value of bond and y_t be its yield to maturity. Then it is straightforward to show that

$$\frac{\mathrm{d}B_t}{B_t} = -D^*\mathrm{d}y_t$$

where D^* is the modified duration of the bond. Hence, the return to the bond is linearly related to the change in yield. By moving funds between a cash account

TABLE 7 Performance of risk-managed interest rate portfolios

		Strategy		
	Yield change	**VAR**	**MLL**	**MLL + SDLL**
One-year Treasury bills				
Mean ($\times 10^{-2}$)	0.1131	0.1108	0.1107	0.1120
Standard deviation	0.0501	0.0386	0.0374	0.0373
Maximum yield change	0.2804	0.1312	0.1302	0.1427
Minimum yield change	−0.2622	−0.1921	−0.1888	−0.1888
Yield decline				
>0.25%	0.22	0	0	0
>0.15%	1.42	0.45	0.39	0.21
>0.05%	10.43	8.22	7.81	7.62
10-year Treasury notes				
Mean ($\times 10^{-2}$)	0.1078	0.1461	0.1043	0.1034
Standard deviation	0.0562	0.0422	0.0416	0.0415
Maximum yield change	0.3166	0.1408	0.1478	0.1644
Minimum yield change	−0.2955	−0.2125	−0.2125	−0.2125
Yield decline				
>0.25%	0.43	0	0	0
>0.15%	1.83	0.65	0.43	0.39
>0.05%	12.29	10.85	10.26	9.98

This table reports the performance of risk-managed interest rate portfolios using various left-tail measures from the AR(1)–asymmetric Garch(1,1) process. "Yield change" indicates that the portfolio is fully exposed to the simulated yield change. The other three portfolios limit exposure to yield changes at, respectively, for one-year Treasury bills (top panel): 5% VAR at 0.06561%; 5% MLL at 0.07484%; and 5% MLL + 5% SDLL at 0.08561%; and for 10-year Treasury notes (bottom panel): 5% VAR at 0.06939%; 5% MLL at 0.08242%; and 5% MLL + 5% SDLL at 0.09195%.

and a bond fund, a risk manager can change the exposure of the bond portfolio to interest rate change.

Table 7 reports the performance of the left-tail measures in limiting interest rate risks. All are able to reduce the exposure to large declines in yield.[7] For the one-year Treasury bills, all strategies reduce exposure to yield declines larger than 25 basis points (bp) from 0.22% of all yields to zero. For declines in yields larger than 15bp, the VAR measure can reduce the occurrence losses from 1.42% to 0.45%, while the MLL and MLL + SDLL measures can reduce the occurrence further to 0.39% and 0.21%, respectively. Similar results hold for the 10-year Treasury notes. In summary, the MLL and SDLL measures are marginally useful in reducing exposure to large interest rate changes.

6 Conclusions

Several recent articles on risk management indicate that a quantile measure of losses such as value-at-risk may not contain enough, or the right, information for risk managers. Basak and Shapiro (2000) showed that risk managers using VAR methods often optimally choose a larger exposure to risky assets than non-risk managers. Consequently they incur larger losses when losses occur. In this paper we examined a set of left-tail measures (LTMs): the mean and standard deviation of a loss larger than the VAR (MLL and SDLL), as well as the VAR. We investigated the empirical dynamics of the LTMs and found that the quantile regression method provides an easy-to-use framework for estimating all the LTMs. Our Monte Carlo simulation showed that the VAR is appropriate for risk management when returns follow Gaussian processes, but that the MLL strategy and strategies taking into account the SDLL are useful in reducing the risk of large losses under non-normal distributions and when there are jumps in asset prices.

1 Duffie and Pan (1997) provide an excellent survey on this topic. The most common approaches in estimating VAR are sample quantile methods, eg, historical simulation, and those based on the assumption of a conditionally normal stock return distribution. For the latter approach, the estimation of VAR is equivalent to estimating the conditional volatility of returns, often by utilizing the Arch-class of models. Other approaches to estimating VAR include the hybrid method of Boudoukh, Richardson and Whitelaw (1997, 1998), the approach based on the stable Paretian distributions of Khindanova, Rachev and Schwartz (2000), the method based on extreme value theory (Boos, 1984; McNeil, 1998; and Neftci, 2000), and the quantile regression approach of Engle and Manganelli (1999).

2 See Koenker and Bassett (1982), Powell (1986), Gutenbrunner and Jureckova (1992), and Buchinsky (1994) among others for subsequent development in quantile regression theory.

3 In the method (quantile regression) that we describe below, we estimate $\gamma(\tau)$ instead of γ (ie, $\gamma_j Q_\varepsilon(\tau)$, $j = 0, 1, \ldots, q$ are estimated instead of γ_j) with no specific assumption on the distributional form for ε.

4 Just as in the median regression – since the quantile regression estimation is based on a linear regression technique – corner solutions are possible. In the special case with a constant regressor, if τn is an integer, there will be an interval of τth sample quantiles between two adjacent-order statistics. If τn is not an integer, the τth sample quantile is unique.

5 Computation of the regression quantiles by linear programming is efficient, and it is straightforward to impose non-negativity constraints on all elements of γ. Barrodale and Roberts (1974) proposed the first efficient algorithm for L_1 estimation problems based on modified simplex method, and Koenker and d'Orey (1987) modified their algorithm to solve quantile regression problems. Portnoy and Koenker (1997) describe a modified simplex approach which combines statistical preprocessing with interior point methods to make computation faster.

6 We are being conservative here. A positive risk-free return can only improve the performance of the risk-managed portfolios. In performance measures such as the Sharpe ratio, a small positive risk-free return will not change the relative rankings of various risk-managed portfolios.

7 Similar results can be derived if we are mainly concerned with a yield increase.

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