Hidden and Displayed Liquidity in Securities Markets with Informed Liquidity Providers

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Abstract

We examine the impact on the quality of a securities market of hiding versus displaying orders that provide liquidity. Display expropriates informational rents from informed agents who trade as liquidity providers. The informed then exit liquidity provision in favor of demanding liquidity where they trade less aggressively. Trading costs to uninformed liquidity demanders are higher, bid-ask spreads are wider and midquotes are less informationally efficient when orders that provide liquidity are displayed. Our analysis suggests that market innovations, which might seem to favor the informed over the uninformed, can enhance market quality by intensifying competition among the informed.

Introduction

A sizeable fraction of all liquidity provided in securities markets is hidden. The SEC estimates that nearly 30% of transaction volume in listed stocks is traded in venues that neither display liquidity nor make it available for execution against public orders.[1] Hidden orders are also important on public exchanges. The Nasdaq website reports that more than 15% of Nasdaq's order flow is non-displayed.[2] Frey and Sandas (2008) report that 16% of shares executed in their sample of Frankfurt stocks have a hidden component to the order, and Bessembinder, Panayides and Venkataraman (2009) report that 44% of the volume in their sample of Euronext-Paris stocks is hidden. Almost all exchanges allow traders to hide limit orders that provide liquidity either partially, as reserve or iceberg orders, or in full (e.g., dark reserve orders on the NYSE). However, hidden limit orders are often given lower priority than displayed orders, suggesting that exchanges view hidden orders that provide liquidity (hereafter "hidden liquidity") as less desirable than orders that are displayed ("displayed liquidity").

How to regard hidden liquidity is a challenge for regulators as well. In addressing the fragmentation of trading between exchanges and off-exchange systems where liquidity is almost uniformly hidden, the SEC Chairman warns, "...we must be careful that the short-term advantages to individual traders from non-transparent trading does not undermine all investors in the long run by compromising the essential price discovery function of the public markets."[3]

This perspective has also been expressed forcefully in relation to hidden liquidity on exchanges. In November of 2011, the SEC proposed rule changes to accommodate a pilot program that allows hidden orders within the one-penny tick size to step ahead of displayed quotes on the NYSE. This applies to order flow the exchange deems as having a retail source. The proposal has the potential to make hidden orders the counterparty to the vast majority of qualifying retail transactions. In a comment letter that mirrors several

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[1] Mary L. Shapiro, Speech by the SEC Chairman: Strengthening our Equity Market Structure, Economic Club of New York, September 7, 2010.

[2] March 2012 headline on *www.nasdaqtrader.com/Trader.aspx?id=DarkLiquidity*.

[3] Shapiro op. cit.

of those submitted, the CFA Institute raises concerns that the relative disincentive this provides to displayed liquidity will degrade market quality, having even "...wider adverse consequences for market integrity beyond the initial loss of transparency."[4] The concern is that the strategic use of hidden orders confers advantages over displayed orders, which might deter competition and reduce market quality. A more benign view is taken by Angel, Harris and Spatt (2011) who argue that hidden orders are merely the modern, computerized equivalent of the unexpressed orders in the trading crowd that competed with orders in the book when trading was conducted in person on trading floors.

The inescapable fact is that market participants have revealed a preference to trade dark both on and off exchanges. Competition among trading systems has made supplying richer means by which to hide liquidity a key element of how exchanges attract orders. The concerns of regulators and market professionals that this might result in a race to the bottom are not totally unfounded. A large body of microstructure research indicates that market quality is impaired by informed traders who camouflage or hide their market-order trading among that of the uninformed. Hidden limit orders would seem to reinforce such opportunities by allowing the informed both to provide liquidity and to profit from their information, suggesting that hidden limit orders degrade market quality. Therefore, a question that is very relevant to markets and regulators is whether hidden orders are an innovation that enhances market quality or merely a way for informed traders to leverage their advantage to the detriment of other market participants.

We address this question in a model where informed traders' strategies adapt to whether liquidity is hidden or displayed. An important feature of our analysis is accounting *jointly* for informed traders' selections between providing or demanding liquidity and the intensity with which each pursues the type of strategy he chooses. Both aspects determine the degree of competition among traders, and the impact of hiding or displaying liquidity on market quality depends on the interaction between these effects.

The specific questions we address are how hidden liquidity affects the execution costs

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[4]The proposal (SR-NYSE-2011-55 dated November 2, 2011) and comment letter (dated November 30, 2011) are available under the NYSE rulemaking archive of the SEC at *www.sec.gov*. The proposal was approved in July, 2012.

of uninformed traders and the informational efficiency of prices. The argument in favor of hiding orders is that doing so attracts traders who would not provide liquidity in displayed markets, thus increasing competition among liquidity providers. The argument against is that hidden liquidity enhances the advantage of informed traders over the uninformed by enabling the informed explicitly to conceal their orders.

Our principal finding is that hidden liquidity has a favorable impact on market quality. The increase in competition resulting from the migration of informed agents toward liquidity provision dominates the profit enhancement associated with the concealment of their orders. This happens because all the informed agents trade more aggressively on their information when more of them trade as liquidity providers. Since hiding liquidity draws more informed agents into liquidity provision, competition intensifies overall and market quality improves.

When liquidity is displayed, less informed agents are drawn into liquidity provision and more of them trade as liquidity demanders. The informed trade less aggressively and market quality is lower than when liquidity is hidden. The quality of the market with hidden liquidity is better because more informed agents are drawn into liquidity provision *and* they trade more aggressively as liquidity providers than they would as liquidity demanders in the displayed market—i.e., the choices of strategy type and the aggressiveness of trading *jointly* determine market quality.

Why are the informed drawn in when liquidity is hidden? The choice to trade as a liquidity provider in a hidden market is analogous to the choice of a cartel member to exceed his production quota. With strategic liquidity providers, there are rents to providing liquidity in equilibrium. When liquidity is hidden, each informed trader is attracted to liquidity provision to capture this rent even though, collectively, their profits would be higher if some resisted and traded as liquidity demanders.

In contrast, when liquidity is displayed, this decision involves a tradeoff. Providing liquidity captures the rent but also conveys some of the trader's private information to liquidity demanders who observe the liquidity being offered before submitting their orders. The existence of this tradeoff results in less informed traders being drawn into liquidity provision when liquidity is displayed. Using the cartel analogy, displaying liquidity better

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aligns individuals' incentives with the interest of the cartel—it acts as a coordination device that deters entry. The expected profit of informed traders, both those who supply and those who demand liquidity, is greater if liquidity is displayed than if it is hidden. These higher profits are realized at the expense of the uninformed traders.

The broader implication of our results is that trading innovations that convey an advantage to the informed need not be harmful to the uninformed. If an innovation also intensifies competition among the informed, the uninformed could benefit with lower trading costs and more efficient prices as they do in our model when liquidity is hidden. This is a useful perspective for brokers, markets and regulators who are concerned with how innovations in trading technology affect their clients and the market environment. Our model suggests that low trading costs and efficient pricing are ultimately a consequence of intense competition—by trading more aggressively on their information, traders compete away its rents. Innovations that appear to favor one group of traders over another can be beneficial if the innovation also intensifies competition among traders within the advantaged group. We discuss in the conclusion how this relates to the controversy over whether high-frequency trading impairs market quality.

Our paper contributes to the literatures on informed liquidity provision and hidden liquidity. In classic models of securities trading, competitive uninformed agents provide liquidity to strategic informed agents [see Glosten and Milgrom (1985), Kyle (1985), Easley and O'Hara (1987) and Glosten (1994)]. However, the evidence from markets and experiments supports the hypothesis that informed traders do participate in securities markets as liquidity providers [e.g., Kavajez and Odders-White (2001), Bloomfield, O'Hara and Saar (2005), Kaniel and Liu (2006) and Rourke (2009)].

Several theoretical papers consider markets with a *single* informed liquidity provider and show that liquidity provision is desirable as a strategy for exploiting an informational advantage [see Kumar and Seppi (1993), Chakravarty and Holden (1995), Kaniel and Liu (2006) and Foucault, Moinas and Theissen (2007)]. These studies do not capture the strategic competition among informed liquidity providers, and between liquidity providers and liquidity demanders, that is key to our results. Bernhardt and Taub (2006) compare markets in which multiple informed agents *all* trade as either liquidity providers as in Kyle

(1989) or liquidity demanders as in Kyle (1985). They show that informational rents are smaller when the informed trade as liquidity providers. However, they do not consider settings in which informed liquidity providers and informed liquidity demanders *coexist.* Goetler, Parlour and Rajan (2009) simulate a model in which informed liquidity providers and demanders arrive sequentially and interact through the evolving contents of the limit order book. They examine the dynamics of prices and spreads, but they do not compare markets with hidden and displayed liquidity.

Transparency of the order book is examined by Madhavan, Porter and Weaver (1999) and Baruch (2005). They compare a market structure in which liquidity is displayed to one in which liquidity is hidden. In their models, a specialist observes the contents of the order book and exercises greater monopoly power when liquidity is hidden from other traders. Whether liquidity is hidden or displayed affects transaction costs through its effect on the monopoly power of the specialist. The order book does not contain information about the security's value, however, because informed traders do not provide liquidity. Butti and Rindi (2011) model the choices of uninformed traders either to hide or display orders. There is no private information about the security's value, however; so the informational efficiency of prices and adverse selection costs are not addressed.

Moinas (2011) models a setting with adverse selection, in which an informed trader is assumed to trade as a liquidity provider. Moinas shows that hiding liquidity enhances the opportunity for the informed trader to camouflage himself among other liquidity providers who are uninformed. This results in a deeper book over a larger range of the severity of adverse selection than when the display of liquidity is mandated. Moinas's result is similar in spirit to ours. However, the enhancement to market quality exists for all levels of adverse selection in our model because we capture the *migration* of informed traders toward liquidity provision that occurs when liquidity is hidden versus displayed and its impact on the intensity of competition.

The evidence is mixed as to whether displaying or disseminating information about liquidity enhances market quality. Among the studies that examine natural experiments in which a market's rules change, Madhavan, Porter and Weaver (2005) and Hendershott, Jones and Menkveld (2011) find that measures of market quality decrease with additional

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transparency. Alternatively, Boehmer, Saar and Yu (2005) and Hendershott and Jones (2005) find that market quality improves.[5] Frey and Sandas (2008) and Bessembinder, Panayides and Venkataraman (2009) examine cross-sections of stocks in European markets, and they compare executions involving hidden and displayed liquidity where both exist. Both studies find that hidden orders are an important source of liquidity in creating transaction volume, and that the presence of hidden liquidity contributes to market quality.

Some very recent work examines theoretically how market quality is affected by crossing networks that clear trades at prices determined on exchanges—so-called "dark pools" [see Butti, Rindi and Werner (2011), Ye (2011) and Zhu (2011)]. These papers address how *adding* a satellite venue with hidden liquidity that does not contribute to price discovery affects traders' opportunities and market quality. Our model examines hidden versus displayed liquidity in the primary market where price discovery occurs.

The next section presents the general setting of our model. Sections 2 and 3 compare markets with hidden and displayed liquidity. Section 4 examines an interesting special case in which hidden and displayed markets are equivalent. Section 5 concludes. Proofs of propositions appear in the Appendix.

1. Model

We model a market in which a single security is traded. The security is in zero net supply, and its terminal payoff is $\tilde{v} = \tilde{v}_1 + \tilde{v}_2 + \cdots + \tilde{v}_M$, where the $\tilde{v}_i$s are iid $N(0, \sigma_{vi}^2)$. There are M risk-neutral informed agents who each observes a private signal s_i, where $s_i = v_i$ in most of the settings we analyze. Of these, $J \leq M$ agents trade as liquidity providers by submitting supply schedules to a centralized market. For simplicity of exposition we refer to these agents as "dealers." The price contingent schedule of orders of dealer j is denoted by $y_j(s_j, p)$ where $y > 0$ indicates a quantity to sell.[6] The collection of such schedules submitted by the J dealers constitutes the liquidity on offer in the order book. The remaining $N = M - J$ agents trade as liquidity demanders ("non-dealers"). They

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[5]Experimental evidence is similarly mixed [see Gozluklu (2010) and Bloomfield, O'Hara and Saar (2011)].
[6]Such schedules can be implemented as collections of limit orders.

submit market orders that are executed against the book at a uniform price. The market order of non-dealer i is denoted by $x_i(s_i)$ where $x > 0$ indicates a quantity purchased. Finally, there are uninformed traders who submit net market orders of $\tilde{u} \sim N(0, \sigma_u^2)$, which is uncorrelated with the security's value.

Dealers and non-dealers are assumed to maximize expected profit. The distinction between them is simply that the dealers submit supply schedules that build the book, and the non-dealers place market orders that execute against the book. If the order book is displayed, the schedule of dealers' orders is visible to the non-dealers before they submit their market orders. If there is useful information in the order book, the non-dealers' orders will condition upon it. Alternatively, if the order book is hidden, non-dealers cannot condition their orders on information in the order book. An equilibrium assignment of agents between supplying and demanding liquidity is defined as the value J_* so that when J_* agents trade as dealers, no individual dealer or non-dealer expects greater profit by switching order types.[7]

We focus on symmetric Bayesian Nash equilibria. Therefore, we seek a pair of strategies $\{y(\cdot), x(\cdot)\}$ that maximizes the expected profit of dealer j and non-dealer i, respectively, given that: other dealers and informed non-dealers are conjectured to follow these strategies, beliefs are consistent with Bayes rule and markets clear. The market clearing condition evaluated at the equilibrium strategies implicitly defines the uniform price at which orders clear.[8] We write the market clearing price as $\tilde{P} = \tilde{P}(\tilde{\omega})$, where the tilde on the functional $P(\cdot)$ emphasizes that the price is random for reasons apart from the randomness in total market orders $\omega \equiv \sum_{i=J+1}^{M} x_i + u$. Since the dealers' orders are condi-

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[7]In unreported results, we consider a setting in which *dealers* also can condition on the order book when it is displayed. Liquidity provision is deterred by display to an even greater degree in this setting because dealers' information is disseminated to the non-dealers *and to other dealers* when the book is displayed. The results reported below may therefore provide a conservative assessment of the deterrent effect of display on liquidity provision. Details are available from the authors.

[8]Back and Baruch (2007) show that when liquidity provision is perfectly competitive, there is an equivalence between dynamic markets that utilize uniform and discriminatory pricing—i.e., whether all orders clear at a single price, or whether orders walk up or down the book executing at distinct prices. Our liquidity providers are not perfectly competitive, so we do not know whether an equilibrium in a model with discriminatory pricing is equivalent to ours. However, the intuition behind our results depends on the existence of rents to providing liquidity, and not on the pricing convention by which those rents are captured.

tioned on their private information, the price *schedule* $\tilde{P}(\cdot)$ can depend on the realizations of the dealers' information signals. All agents take this into account when choosing their strategies.

Our model builds on Kyle (1985) and Dennert (1993). As in Dennert, we relax the assumption that liquidity providers are perfectly competitive.[9] In contrast to Kyle's and Dennert's models, dealers are allowed to be informed in our model and multiple informed non-dealers exist as counterparties to dealers' orders. The model can be solved in closed form. In each setting that we consider, a pure strategy equilibrium exists in linear strategies for any allocation of informed agents between liquidity provision and liquidity demand, except those where liquidity providers are near monopolists.[10] Because the dealers' strategies are linear, the market clearing condition can be inverted to obtain a simple expression for the price schedule.

2. A Market with Hidden Liquidity

We consider first a market in which the non-dealers cannot observe the order book before submitting their orders. In the next section we consider a market in which liquidity is displayed, and there we compare the two markets.

Dealer j selects his supply schedule $y_j(s_j, p)$ to maximize expected profit

$$\max_{y_j(\cdot)} E\left[(\tilde{P} - \tilde{v})\, y_j(v_j, \tilde{P}) \,|\, v_j\right], \tag{1}$$

taking as given his signal $s_j = v_j$, the strategies of all other agents and market clearing. Because liquidity is hidden, non-dealers do not observe the order book, and they condition their beliefs only on their private signals. Non-dealer i selects his market order $x_i(s_i)$ by solving

$$\max_{x_i} E\left[(\tilde{v} - \tilde{P})\, x_i \,|\, v_i\right], \tag{2}$$

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[9]Other models that relax the assumption of perfect competition are Glosten (1989), Biais, Martimort and Rochet (2000) and Bondarenko (2001). However, they retain the assumption that dealers are uninformed and, as in Dennert, a single informed trader demands liquidity.

[10]We do not constrain strategies to be linear. We show that linear strategies are optimal when agents conjecture that others follow linear strategies.

taking as given his signal $s_i = v_i$, the strategies of all other agents and market clearing. The market clearing condition is

$$\sum_{j=1}^{J} y_j(\tilde{v}_j, \tilde{P}) = \sum_{i=J+1}^{M} x_i(\tilde{v}_i) + \tilde{u} \tag{3}$$

whatever are the realizations of $\tilde{v}_1, \ldots, \tilde{v}_M$ and $\tilde{u}$. Our first result characterizes the equilibrium for any number of dealers, J. Later, we use traders' expected profit functions conditional on J to solve for J_*.

PROPOSITION 1. *Assume that $J > 2$ traders provide liquidity and $M - J$ demand liquidity.[11] There exists a symmetric equilibrium in which order strategies are linear, and this is the unique symmetric equilibrium in which strategies are linear. Explicit expressions for the strategies are*

$$y(v_j, P) = \gamma v_j + \delta P \qquad \text{for } j = 1, \ldots, J$$

$$x(v_i) = \beta v_i \qquad \text{for } i = J+1, \ldots, M$$

where

$$\gamma = -\beta, \qquad \beta = \frac{J\delta}{2}, \qquad \text{and} \qquad J\delta = \frac{2\sigma_u}{\sigma_{vi}}\sqrt{\frac{(J-2)}{J(M-1)}}.$$

The dealers' and non-dealers' equilibrium expected profits, denoted by π_j and π_i respectively, satisfy

$$J\pi_j + (M-J)\pi_i = \frac{\sigma_u^2}{J\delta} \qquad \text{and} \qquad \pi_i = \frac{\beta\sigma_{vi}^2}{2}.$$

The dealers submit orders to sell more at high prices than at low prices ($\delta > 0$), and the non-dealers submit orders that are increasing in their private signals ($\beta > 0$) as in Kyle (1985). However, because the dealers are informed, they shift their supply schedules in accordance with their private signals—the greater is the signal, the less the dealer supplies at each possible price ($\gamma < 0$).

The orders of dealers and non-dealers are equally aggressive in exploiting their information signals ($|\gamma| = |\beta|$). This is unique to the case in which liquidity is hidden. When

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[11] The condition $J>2$ is necessary because, if $J\leq 2$, there is insufficient competition among the dealers to produce a price schedule whose slope is finite. Two or less dealers can earn infinite expected profit from the uninformed whose trading is price inelastic. This is discussed in Kyle (1989). Bernhardt and Hughson (1997) study the case of two uninformed dealers in detail and show that an equilibrium can exist with two dealers if uninformed market-order trading is price elastic.

liquidity is displayed, dealers hold back to prevent their information from being revealed via the order book to the non-dealers who observe it.

The aggregate expected profit of the dealers and non-dealers equals the expected loss of the uninformed traders, which is equal to $\sigma_u^2/J\delta$ in equilibrium. This is increasing in the expected magnitude of uninformed trading, σ_u, and decreasing in the aggressiveness with which the dealers and non-dealers trade on their signals, $J\delta$, because the magnitudes of both β and γ are proportional to $J\delta$. The more aggressively the informed trade, the less the price will deviate from the security's terminal payoff, and the smaller will be the uninformed traders' expected loss.

What determines how aggressively the informed trade on their signals? Fixing M, it is easy to show that $J\delta$ is increasing in J. This means that both the dealers and the non-dealers trade more aggressively on their private signals the greater is the concentration of informed traders on the liquidity provision side of the market. This also reduces the expected loss of the uninformed traders because $\sigma_u^2/J\delta$ is decreasing in J. In contrast, if the concentration of informed traders shifts toward the liquidity demand side of the market, both the dealers and the non-dealers trade less aggressively on their signals, and the expected loss to the uninformed is larger.

The degree to which competition among the informed is concentrated on the liquidity provision side of the market also determines important properties of prices, specifically how fully dealers' information is impounded into prices and how sensitive the market clearing price is to order flow. The price schedule is obtained by substituting the equilibrium orders of dealers into the market clearing condition,

$$\gamma \sum_{j=1}^{J} v_j + J\delta P = \omega, \tag{4}$$

where $\omega \equiv \sum_{i=J+1}^{M} x_i + u$ is the flow of market orders. The price schedule (written as a function of market-order flow) is therefore

$$P(\omega) = \frac{1}{2} \sum_{j=1}^{J} v_j + \frac{1}{J\delta}\, \omega. \tag{5}$$

Because dealers are informed, the realized price is affected by two sources of randomness—the information of dealers, $\sum_{j=1}^{J} v_j$, and the flow of market orders, ω.

The factor 1/2 in the intercept is important. It implies that the information observed by dealers is only partially impounded into the price schedule—the intercept is half way between the unconditional expectation of $\tilde{v}$ (which is zero) and the information collectively observed by the dealers. If $\sum_{j=1}^{J} v_j$ were common knowledge, the factor would be unity rather than 1/2, and the intercept of the price schedule would fully reflect the information collectively observed by the dealers. The factor is 1/2 because dealers strategically moderate their trading to prevent the rents to their information from being dissipated by fully impounding their private information into the price schedule.

Nevertheless, if an agent were able to observe directly the intercept of the price schedule, he could infer the dealers' aggregate signal. This would be useful in formulating his own trading strategy because the aggregate signal is only partially impounded into the price schedule. This is why, when liquidity is displayed, non-dealers can *potentially* use information gleaned from the order book in formulating their market-order strategies. We write *potentially* because Equation (5) is the equilibrium price schedule in a hidden market. It remains to be shown that a similar situation occurs in a market where the order book is visible to non-dealers—i.e., we have to verify that the order book contains useful information when it is common knowledge that the non-dealers can observe and use information from the order book in formulating their trading strategies.

Even though the factor 1/2 is invariant to J, the number of signals in the sum depends on J. As more agents elect to trade as dealers, more of the security's payoff will be reflected (by half) in the price schedule. In addition, the slope of the price schedule is flatter when J is greater because $1/J\delta$ is decreasing in J—the more dealers there are, the smaller is the price impact of each unit of market order flow. As noted earlier, this reduces the expected loss of the uninformed.

The bid-ask spread is narrower as well. In fact, the bid-ask spread and the expected loss of the uninformed are equivalent measures of market quality in all the settings of our model. The spread, defined as the difference between the prices to buy and sell small quantities, such as $P(+1)-P(-1)$, is given by $2/J\delta$. This is proportional to the uninformed traders' expected loss of $\sigma_u^2/J\delta$. If the expected loss of the uninformed is larger in one setting of our model than another, the bid-ask spread is wider as well.

2.1 Equilibrium Liquidity Provision and Demand

The foregoing discussion establishes that the number of informed agents who trade as liquidity providers, J, is key to determining how aggressively the informed trade on their signals, how much information is reflected in the price schedule, the slope of the price schedule, and the expected loss suffered by the uninformed. To identify the equilibrium value J_*, we let $\pi_j(J)$ and $\pi_i(J)$ denote the expected profit of individual dealer j and non-dealer i, respectively, given that J agents trade as dealers.

It turns out that the expected profit of individual dealers is greater than that of non-dealers over the entire feasible range of J when liquidity is hidden—i.e., $\pi_j(J) > \pi_i(J)$ over $2 < J \leq M$. This is because, at the margin, trading as a dealer enables the agent to earn a rent from providing liquidity. This implies that $J_{*,hidden} = M$—at the equilibrium of a market with hidden liquidity, all M informed agents trade as dealers.

When more informed agents trade as dealers, it benefits the uninformed—their expected loss is decreasing in J over $2 < J \leq M$. So the most favorable pricing possible for uninformed traders is achieved when $J = M$. This, and the fact that $J_{*,hidden} = M$, imply that competition on the basis of order *type* (in addition to the usual competition in quantities given order type) gives rise to an equilibrium that *minimizes* the expected loss sustained by the uninformed. These assertions are summarized in the following proposition and proved in the Appendix.

PROPOSITION 2. *When liquidity is hidden, $J_{*,hidden} = M$; i.e., all the informed agents provide liquidity. This outcome minimizes the expected loss sustained by the uninformed across all possible assignments of informed traders to order types.*

This complements a result that Glosten (1994) obtains in a very different setting—one where dealers are uninformed, and the informed are constrained to trade via market orders. Glosten shows that at the limit of perfect competition, the price schedule arising from uninformed dealers is maximally competitive in the following sense. If an uninformed dealer or trading venue were to improve on this price schedule, for any quantity, it would earn unconditional expected losses, holding fixed the composition of traders generating market orders. In other words, no other "uninformed method" of liquidity provision can

survive if it prices more aggressively than Glosten's, when faced with the adverse-selection risk posed by a fixed set of market order traders.

Our result relates to a different comparison, but the qualitative conclusion is similar. We consider explicitly that informed agents can trade either as liquidity providers or liquidity demanders, and we account for the optimal strategies (within the linear class) that they adopt in each role. We show that among all possible assignments of informed agents to roles, the assignment that emerges at the equilibrium of a market with hidden liquidity is maximally competitive.[12] Thus, the spirit of Glosten's conclusion does not require uninformed liquidity providers or perfect competition. It extends to a setting where *informed* traders can demand or supply liquidity, and competition among liquidity providers is *imperfect*.[13]

As described in the introduction, endogenizing liquidity provision gives rise to a "prisoners' dilemma" that applies to order *types*. It is individually rational for the informed agents to trade as dealers to capture rents to liquidity provision, but doing so leads all the informed to trade aggressively and reduces their expected profit. This benefits the uninformed traders. In a market where liquidity is hidden, *all* the informed agents choose to trade as dealers. This means that across all possible assignments of informed agents to order types, the expected loss suffered by the uninformed is minimized at the assignment achieved *at equilibrium* when liquidity is hidden.

Even though the rents to providing liquidity are minimized at equilibrium, they are still positive because the dealers are not perfectly competitive. This implies that the price schedule in the order book is steeper than in a Kyle (1985)-type market with perfect competition among uninformed liquidity providers and informed traders who submit only market orders. It is precisely the non-zero rents to liquidity provision that draw the

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[12]Since liquidity provision is not perfectly competitive in our model, it would obviously be possible to undercut the price schedule and still earn positive expected profit. However, doing so is not optimal for the agents in our model. See Sandas (2001) for evidence that liquidity supply is not perfectly competitive in equities trading.

[13]Our results would be unaffected by also including a finite number of uninformed liquidity providers. Such agents would compete against the informed dealers. The scale of the rents associated with providing liquidity would be smaller. However, the informed agents would still perceive an incremental rent to providing liquidity and all trade as dealers.

informed agents into trading as liquidity providers in our model. This, in turn, causes information to be incorporated into *quoted* prices. While those who demand liquidity pay greater transaction costs relative to the midquote, the midquote itself more closely approximates the security's true value. This is summarized in the next result, which uses the notation $\sigma_v^2 = \text{Var}\,[\tilde{v}]$.

PROPOSITION 3. *In the equilibrium of Proposition 2, the expected loss of the uninformed is given by $\frac{\sigma_u \sigma_v}{2}\sqrt{\frac{(M-1)}{(M-2)}}$, which is greater than the expected loss of the uninformed in a Kyle (1985)-type market in which M informed traders submit market orders to a perfectly competitive, uninformed, risk-neutral market maker: $\frac{\sigma_u \sigma_v}{2}$. In addition, the bid-ask spread, $P(1) - P(-1)$, is wider, and transaction prices are more volatile, in the equilibrium of Proposition 2 than in a Kyle-type market. However, the midquote better approximates the security's value (in a mean-square sense) in the equilibrium of Proposition 2 than in a Kyle-type market.*

3. A Market with Displayed Liquidity

The informed are drawn into liquidity provision by a desire to capture rents to providing liquidity. When liquidity is displayed, a portion of the dealers' information is conveyed to the non-dealers because the non-dealers observe the order book before selecting their orders. This creates a trade off between capturing the rent and giving up profit from private information, which deters some informed agents from choosing to trade as dealers when liquidity is displayed.

This deterrent has important consequences for market quality. The expected loss suffered by the uninformed traders is not minimized across all possible assignments of agents to order types at the equilibrium of a market with displayed liquidity. Moreover, this loss is greater, and the midquote is also less informationally efficient, in the equilibrium of the market with displayed liquidity than in the equilibrium of the market with hidden liquidity.

To justify these assertions, we now consider a market in which the order book is observed by the non-dealers before they submit their orders. This means the non-dealers' orders can be conditioned on both their private signals and on the information conveyed

by the order book. Equation (5) indicates that in the equilibrium of Propositions 1 and 2 (where liquidity is hidden), the dealers' information is aggregated into the intercept of the price schedule, $P(0)$. Using this as a guidepost, we examine whether an equilibrium with this feature exists when non-dealers' strategies also condition on this particular attribute of the order book.

Specifically, we examine whether there is an equilibrium in which (i) non-dealers condition on the intercept of the price schedule *and* (ii) the intercept alone reflects the information of dealers, thus making it rational for non-dealers to focus on only this statistic when they observe the order book. We show that such an equilibrium exists provided, as before, there is sufficient competition among the dealers.

<u>PROPOSITION 4</u>. *Assume that $J > 1 + \psi(J, M)$ traders provide liquidity and $M - J$ demand liquidity. There exists a symmetric equilibrium in which order strategies are linear, and this is the unique symmetric equilibrium in which strategies are linear. Explicit expressions for the strategies are*

$$y(v_j, P) = \gamma v_j + \delta P \qquad \text{for } j = 1, \ldots, J$$

$$x(v_i, P(\cdot)) = \beta_1 v_i + \beta_2 P(0) \qquad \text{for } i = J + 1, \ldots, M$$

where $P(0)$ is the intercept of the equilibrium price schedule $P(\cdot)$, and

$$\beta_1 = \frac{J\delta}{2} \qquad\qquad \beta_2 = -\frac{J\delta}{M - J + 1}\left(1 + \frac{J\delta}{\gamma}\right)$$

$$\gamma = -\left(\frac{J - 1 + \psi(J, M)}{2\ \psi(J, M)}\right)\delta \qquad \delta = \frac{\sigma_u\sqrt{2}}{\sigma_{vi}}\ F(J, M).$$

The functions $\psi(\cdot)$ and $F(\cdot)$ depend only on M and J and are defined in the Appendix. The dealers' and non-dealers' equilibrium expected profits, denoted by π_j and π_i respectively, satisfy

$$J\pi_j + (M - J)\pi_i = \frac{\sigma_u^2}{J\delta},$$

where explicit expressions for π_j and π_i are given in the Appendix.

The differences between this equilibrium and that of Proposition 1 are that $\beta_2 > 0$ and $\psi(\cdot) \geq 1$. If we set $\beta_2 = 0$, then $\psi(\cdot) = 1$ and the expressions here reduce to those in Proposition 1. The β_2 parameter captures the reliance of non-dealers' strategies on the

information they glean from observing the order book. If the dealers' signals are optimistic on average, then this is conveyed to the non-dealers by the fact that $P(0) > 0$. Accordingly, the non-dealers' demand $\beta_2 P(0)$ more units of the security than if $P(0)$ were equal to zero.

The ψ function captures the dealers' strategic response to having their aggregated signals transmitted to the non-dealers via the order book. Equation (4.6) in the Appendix shows that the dealers' strategies can be written as

$$y(v_j, P) = \frac{(J-1)\delta}{\psi} \left\{ P - E\left[\tilde{v} | v_j, \tilde{P} = P \right] \right\} \tag{6}$$

in the market with displayed liquidity. In a market with hidden liquidity, the dealers' strategies can be written in exactly this form with $\psi = 1$. The extent to which ψ exceeds unity is therefore the degree to which dealers scale back their orders because liquidity is displayed rather than hidden. Thus, ψ measures the deterrent effect of displaying liquidity on how aggressively the informed who trade as dealers trade on their perception of mispricing.

If $\psi > 1$ when liquidity is displayed, dealers trade (strictly) less aggressively than when liquidity is hidden, so competition among more of them is needed to produce a price schedule with finite slope (see footnote 11). Thus, the condition $J > 1 + \psi$ for an equilibrium to exist in a market with displayed liquidity is analogous to the condition $J > 2$ in the market with hidden liquidity of Proposition 1.[14] This poses no problem for characterizing equilibrium, however. We illustrate below that the equilibrium $J_{*,displayed}$ is always greater than $1 + \psi$ when $M > 2$.

3.1 Comparing Markets with Hidden and Displayed Liquidity

To explain how displaying liquidity affects the market, we begin with the observation that the hidden-liquidity equilibrium is one of the feasible outcomes in the displayed market. This follows from the fact that $\psi(M, M) = 1$ for all M. If all the informed just happen to trade as dealers ($J = M$), then there are no informed non-dealers—none to glean information from observing the price schedule. In this case, $\psi = 1$ because there is no reason for dealers to scale back their orders. This implies that if $J = M$ in a market

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[14]It turns out that $J > 1 + \psi(J,M)$ is equivalent to $J > \frac{2}{3}(M+1)$.

with displayed liquidity, then traders' strategies, expected profits, and the price schedule are all the same as in the *equilibrium* of an otherwise identical market with hidden liquidity ($J_{*,hidden} = M$).

We show below that some informed traders will in fact migrate away from trading as dealers when liquidity is displayed—i.e, in equilibrium $J_{*,displayed} < M$. The informed back away from trading as dealers because displaying liquidity publicizes their information and expropriates its value. The threat this poses to each dealer extends beyond losing control of his individual piece of information because the price schedule *aggregates* information. If there are many dealers, displaying liquidity conveys a substantial informational advantage to the non-dealers. The threat can therefore be quite large. The potential harm to each individual dealer depends on how many dealers there are, how aggressively they trade, and how aggressively non-dealers trade on the aggregated information they glean from the price schedule (i.e., β_2), all of which are endogenous.

There is a subtle effect at work as well. When the dealers' information is conveyed via the price schedule to the non-dealers, dealers' and non-dealers' information sets overlap, which enables the dealers to predict better how the non-dealers will trade. Although the dealers lose profit from having their information conveyed, the loss is tempered by their ability to price discriminate against order flow they can predict. For example, the non-dealers buy aggressively when they infer a high signal from the order book, so dealers optimally reduce their quantity conditional on a high (private) signal to extract a larger price concession from the non-dealers. This steepens the price schedule and deters the non-dealers from trading "too" aggressively on the information they glean from the book, which is one of the reasons why rents exist at equilibrium to the common signal the non-dealers glean from the order book.

3.1.1 Strategies and Profit

The effects of these forces on strategy choices depend on the impact that individual traders perceive their orders to have on their own expected profit at the margin. For example, if J is close to M, there are many dealers and only a few non-dealers. No single dealer's strategy is important to the aggregate information conveyed by the price sched-

ule. This limits the harm each individual dealer does to himself by trading aggressively. Consequently, dealers scale back their trading only slightly in comparison to their strategy in a market with hidden liquidity—i.e., ψ remains close to one. Non-dealers receive a *lot* of information from the price schedule and, because they are few, they have nearly monopolistic control of it. They do not trade very aggressively, they have a significant informational advantage over the dealers, and their expected profit is high. In this situation, some dealers clearly would be better off trading as non-dealers instead.

As traders migrate away from dealing, J decreases and the strategy of each of the fewer remaining dealers becomes more influential in determining the aggregate information conveyed by the price schedule. Competition among a larger number of non-dealers intensifies, which dissipates the value (to both dealers and non-dealers) of the dealers' information conveyed by the price schedule. Anticipating this, dealers scale back their orders to protect the value of their information and to restrain the supply of liquidity in order to profit from their ability to predict order flow.

The foregoing discussion suggests that ψ increases as J decreases, and that when J is strictly less than M, ψ is greater than its hidden-market value of unity. Figure 1 illustrates this by graphing $\psi(J, M)$ as a function of J for $M = 40$ over the feasible range of J (i.e., $J > 1 + \psi(J, M)$). The graph is similar for other values of M.

The forces that determine ψ also determine expected profit. When there are few dealers, each dealer's information is aggregated into the price schedule with that of only a few other dealers, and each one exercises a great deal of control over the collective information conveyed. Many non-dealers then compete aggressively on the information conveyed by the price schedule and their private signals. Since dealers are few, each dealer can also forecast order flow with high precision relative to the situation in which there are many dealers. Dealers therefore perceive their individual decisions as important at the margin, so they trade conservatively to prevent too much information from being revealed and to price discriminate against the non-dealers. Thus, dealers' profit is high and non-dealers' profit is low when J is small.

In contrast, when J is large, a few non-dealers infer a great deal of aggregated information from the price schedule and trade on it as near monopolists. Individual dealers

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cannot forecast order flow very well, and they do not perceive their trading as important at the margin, so they trade aggressively. More information is conveyed via the price schedule, which dissipates dealers' profit. Thus, when J is large, dealers' profit is low and non-dealers' profit is high. Figure 2 illustrates, by graphing the expected profit of individual non-dealers and dealers for the case $M = 40$. Again, the graph is similar for other values of M.[15]

3.1.2 Equilibrium Liquidity Provision and Demand

When liquidity is hidden, $J_{*,hidden} = M$ because the expected profit to providing liquidity dominates the expected profit to demanding liquidity for all J. It is apparent from Figure 2 that this is not true when liquidity is displayed. If there are $J < \hat{J}$ dealers, some liquidity demanders are better off switching to a strategy of supplying liquidity. Conversely, if $J > \hat{J}$, some liquidity providers are better off switching to a strategy of demanding liquidity instead. Therefore, the equilibrium number of liquidity providers for the market with displayed liquidity depicted in Figure 2 is $J_{*,displayed} = \hat{J}$. When $M = 40$, $J_* \approx 37$. In markets with displayed liquidity, the endogenous assignment of informed traders to order types is interior—informed agents both supply and demand liquidity in equilibrium.

Upon substituting the equilibrium value of δ into the profit functions, the condition $\pi_j(J_*) = \pi_i(J_*)$ simplifies to an expression that defines $J_{*,displayed}$ implicitly as a function of M alone. We denote this expression by $\mathcal{G}(J_*, M) = 0$, which is given in Equation (5.1) in the Appendix. The solid line in Figure 3 plots the locus of $J_{*,displayed}$ for different values of M. The number of informed agents who trade as liquidity providers is *strictly* less than M and increases as the total number of informed agents in the market increases. Informed traders both supply and demand liquidity in the equilibrium of a displayed market of any "size" M. The dashed line in Figure 4 is the boundary $1 + \psi(J, M)$ for the existence of equilibrium discussed in connection with Proposition 4. The fact that $J_{*,displayed}$ lies

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[15]This graphical illustration is without loss of generality. Equations (4.20) and (4.22) in the Appendix show that the expected profit functions of the informed (and therefore the expected loss of the uninformed) can be written as $\sigma_u \sigma_{vi}$ times functions that depend only on M and J, just as in the market with hidden liquidity. The shapes of these graphs, and the conclusions we draw from them, are invariant to the σ parameters of the model.

everywhere above the dashed line (for $M > 2$) means an equilibrium in linear strategies always exists in a market with displayed liquidity provided that $M > 2$.

The endogenous assignment of informed traders between liquidity demand and supply in our model is obviously quite different from the assignment *assumed* in the classic microstructure models wherein liquidity providers are uninformed and informed traders only demand liquidity. Even when liquidity is displayed, the number of informed agents who trade as liquidity providers remains sizable. Individual liquidity providers adapt their strategies to the market structure by not trading so aggressively as to compete the rents to their information to zero. The incremental rents they can earn still draw them into trading as liquidity providers.

This has implications for the design and interpretation of empirical studies of price discovery. If information flows into prices when orders that provide liquidity are placed in the book *and* when market orders arrive, then the arrival of both types of orders should be included in measures of the impact of private information on prices. As noted earlier, some empirical studies adopt this perspective and conclude that significant information does flow into prices through (limit) orders that provide liquidity [see Kavajez and Odders-White (2001), Kaniel and Liu (2006) and Rourke (2009)].

3.1.3 Market Quality

Figure 4 plots the aggregate expected loss of the uninformed traders in a market with displayed liquidity as a function of J. Similar to when liquidity is hidden, uninformed trader losses are decreasing in J and minimized when all the informed trade as liquidity providers. Since the equilibrium assignment of traders to order types is interior (i.e., $J_{*,displayed} < M$), the equilibrium in a displayed market does *not* minimize uninformed trader losses across all possible assignments of agents to order types. If it were possible to constrain the informed agents exclusively to supplying liquidity, doing so would lessen the expected loss of the uninformed.

Since allocations in the case of $J = M$ in a market with displayed liquidity correspond to the *equilibrium* in a market with hidden liquidity, it also follows that uninformed trader losses are greater at the equilibrium of a displayed market than at the equilibrium of an

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otherwise identical hidden market. Displaying liquidity leads informed traders to compete less aggressively overall, and their expected profit is greater, than in an otherwise identical market in which liquidity is hidden. This suggests that market architectures that allow liquidity to be hidden are friendlier to uninformed traders than are architectures that mandate the display of liquidity.

The implication for bid-ask spreads follows directly from the observation that the uninformed traders' expected loss is smaller at equilibrium when liquidity is hidden than when it is displayed. The slope of the price schedule is $1/J\delta$, which is proportional to the expected loss of the uninformed traders, $\sigma_u^2/J\delta$. The bid-ask spread, $P(1) - P(-1)$, and the price impact of market orders, are therefore less at the equilibrium of a market with hidden liquidity than a market with displayed liquidity.

Hidden liquidity favors the informational efficiency of midquotes as well. This follows from the facts that $J_{*,hidden} > J_{*,displayed}$, and $\psi = 1$ in a hidden market whereas $\psi > 1$ in a displayed market. A larger number of informed traders act as dealers, and each of them trades more aggressively, when liquidity is hidden than when it is displayed. Consequently, the midquote incorporates more information and is a better predictor (in a mean-square sense) of the true security value in a market with hidden liquidity than in a market with displayed liquidity.

Summarizing, a market in which informed traders can hide their orders that provide liquidity has lower transaction costs *and* more informationally efficient midquotes than an otherwise identical market in which liquidity provided is displayed. One might expect that displaying liquidity improves market quality by publicizing private information that traders would otherwise exploit for profit. However, it is precisely because display robs the informed of their informational advantage that they alter their strategies by trading less aggressively and by shifting to demanding liquidity rather than supplying it. This reduces the intensity of competition that exists in filling the order book and in clearing the market—less information is incorporated into the midquote, bid-ask spreads are wider and greater adverse-selection costs are borne by the uninformed.

3.1.4 Large Markets

Displaying liquidity degrades market quality because it deters the informed from providing liquidity. In this section, we examine whether this deterrent disappears as the market becomes large—i.e., as $M \to \infty$. Increasing the number of informed agents creates additional competitors. It is therefore possible that the relevance to market quality of whether liquidity is hidden or displayed might disappear as the market grows. Alternatively, if the market quality differences persist, it suggests that the display features of a market's architecture are relevant to small and large markets alike.[16]

We show that the differences in market quality persist even in large markets. Growth in a market's size does not eliminate the deterrent effect of display on the informed agents' willingness to provide liquidity and to trade aggressively on their perception of mispricing. Even if a market is large, display still robs the informed of informational rents when compared to a large hidden market. This, in turn, deters the informed from providing liquidity and from trading aggressively, as illustrated in the following result.

PROPOSITION 5. *In a market with displayed liquidity, $\mathcal{G}(J_*, M) \to 0$ if and only if $J_* = M - M^{\frac{1}{3}} + O(1)$ where $O(1)$ denotes terms of order zero and lower in M (which remain bounded as $M \to \infty$). Moreover, $J_* = M - M^{\frac{1}{3}} + O(1)$ implies that $\psi(J_*, M) = M - M^{\frac{1}{3}} + O(1)$ when M is large.*

In a hidden market of any size, $J_{*,hidden} = M$. This is not true in a displayed market, even if it is large. The proposition says the asymptotic solution for $J_{*,displayed}$ grows with the size of the market in such a way that the *gap* between M and $J_{*,displayed}$ also grows. Even in large markets, display deters informed agents numbering $M^{\frac{1}{3}}$ from providing liquidity. Figure 5 plots the ratio of the equilibrium number of liquidity demanders to its asymptotic limit: $\frac{J_* - M}{M^{\frac{1}{3}}}$. The graph illustrates that the asymptotic approximation is rather close even for moderate M—the ratio is about 0.90 when $M = 40$.

Also important to market quality is the asymptotic behavior of ψ, which describes how the informed who do trade as dealers hold back in trading on their perception of security mispricing. The ψ parameter equals unity in a hidden market of *any* size. However, $\psi_{displayed} \sim J_* >> 1$ in a large market. When liquidity is displayed, those who do

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trade as dealers hold back more as the market grows larger. For M in the range of 100, for example, $\psi_{displayed} \approx 95$. Those who trade as dealers in a displayed market trade only 1/95th as aggressively per unit of perceived mispricing as their counterpart dealers in a hidden market. The implications for the expected loss of the uninformed and the informativeness of midquotes are discussed below.

Both J_* and ψ are independent of the σ parameters of the model. This means the deterrent is robust in the sense that it persists independently of how the quality of individual traders' information and the magnitude of uninformed trading is modeled to scale with M. For example, if σ_{vi}^2 is held fixed as M grows, the overall informational advantage of the informed over the uninformed grows with M. Alternatively, one could specify that $\sigma_{vi}^2 = \sigma_v^2/M$ for fixed σ_v^2. This distributes smaller bits of information among ever greater numbers of traders as M grows, and keeps their overall informational advantage constant. Similarly for uninformed trading, one could model σ_u^2 as fixed, so only the population of informed traders grows. Alternatively, if σ_u^2 scales up with M, then the magnitude of uninformed trading also increases as the number of informed traders grows. Regardless of these modeling choices, the deterrent persists in large markets.

The next result identifies the implications of the deterrent for the expected loss of the uninformed. In view of the results above, one would expect the uninformed to lose more when liquidity is displayed than when it is hidden even in a large market. This is indeed what we find.

PROPOSITION 6. *As the market grows large, the expected loss of the uninformed asymptotes to $\frac{1}{2}M^{\frac{1}{2}}\sigma_u\sigma_{vi}$ when liquidity is hidden and to $M^{\frac{2}{3}}\sigma_u\sigma_{vi}$ when liquidity is displayed.*

The expected loss of the uninformed is therefore greater by a factor of $2M^{\frac{1}{6}}$ in a large market with displayed liquidity than in a large market with hidden liquidity. When $M = 100$, the uninformed lose about four times as much in a large displayed market than in a large hidden market. This also is independent of how the σ parameters are modeled to scale with M.

The relation between price and order flow variability is interesting as well. Variation in transaction prices arises from two sources. The first is information impounded into the

price schedule because the informed place orders *into* the book. The second is information and noise associated with the clearing of informed and uninformed market orders *against* the book. The relative importance of these sources of price variability can be measured by the squared correlation (R^2) between prices and market-order flow. The smaller is the R^2, the more important is information impounded into the order book by the informed traders who provide liquidity.

Since display deters informed traders from providing liquidity, we expect the R^2 to be larger when liquidity is displayed than when it is hidden. The next proposition confirms this intuition.

<u>PROPOSITION 7</u>. *As the market grows large, R^2_{hidden} asymptotes to $\frac{1}{2}$ and $R^2_{displayed}$ asymptotes to 1.*

When liquidity is hidden, half the variation in the transaction price is explained by dealers' information moving the price schedule as their orders fill the book. The other half is explained by order flow, which has no information content because all the informed trade as liquidity providers, $J_{*,hidden} = M$. The trading strategies pursued by the informed induce information-related variation into the midquote that exactly matches the variation associated with uninformed market-order flow.

The situation is very different when liquidity is displayed. The relative importance of dealers' information vanishes completely because the incentive for dealers to hold back grows so strongly as the market grows. The information of non-dealers and noise from uninformed trading come to dominate the impact of the dealers' orders on the price. This is important because the impact of dealers' information on the price is what creates informationally efficient midquotes. Display impedes this in small markets, and causes it to die out completely in large markets. As with Propositions 5 and 6, these results are invariant of how the σ parameters are modeled to scale with M.

4. When are Markets with Hidden and Displayed Liquidity Equivalent?

Hiding liquidity enhances market quality because the informed compete more aggressively when they trade as liquidity providers than when they trade as liquidity demanders.

Display threatens informational rents, which causes some informed traders to migrate away from liquidity provision. This happens in small and large markets. An interesting question is whether there are circumstances under which this does not happen, and market quality is unaffected by whether liquidity is hidden or displayed.

If agents have no private information to begin with, informational rents are necessarily zero, and display poses no threat that would cause a migration of agents from providing toward demanding liquidity. Display also poses no threat if those who trade as dealers possess private information, yet earn zero expected profit on it *at equilibrium* when liquidity is hidden. In this case also, display itself causes no incremental loss of informational rents because such rents are totally competed away even when liquidity is hidden.

In this section, we show that this happens when dealers have identical signals. One would expect this result if dealers were modeled as Bertrand-style competitors because identical signals implies that dealers have the same valuation of the security–i.e., perfect competition with common values would imply zero profit. However, we show that it happens even though dealers engage in strategic competition.

Dealers still earn profit from providing liquidity (by capturing the bid-ask spread), but they do not profit from their signal about the security's payoff even when liquidity is hidden. Whether liquidity is hidden or displayed has no impact on their strategies and the quality of the market is unaffected by whether liquidity is hidden or displayed when the dealers' signals are identical.

Interestingly, such aggressive trading on a common signal is unique to the dealers. If the non-dealers share a signal in common, they do not trade so aggressively as to compete away totally the rents to their signal. The non-dealers moderate their trading and, at equilibrium, their expected profit depends on how much of the security's payoff their information explains. They earn expected profit too, however its source is their common informational advantage over the uninformed agents.

This illustrates again that agents trade on information more aggressively as liquidity providers than as liquidity demanders. What is remarkable is that the intensity of competition is so extreme as to render common information worthless to dealers. The results in the previous section for large markets show that growing the number of informed competi-

tors does not intensify competition in a manner that extinguishes the impact of whether liquidity is hidden or displayed on market quality. The results in this section identify what does. Eliminating *heterogeneity* in the dealers' information intensifies competition in a manner, and to a degree, that whether liquidity is hidden or displayed is irrelevant to market quality. This is true in both small and large markets.

To model dealers with a common signal and also to contrast them with non-dealers who have a common signal in a single setting, we assume that the numbers of agents who trade as dealers and as non-dealers are already determined. The security's payoff has two components: $\tilde{v} = \tilde{v}_1 + \tilde{v}_2$, where $\tilde{v}_1$ and $\tilde{v}_2$ are independent normal random variables with zero means and variances $\sigma_{v_1}^2$ and $\sigma_{v_2}^2$, respectively. All dealers j observe the same signal $s_j = v_1$ before submitting their supply schedules. All non-dealers i observe $s_i = v_2$ prior to submitting their market orders.[17] Consider first a market in which liquidity is hidden.

PROPOSITION 8. *Assume that $s_j = v_1$ for $j = 1, \ldots, J$ and $s_i = v_2$ for $i = 1, \ldots, N$. If $J > 2$, there exists a symmetric equilibrium in which dealers' and non-dealers' strategies are linear, and this is the unique symmetric equilibrium in which strategies are linear. Explicit expressions for the strategies are*

$$y(v_1, P) = \gamma v_1 + \delta P \qquad and \qquad x(v_2) = \beta v_2$$

where

$$\gamma = -\delta \qquad \beta = \frac{J\delta}{N+1}, \qquad and \qquad J\delta = \frac{\sigma_u}{\sigma_{v_2}} \left(\frac{1}{N} + 1 \right) \sqrt{\frac{J-2}{\frac{1}{N}(J-1)+1}}.$$

The dealers' and non-dealers' equilibrium expected profits, denoted by π_j and π_i respectively, satisfy

$$J\pi_j + N\pi_i = \frac{\sigma_u^2}{J\delta} \qquad and \qquad \pi_i = \frac{\beta \sigma_{v_2}^2}{N+1}.$$

Substituting dealers' equilibrium strategies into the market clearing condition, solving for the price schedule, and noting that $\gamma = -\delta$, yields

$$P(\omega) = v_1 + \frac{1}{J\delta} \omega, \tag{7}$$

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[17]The primary dealers of government securities might be an example of a network of dealers that is predetermined and observes nearly identical information that non-dealers do not have. In many countries, primary dealers are the exclusive counterparties to central banks' open market operations. This gives primary dealers information about the central bank's trading strategy that other market participants do not have [see Arnone and Iden (2003)].

where, as before, $\omega \equiv \sum_{i=1}^{N} x_i + u$ is the net flow of market orders. Dealers' private information, v_1, anchors the price schedule as if it were the *unconditional* expected security value. Thus, information held in common among the dealers affects prices as though it were *public* information—dealers' common private information is *fully* priced into the order book. Accordingly, it would be useless for non-dealers to condition their strategies on information in the order book if it were visible because the price schedule fully reflects the information.

Now consider the situation in which the book is visible to the non-dealers before they choose their orders. Suppose the non-dealers conjecture that the dealers' common signal is fully reflected in prices at equilibrium, and they therefore ignore the order book as a source of information. The non-dealers then adopt the same trading strategies they would follow if the order book were hidden. This implies that the equilibrium price schedule is given by Equation (7), and the non-dealers' conjecture is correct. The equilibrium in Proposition 8 is therefore also an equilibrium when the order book is displayed. So whether the order book is hidden or displayed is irrelevant to the allocation of profit across traders and also irrelevant to measures of market quality.

Substituting for β and $J\delta$ yields explicit expressions for the non-dealers' and dealers' expected profit in terms of exogenous variables:

$$
\begin{aligned}
\pi_i &= \left(\frac{\sigma_{v_2} \sigma_u}{N+1} \right) \sqrt{\frac{J-2}{N(J+N-1)}} \\
\pi_j &= \left(\frac{\sigma_{v_2} \sigma_u}{J} \right) \sqrt{\frac{N}{(J-2)(J+N-1)}}.
\end{aligned}
\tag{8}
$$

The non-dealers' expected profit depends on the quality of their private information, σ_{v_2}, but dealers' expected profit is independent of σ_{v_1}. Dealers do not profit from their signal.

Despite being strategic, dealers submit price contingent orders that compete away all the rents to the information they hold in common. In contrast, non-dealers do not compete their informational rents to zero, even though they also have common information. This asymmetry between dealers and non-dealers is attributable entirely to how order *type* affects the intensity of competition, as observed in the sections above [see also Bernhardt and Taub (2006)]. This is separate from the effect of traders competing more aggressively

on the basis of common versus diverse information [see Foster and Viswanathan (1994) and Holden and Subrahmanyam (1994)].

The dealers' strategies are price contingent and non-dealers' strategies are not. Allowing a monopolist (or duopolist) dealer the degree of control that a price contingent strategy provides enables him to make infinite expected profit in our model because uninformed trading is not price elastic (see footnote 11). However, if there are more than two dealers, the additional control that price contingent strategies provide for earning profit is dominated by how such control elevates the aggressiveness of competition between the dealers. As few as $J = 3$ dealers with common signals are enough that price contingent strategies intensify competition to the point of totally eliminating their informational rents.

Dealers do earn positive expected profit by providing liquidity to the non-dealers and the uninformed. The rents to liquidity provision captured by the dealers behave as a tax on the non-dealers for access to the market. The aggregate expected profit dealers capture, $J\pi_j$ is a fraction of that earned by the non-dealers:

$$J\pi_j = \left(\frac{N+1}{J-2} \right) N\pi_i. \tag{9}$$

The dealers' profit is increasing in N, and when the number of non-dealers approaches zero, so does the dealers' expected profit (i.e., $J\pi_j \to 0$ as $N \to 0$). For a given number of non-dealers, the higher is the quality of their information or the more uninformed trading there is, the larger is $\sigma_{v_2}\sigma_u$ and the greater are the informed non-dealer *and* dealer profits. Thus, dealers literally capture a share of the informational rents that the non-dealers extract from the uninformed traders. The share dealers capture is increasing in the number of non-dealers and decreasing in the number of dealers, as intuition would suggest.

5. Conclusion

The desire by market participants to trade dark on off-exchange venues has pressured exchanges to offer trade types that allow the placement of hidden orders into exchange order books. Exchanges, regulators and market professionals seem divided on whether allowing liquidity to be hidden is good or bad for market quality. Ambivalence and even skepticism about allowing hidden orders on exchanges seems well founded in the intuition

of classic microstructure theory. Informed traders impose adverse selection costs on the uninformed by camouflaging their orders among those of the uninformed. Allowing the informed literally to hide their orders would seem to exacerbate the adverse selection costs faced by the uninformed, which will drive them away from the market.

We study the impact of hidden liquidity on market quality using a model of securities trading in which informed agents coexist as liquidity demanders as in Kyle (1985) and liquidity providers as in Kyle (1989). Our model is well suited to this question because it enables us to endogenize the two key elements of traders' strategies that affect the level of competition and market quality. The first is the attraction of informed agents to trading as liquidity providers or demanders according to whether liquidity is hidden or displayed. The second is the greater intensity with which the informed compete when more of them provide liquidity. Several studies extend Kyle's models, but none considers the situation where informed agents coexist as liquidity demanders and liquidity providers.

If the informed must display orders that provide liquidity, they lose a portion of their informational advantage because the order book discloses information in their orders to other traders. This does make liquidity demanders better informed than they would have been otherwise. However, it also causes informed agents to back away from liquidity provision, which weakens competition among liquidity providers and also among liquidity demanders. We show that the reduction in competition dominates. The net effect is that the trading costs of uninformed agents are higher, bid-ask spreads are wider and the informational efficiency of midquotes is lower in markets that require liquidity to be displayed than in markets where liquidity is hidden.

Our model considers situations in which liquidity is either hidden or displayed. Some exchanges take a hybrid approach wherein orders can only be partially hidden (reserve or iceberg orders), and displayed liquidity has priority over liquidity that is hidden. Although less extreme than a mandate of full display, these policies still weaken the incentives of the informed to trade as liquidity providers. Such incentives are what deliver the improvements in market quality associated with hidden liquidity in our model. Our results suggest the better policy is to allow fully hidden liquidity with no requirement of partial display and no reduction in priority.

There is nothing in the logic behind our results that suggests traders should be prevented from displaying orders if they want their orders to be displayed. In fragmented markets, displaying orders can attract potential counter parties away from other venues [see Hendershott and Jones (2005)]. In our model, orders that supply liquidity are consolidated and there is no reason for traders to advertise to attract counter parties.

Our model suggests that innovations, which at first might seem to advantage one group of traders over another, can improve market quality by encouraging more aggressive competition. This perspective could be useful in addressing the controversy over whether high-frequency traders (HFTs) are good for markets or not. The academic studies of HFTs suggest that, as a group, HFTs primarily provide liquidity, and they seem to be able to predict price changes a tick or two ahead [see Boehmer, Fong and Wu (2011), Hasbrouck and Saar (2011) and Kirilenko, Kyle, Samadi and Tuzun (2011)].[18] There have been allegations of front-running.[19] However, the broader concerns of markets, regulators and market professionals seems to be that the speed advantage of HFTs over other traders harms market quality through the submission and nearly immediate cancelation of large numbers of orders. Remedies proposed include (i) charging order cancelation fees, (ii) requiring a minimum duration of orders placed in the book, (iii) fines for exceeding a set order-to-trade ratio, and (iv) requirements to post at the NBBO a certain percentage of the time.[20]

Liquidity providers' strategies, in our model and in others, are price contingent order *schedules.* These are implemented by placing collections of limit orders into the book, the vast majority of which will not be executed. Any change in the state of the market, such as the arrival or execution of market orders, will lead liquidity providers to revise their order schedules. This naturally leads to cancelations of large numbers of orders in relation to orders executed. The faster liquidity providers perceive the state of the market to be

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[18]See also Chester Spatt's comments in "Does High Speed Trading Hurt the Small Investor?" *Wall Street Journal,* October 10, 2011.

[19]See "SEC Probes Ties to High-Speed Traders," *Wall Street Journal,* April 3, 2012.

[20]Item (iii) is already in force in some exchanges, though the fines are small—e.g. 0.01 - 0.03 Euro per order exceeding a ratio of 100-to-1; see "Bourses Play Nice Cop to Head Off Speed-Trade Rules," *Reuters,* April 10, 2012.

changing (e.g., because the flow orders or news is rapid), the greater will be the intensity of cancelations.

From this perspective, the "remedies" above raise the cost of providing liquidity versus demanding liquidity in a manner similar to the effect of mandating the display of liquidity in our model. It deters traders who would otherwise provide liquidity from doing so, which reduces competition and degrades market quality. Our analysis cautions that measures to level the playing field between fast and slow traders might actually deny slow traders the greater benefits of robust competition among those who trade fast.

APPENDIX

Proof of Proposition 1: The proof follows exactly the steps of the proof of Proposition 4, except non-dealers do not condition on the intercept of the price schedule. The result turns out to be the same as the expressions in Proposition 4 with $\beta_2 = 0$ and $\psi(\cdot) = 1$. ‖

Proof of Proposition 2: An *interior* equilibrium assignment of agents between dealing and trading via market orders is $\hat{J} \in \{2, \ldots, M\}$ such that:

(i) Agents i trading via market order have no incentive to switch to supply schedules

$$\pi_i(\hat{J}) \geq \pi_j(\hat{J} + 1), \quad \text{and}$$

(ii) agents j trading via supply schedules have no incentive to switch to market orders

$$\pi_j(\hat{J}) \geq \pi_i(\hat{J} - 1).$$

Ignoring the integer constraint, an interior equilibrium $\hat{J}$ is a point at which π_i and π_j cross when both are regarded as functions of J.[1]

Substituting from the expected profit expressions in Proposition 1, $\pi_j(\hat{J}) = \pi_i(\hat{J})$ is equivalent to

$$\hat{J}\pi_j(\hat{J}) = \frac{\sigma_u^2}{\hat{J}\delta} - (M - \hat{J})\pi_i(\hat{J}) = J\pi_i(\hat{J})$$

$$\frac{\sigma_u^2}{\hat{J}\delta} = M\frac{\hat{J}\delta}{4}\sigma_{vi}^2$$

$$(\hat{J}\delta)^2 = \frac{4}{M}\frac{\sigma_u^2}{\sigma_{vi}^2}.$$

Substituting the expression for $J\delta$ in Proposition 1 yields

$$\left(\frac{4\sigma_u^2}{\sigma_{vi}^2}\right)\left(\frac{\hat{J} - 2}{\hat{J}\left[(M - \hat{J}) + (\hat{J} - 1)\right]}\right) = \frac{4}{M}\frac{\sigma_u^2}{\sigma_{vi}^2}$$

or

$$\hat{J} = 2M. \tag{2.1}$$

Thus, there is a *single* crossing point of π_i and π_j in the domain $J \in (2, \infty)$ at $\hat{J} = 2M$. Furthermore, π_j crosses π_i from above because

$$\lim_{J \to 2} \pi_i(J) = 0 \quad \text{and} \quad \lim_{J \to 2} \pi_j(J) = \infty.$$

Since the crossing point exceeds M, all informed traders elect to trade as dealers and equilibrium J_* is at the boundary, $J_* = M$. ‖

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[1]It is easy to show that if an equilibrium exists while ignoring the integer constraint, then an equilibrium also exists under the constraint that J take only integer values. Details are available from the authors.

Proof of Propositions 3: Substituting $J = M$ from Proposition 2 into the expression for uninformed trader expected losses in Proposition 1 yields

$$\frac{\sigma_u^2}{M\delta} = \frac{\sigma_u \sigma_{vi}}{2} \sqrt{\frac{M(M-1)}{M-2}} = \frac{\sigma_u \sigma_v}{2} \sqrt{\frac{M-1}{M-2}},$$

where the last equality follows from $\sigma_v^2 = M\sigma_{vi}^2$. This is the first expression in the statement of Proposition 3. To obtain the second expression, note that if liquidity is provided by a perfectly competitive, uninformed, risk-neutral market maker, the price schedule is $\hat{P}(\hat{\omega}) = E[\tilde{v}|\hat{\omega}]$, where $\hat{\omega} = \sum_{i=1}^{M} \hat{x}_i + u$ because informed traders submit market orders only. The derivation of a linear equilibrium follows the steps in Kyle (1985) with minor modifications, so we do not reproduce them here. The solution is:

$$\hat{x}_i = \hat{\beta} v_i \qquad \text{for } i = 1, \ldots, M$$

$$\hat{P}(\hat{\omega}) = \hat{\lambda}\hat{\omega}$$

where

$$\hat{\beta} = \frac{\sigma_u}{\sigma_v} \qquad \text{and} \qquad \hat{\lambda} = \frac{\sigma_v}{2\sigma_u}.$$

Expected profit of informed trader i is

$$\hat{\pi}_i = E\left[(\tilde{v} - \hat{P}(\hat{\omega}))\hat{x}_i\right] = E\left[(\tilde{v} - \hat{\lambda}\hat{x}_i)\hat{x}_i\right]$$

$$= E\left[v_i\hat{\beta}v_i - \hat{\lambda}\hat{\beta}^2 v_i^2\right] = \sigma_{vi}^2\hat{\beta}(1 - \hat{\lambda}\hat{\beta})$$

$$= \sigma_{vi}^2 \frac{\sigma_u}{\sigma_v}\left(1 - \frac{1}{2}\right) = \frac{\sigma_u \sigma_v}{2}$$

where the last equality follows from $\sigma_v^2 = M\sigma_{vi}^2$.

The equilibrium price schedule of Proposition 1 is easily shown to be

$$P(\omega) = \frac{1}{2}\sum_{i=1}^{M} v_i + \left(\frac{\sigma_v}{2\sigma_u}\sqrt{\frac{M-1}{M-2}}\right)\omega \tag{3.1}$$

where $\omega = u$ because the informed do not use market orders in equilibrium. A standard calculation shows that the price function of a perfectly competitive, uninformed, risk-neutral market maker in our informational environment is given by

$$\hat{P}(\hat{\omega}) = 0 + \left(\frac{\sigma_v}{2\sigma_u}\right)\hat{\omega}. \tag{3.2}$$

where $\hat{\omega} = \sum_{i=1}^{M} \hat{\beta}v_i + u$, $\hat{\beta} = \frac{\sigma_u}{\sigma_v}$, and the "hats" denote the competitive market maker setting. Substituting these into (3.2) yields

$$\hat{P}(\hat{\omega}) = \frac{1}{2}\sum_{i=1}^{M} v_i + \left(\frac{\sigma_v}{2\sigma_u}\right)u. \tag{3.3}$$

Using (3.1) and (3.3), the midpoints of the bid and ask prices are

$$P(0) = \frac{1}{2}\sum_{i=1}^{M} v_i \qquad \text{and} \qquad \hat{P}(0) = 0.$$

Thus,

$$E\left[(\tilde{v} - \tilde{P}(0))^2\right] < E\left[(\tilde{v} - \hat{P}(0))^2\right],$$

midquotes in the equilibrium of Proposition 2 are better predictors of $\tilde{v}$ in a mean-square error sense than are midquotes in a market with an uninformed, perfectly competitive, risk-neutral market maker.

The bid-ask spread is the difference between prices quoted for unit-sized market buy and sell orders. Thus,

$$s \equiv P(1) - P(-1) = \frac{\sigma_v}{\sigma_u}\sqrt{\frac{M-1}{M-2}}$$
$$\hat{s} \equiv \hat{P}(1) - \hat{P}(-1) = \frac{\sigma_v}{\sigma_u},$$

so $s > \hat{s}$. Finally, the difference in the variance of prices is evident from equations (3.1) and (3.3):

$$\mathrm{Var}\left[\tilde{P}(\tilde{u})\right] - \mathrm{Var}\left[\hat{P}(\tilde{u})\right] = \left\{\sqrt{\frac{M-1}{M-2}} - 1\right\}\sigma_u^2.$$

This concludes the proof.‖

Proof of Proposition 4: Consider non-dealers first. Suppose non-dealer i believes that (i) dealers $j \in \mathcal{J} \equiv \{1, \ldots, J\}$ follow symmetric strategies of the form $y_j = \gamma v_j + \delta P$, and (ii) other non-dealers $k \neq i$ follow symmetric strategies of the form $x_k = \beta_1 v_k + \beta_2 \theta$ where $\theta \equiv P(0)$ is the intercept of the consolidated price schedule. Then non-dealer i perceives the market clearing condition as

$$\gamma \sum_{j \in \mathcal{J}} \tilde{v}_j + J\delta\tilde{P} = \tilde{\omega} \qquad \text{where} \qquad \tilde{\omega} = x_i + \beta_1 \sum_{k \in \mathcal{I} \sim i} \tilde{v}_k + (M - J - 1)\beta_2\theta + \tilde{u},$$

$\mathcal{I} \equiv \{J+1, \ldots, M\}$, and $\mathcal{I} \sim i$ is the set of integers in $\mathcal{I}$ with the exception of i. Equivalently, non-dealer i perceives the market clearing price as a realization of the random variable

$$\tilde{P} = \tilde{P}(\tilde{\omega}) = -\frac{\gamma}{\delta}\bar{v}_{\mathcal{J}} + \frac{1}{J\delta}\tilde{\omega} \qquad \text{where} \qquad \bar{v}_{\mathcal{J}} = \frac{1}{J}\sum_{j \in \mathcal{J}} \tilde{v}_j,$$

which implies

$$\theta = \frac{-\gamma}{\delta}\bar{v}_{\mathcal{J}}.$$

Non-dealer i chooses x_i to maximize expected profit conditional on his private information $s_i = v_i$ and his observation of the intercept θ:

$$\max_{x_i} E\left[\left(\sum_{h=1}^{M} \tilde{v}_h + \frac{\gamma}{\delta}\bar{v}_{\mathcal{J}} - \frac{1}{J\delta}\left(x_i + \beta_1 \sum_{k \in \mathcal{I} \sim i} \tilde{v}_k + (M - J - 1)\beta_2\theta + \tilde{u}\right)\right)x_i \,\middle|\, v_i, \theta\right].$$

Since $E\left[\bar{v}_{\mathcal{J}}|\theta\right] = -\frac{\delta}{\gamma}\theta$ and $E\left[\tilde{v}_k|v_i\right] = 0$ for $k \neq i$, the first-order condition implies

$$x_i = \frac{J\delta}{2}v_i - \frac{1}{2}\left\{J(J\delta + \gamma)\frac{\delta}{\gamma} + (M - J - 1)\beta_2\right\}\theta.$$

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The second-order condition is satisfied if $\delta > 0$, which we verify later. Thus, if non-dealer i believes others follow symmetric linear strategies, he also follows a strategy that is linear in his signal and the intercept of the price schedule. Symmetry among non-dealers' strategies implies the coefficients of non-dealer i's strategy are equal to the corresponding coefficients of the other non-dealers' strategies; i.e.,

$$\beta_1 = \frac{J\delta}{2} \tag{4.1}$$

and β_2 equals the coefficient on θ. After simplifying,

$$\beta_2 = \frac{-J\delta}{M - J + 1}\left(1 + \frac{J\delta}{\gamma}\right). \tag{4.2}$$

Now consider dealers. Suppose dealer j believes that (i) other dealers $k \neq j$ follow symmetric strategies of the form $y_k = \gamma v_k + \delta P$ and (ii) non-dealers $i \in \mathcal{I}$ follow symmetric strategies of the form $x_i = \beta_1 v_i + \beta_2 \theta$. Then dealer j perceives the market clearing condition as

$$\gamma \sum_{k \in \mathcal{J} \sim j} \tilde{v}_k + (J-1)\delta\tilde{P} + y_j = \tilde{\omega} \qquad \text{where} \qquad \tilde{\omega} = \beta_1 \sum_{i \in \mathcal{I}} \tilde{v}_i + (M-J)\beta_2\theta + \tilde{u},$$

and $\mathcal{J} \sim j$ is the set of integers in $\mathcal{J}$ with the exception of j. Equivalently, dealer j perceives the market clearing price as a realization of the random variable

$$\tilde{P} = \tilde{P}(\tilde{\omega}) = \frac{-\gamma}{\delta}\overline{v}_{\mathcal{J}\sim j} + \frac{1}{(J-1)\delta}(\tilde{\omega} - y_j) \qquad \text{where} \qquad \overline{v}_{\mathcal{J}\sim j} = \frac{1}{J-1}\sum_{k \in \mathcal{J}\sim j} \tilde{v}_k, \tag{4.3}$$

and the intercept of the price function as the realization of

$$\tilde{\theta} \equiv \tilde{P}(0) = \frac{-\gamma}{\delta}\overline{v}_{\mathcal{J}\sim j} - \frac{1}{(J-1)\delta}y_j. \tag{4.4}$$

Dealer $j's$ strategy is a signal and price contingent supply schedule $y_j(s_j, P)$ that maximizes expected profit for each realization of the market clearing price $\tilde{P} = P$ and signal $\tilde{s}_j = v_j$:

$$\max_{y_j(\cdot)} E\left[\left(\frac{-\gamma}{\delta}\overline{v}_{\mathcal{J}\sim j} + \frac{1}{(J-1)\delta}\left[\beta_1(M-J)\overline{v}_{\mathcal{I}} + \beta_2(M-J)\tilde{\theta} + \tilde{u} - y_j(v_j, P)\right] - \tilde{v}\right)y_j(v_j, P)\right]$$

where $\overline{v}_{\mathcal{I}} = \frac{1}{M-J}\sum_{i \in \mathcal{I}} \tilde{v}_i$. Substituting from (4.4) for $\tilde{\theta}$ and collecting like terms, this problem is equivalent to selecting $y_j(\cdot)$ to maximize

$$E\left[\left(\frac{-\gamma}{\delta}\overline{v}_{\mathcal{J}\sim j} + \frac{1}{(J-1)\delta}\left[\beta_1(M-J)\overline{v}_{\mathcal{I}} - \beta_2(M-J)\frac{\gamma}{\delta}\overline{v}_{\mathcal{J}\sim j} + \tilde{u} - \psi y_j(v_j, P)\right] - \tilde{v}\right)y_j(v_j, P)\right]$$

where $\psi \equiv 1 + \left(\frac{M-J}{J-1}\right)\left(\frac{\beta_2}{\delta}\right)$. Pointwise optimization yields a collection of first order conditions, each relating to a particular realization of the signal and price:

$$E\left[\frac{-\gamma}{\delta}\overline{v}_{\mathcal{J}\sim j} + \frac{1}{(J-1)\delta}\left[\beta_1(M-J)\overline{v}_{\mathcal{I}} - \beta_2(M-J)\frac{\gamma}{\delta}\overline{v}_{\mathcal{J}\sim j} + \tilde{u} - 2\psi y_j\right] - \tilde{v}\,\bigg|\,v_j, \tilde{P} = P\right] = 0.$$

The collection of second-order conditions are satisfied if $\psi/\delta > 0$, which is verified below. Substituting (4.4) for θ in the expression for $\tilde{\omega}$, and substituting the result into (4.3) yields

$$\tilde{P} = \frac{-\gamma}{\delta}\bar{v}_{\mathcal{J}\sim j} + \frac{1}{(J-1)\delta}\left[\beta_1(M-J)\bar{v}_{\mathcal{I}} - \beta_2(M-J)\frac{\gamma}{\delta}\bar{v}_{\mathcal{J}\sim j} + \tilde{u} - \psi y_j\right] \tag{4.5}$$

so the collection of first-order conditions is equivalent to

$$E\left[\tilde{P} - \tilde{v} - \frac{\psi}{(J-1)\delta}y_j\Big|v_j, \tilde{P} = P\right] = 0$$

or

$$y_j = \frac{(J-1)\delta}{\psi}\left\{P - E\left[\tilde{v}|v_j, \tilde{P} = P\right]\right\} \tag{4.6}$$

for each realization $\{v_j, P\}$. From the perspective of dealer j, all variates are jointly normal so

$$E\left[\tilde{v}|v_j, \tilde{P} = P\right] = E\left[\tilde{v}_1 + \cdots + \tilde{v}_J + \tilde{v}_{J+1} + \cdots + \tilde{v}_M|v_j, \tilde{P} = P\right]$$
$$= v_j + (J-1)E\left[\tilde{v}_k|v_j, \tilde{P} = P\right] + (M-J)E\left[\tilde{v}_i|v_j, \tilde{P} = P\right]$$

where $k \in \mathcal{J} \sim j$ and $i \in \mathcal{I}$. Using the expression in (4.5) for $\tilde{P}$, we have

$$E\left[\tilde{v}_k|v_j, \tilde{P} = P\right] = \frac{\mathcal{C}_1}{\mathcal{V}}\left(P - E\left[\tilde{P}|v_j\right]\right) = \frac{\mathcal{C}_1}{\mathcal{V}}\left(P + \frac{\psi}{(J-1)\delta}y_j\right) \tag{4.7}$$

$$E\left[\tilde{v}_i|v_j, \tilde{P} = P\right] = \frac{\mathcal{C}_2}{\mathcal{V}}\left(P - E\left[\tilde{P}|v_j\right]\right) = \frac{\mathcal{C}_2}{\mathcal{V}}\left(P + \frac{\psi}{(J-1)\delta}y_j\right) \tag{4.8}$$

where

$$\mathcal{C}_1 \equiv \mathrm{Cov}\left[\tilde{v}_k, \tilde{P}|v_j\right] = \frac{-\gamma}{(J-1)\delta}\psi\sigma_{vi}^2 \qquad \text{for } k \in \mathcal{J} \sim j$$

$$\mathcal{C}_2 \equiv \mathrm{Cov}\left[\tilde{v}_i, \tilde{P}|v_j\right] = \frac{\beta_1}{(J-1)\delta}\sigma_{vi}^2 \qquad \text{for } i \in \mathcal{I}, \text{ and}$$

$$\mathcal{V} \equiv \mathrm{Var}\left[\tilde{P}|v_j\right] = \psi^2\left(\frac{\gamma}{\delta}\right)^2\frac{\sigma_{vi}^2}{J-1} + \left(\frac{1}{(J-1)\delta}\right)^2\left\{(M-J)\beta_1^2\sigma_{vi}^2 + \sigma_u^2\right\}.$$

Therefore, (4.7) and (4.8) imply

$$E\left[\tilde{v}|v_j, \tilde{P} = P\right] = v_j + \frac{(J-1)\mathcal{C}_1 + (M-J)\mathcal{C}_2}{\mathcal{V}}\left[P + \frac{\psi}{(J-1)\delta}y_j\right].$$

Substituting this into (4.6) and solving for y_j yields the dealer's strategy

$$y_j(v_j, p) = \frac{(J-1)\delta}{\psi}\left\{\left(\frac{1-\phi}{1+\phi}\right)P - \left(\frac{1}{1+\phi}\right)v_j\right\},$$

where $\phi \equiv \dfrac{(J-1)\mathcal{C}_1 + (M-J)\mathcal{C}_2}{\mathcal{V}}$. Thus, if dealer j believes the others follow symmetric linear strategies, he also follows a strategy that is linear in the realizations of his signal and the price. Symmetry among the dealers' strategies implies

$$\delta = \frac{(J-1)\delta(1-\phi)}{\psi(1+\phi)} \qquad \Longleftrightarrow \qquad \phi = \frac{J-1-\psi}{J-1+\psi} \tag{4.9}$$

and

$$\gamma = \frac{-(J-1)\delta}{(1+\phi)\psi}.$$

Substituting for ϕ from (4.9) yields

$$\frac{\gamma}{\delta} = \frac{-(J-1+\psi)}{2\psi}. \tag{4.10}$$

To identify ψ, substitute from (4.2) and (4.10) into the definition of ψ:

$$\psi \equiv 1 + \left(\frac{M-J}{J-1}\right)\left(\frac{\beta_2}{\delta}\right) = 1 + \left(\frac{M-J}{J-1}\right)\left(\frac{-J}{M-J+1}\right)\left[1 + J\left(\frac{-2\psi}{J-1+\psi}\right)\right]$$

This is a quadratic equation whose negative root can be ruled out because it yields a negative ψ, which violates the requirement of the second-order conditions that $\psi/\delta > 0$ (that $\delta > 0$ is verified below). The solution for ψ depends only on J and M:

$$\psi_* \equiv \frac{\mathcal{Z} + \sqrt{\mathcal{Z}^2 - 4\left[J\left(\frac{M-J}{M-J+1}\right) - (J-1)\right]}}{2} \equiv \psi(J, M) \tag{4.11}$$

where

$$\mathcal{Z} = J\left(\frac{2J-1}{M-J+1}\right)\left(\frac{M-J}{J-1}\right) - (J-2). \tag{4.12}$$

Using (4.11) and (4.12) it is easy to show that $J \leq M$ implies $\psi_* \geq 1$ with equality iff $J = M$. Substituting ψ_* back into (4.10) yields

$$\frac{\gamma}{\delta} = \frac{-(J-1+\psi_*)}{2\psi_*}. \tag{4.13}$$

Substituting this into (4.2) and simplifying yields

$$\frac{\beta_2}{\delta} = \frac{J}{M-J+1}\left[\frac{(2J-1)\psi_* - (J-1)}{J-1+\psi_*}\right], \tag{4.14}$$

which is positive because $\psi_* \geq 1$.

To identify δ, equate the definition of ϕ to the expression given for it in (4.9)

$$(J-1)\mathcal{C}_1 + (M-J)\mathcal{C}_2 = \left(\frac{J-1-\psi_*}{J-1+\psi_*}\right)\mathcal{V}.$$

Substituting from above for $\mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{V}$, substituting ψ_* for ψ, and using the solutions for endogenous parameters obtained so far yields the following equation for δ^2:

$$\frac{1}{\delta^2} = \frac{\sigma_{vi}^2}{\sigma_u^2}\frac{(J-1)^2}{2}\left\{(J-1+\psi_*)^2\left[\frac{1}{J-1-\psi_*} - \frac{1}{2}\left(\frac{1}{J-1}\right)\right] + \frac{J(M-J)}{J-1}\left[\frac{J-1+\psi_*}{J-1-\psi_*} - \frac{1}{2}\left(\frac{J}{J-1}\right)\right]\right\}. \tag{4.15}$$

The requirement that $J > 1 + \psi_*$ in the statement of the proposition ensures that $\delta > 0$. To see why, observe that if $J > 1 + \psi_*$, then the term in curly brackets is positive and there is a unique $\delta > 0$ that satisfies (4.15), so an equilibrium exists that is linear and this equilibrium is unique in the linear class. If $J < 1 + \psi_*$

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then the term in curly brackets is negative and there is no real δ that satisfies (4.15). If $J \to 1 + \psi_*$ then $\delta \to \infty$, which means dealers' strategies are undefined, so an equilibrium does not exist in linear strategies.

Equations (4.2),(4.11) - (4.15) constitute the solution stated in the proposition. The function $F(J, M)$ referenced in the proposition is defined as $1/(J-1)$ times the square root of the inverse of the expression in curly brackets in (4.15). Thus,

$$\delta = \frac{\sigma_u \sqrt{2}}{\sigma_{vi}} F(J, M). \tag{4.16}$$

The unconditional expected profit of non-dealer i in equilibrium is

$$\pi_i \equiv E\left[\left(\tilde{v} - \tilde{P}_*\right) x_i(\tilde{v}_i, \tilde{\theta})\right]$$

where $x_i(\cdot)$ is agent i's optimal strategy and $\tilde{P}_*$ is the equilibrium price schedule (i.e., evaluated at dealers' and non-dealers' optimal strategies). Thus,

$$\pi_i = E\left[\left[\tilde{v} - \tilde{\theta} - \frac{1}{J\delta}\left(\beta_1 \sum_{\mathcal{I}} v_i + \beta_2(M-J)\tilde{\theta} + \tilde{u}\right)\right](\beta_1 \tilde{v}_i + \beta_2 \tilde{\theta})\right]$$

$$= \beta_1 \sigma_{vi}^2 \left[1 - \frac{\beta_1}{J\delta}\right] + \beta_2 E\left[\tilde{v}\tilde{\theta}\right] - \beta_2 \left[1 + (M-J)\frac{\beta_2}{J\delta}\right] E\left[\tilde{\theta}^2\right]$$

The fact that $\tilde{\theta} = -\frac{\gamma}{\delta}\overline{v}_{\mathcal{J}}$ implies $E\left[\tilde{v}\tilde{\theta}\right] = -\frac{\gamma}{\delta}\sigma_{vi}^2$ and $E\left[\tilde{\theta}^2\right] = \left(\frac{\gamma}{\delta}\right)^2 \frac{\sigma_{vi}^2}{J}$. In addition, $\beta_1/J\delta = 1/2$, so

$$\pi_i = \frac{\beta_1 \sigma_{vi}^2}{2} - \beta_2 \sigma_{vi}^2 \frac{\gamma}{\delta} \left\{\frac{\gamma}{J\delta}\left[1 + (M-J)\frac{\beta_2}{J\delta}\right] + 1\right\}. \tag{4.17}$$

Substituting for γ/δ from (4.13), and β_2/δ inside the curly brackets from (4.14),

$$\pi_i = \frac{\beta_1 \sigma_{vi}^2}{2} - \beta_2 \sigma_{vi}^2 \frac{(J-1+\psi_*)}{2\psi_*} G(J, M)$$

where

$$G(J, M) \equiv \frac{(J-1+\psi_*)}{2J\psi_*}\left\{1 + \frac{M-J}{M-J+1}\left[J\left(\frac{2\psi_*}{J-1+\psi_*}\right) - 1\right]\right\} - 1. \tag{4.18}$$

Substituting for the remaining β_1 and β_2 using (4.1) and (4.14) yields

$$\pi_i = J\delta\sigma_{vi}^2 H(J, M),$$

where

$$H(J, M) = \frac{1}{4} - \frac{1}{M-J+1}\left[J\left(\frac{2\psi_*}{J-1+\psi_*}\right) - 1\right]\frac{(J-1+\psi_*)}{2\psi_*} G(J, M). \tag{4.19}$$

Finally, substituting for δ from (4.16) yields

$$\pi_i = \sigma_u \sigma_{vi} \times J\sqrt{2}\, F(J, M)\, H(J, M). \tag{4.20}$$

This shows that π_i can be factored into the product $\sigma_u \sigma_{vi}$, and an expression involving only J and M.

The unconditional expected profit of informed dealer j in equilibrium is

$$\pi_j \equiv E\left[\left(\tilde{P}_* - \tilde{v}\right) y_i(\tilde{v}_j, \tilde{P}_*)\right]$$

where $y_i(\cdot)$ is dealer j's optimal strategy. Thus,

$$\pi_j = \delta E\left[(\tilde{P}_* - \tilde{v})\tilde{P}_*\right] + \gamma E\left[(\tilde{P}_* - \tilde{v})\tilde{v}_j\right].$$

Since the market clears,

$$\tilde{P}_* = \frac{-\gamma}{\delta}\overline{v}_{\mathcal{J}} + \frac{1}{J\delta}\left(\beta_1 \sum_{i=1}^{M-J} v_i - \beta_2(M-J)\frac{\gamma}{\delta}\overline{v}_{\mathcal{J}} + \tilde{u}\right).$$

By (4.1), $\beta_1/J\delta = 1/2$ so

$$\tilde{P}_* - v = -\left\{\frac{\gamma}{J\delta}\left[1 + (M-J)\frac{\beta_2}{J\delta}\right] + 1\right\}\sum_{j\in\mathcal{J}}\tilde{v}_j - \frac{1}{2}\sum_{i=1}^{M-J}\tilde{v}_i + \frac{1}{J\delta}\tilde{u}.$$

Multiplying and taking expectations yields

$$E\left[(\tilde{P}_* - \tilde{v})\tilde{P}_*\right] = -\left\{\frac{\gamma}{J\delta}\left[1 + (M-J)\frac{\beta_2}{J\delta}\right] + 1\right\}\left(\frac{-\gamma}{J\delta}\right)\left[1 + (M-J)\frac{\beta_2}{J\delta}\right]J\sigma_{vi}^2$$

$$- \frac{1}{4}(M-J)\sigma_{vi}^2 + \left(\frac{1}{J\delta}\right)^2\sigma_u^2$$

$$E\left[(\tilde{P}_* - \tilde{v})\tilde{v}_j\right] = -\left\{\frac{\gamma}{J\delta}\left[1 + (M-J)\frac{\beta_2}{J\delta}\right] + 1\right\}\sigma_{vi}^2 \equiv -\{\cdot\}\sigma_{vi}^2.$$

After simplifying and using the fact that $\delta/2 = \beta_1/J$,

$$\pi_j = \frac{\sigma_u^2}{J^2\delta} - \frac{\delta}{4}(M-J)\sigma_{vi}^2 + \{\cdot\}(M-J)\frac{\beta_2}{J\delta}\gamma\sigma_{vi}^2$$

$$= \frac{\sigma_u^2}{J^2\delta} - \frac{\beta_1}{2J}\sigma_{vi}^2(M-J) + \{\cdot\}(M-J)\frac{\beta_2}{J\delta}\gamma\sigma_{vi}^2$$

$$= \frac{\sigma_u^2}{J^2\delta} - \frac{1}{J}\left[\frac{\beta_1}{2} + \beta_2\frac{(J-1+\psi_*)}{2\psi_*}\{\cdot\}\right]\sigma_{vi}^2(M-J)$$

$$J\pi_j = \frac{\sigma_u^2}{J\delta} - (M-J)\pi_i \tag{4.21}$$

the last line follows from (4.13) and the expression for π_i in (4.17). To see that dealer profit can be factored just as π_i in (4.20) above, substitute for δ from (4.16) and for π_i from (4.20) into (4.21)

$$J\pi_j = \sigma_u\sigma_{vi}\frac{F(J,M)^{-1}}{J\sqrt{2}} - (M-J)\sigma_u\sigma_{vi}J\sqrt{2}\,F(J,M)\,H(J,M). \tag{4.22}$$

Dividing by J shows that π_j can be written as the product of $\sigma_u\sigma_{vi}$ and an expression involving only J and M. ∥

Proof of Proposition 5: First we derive an equation that defines J_* implicitly as a function of M alone. Then we show that the functional form $J_*(M) = M - M^{\frac{1}{3}}$ uniquely satisfies this equation as $M \to \infty$.

At the equilibrium J_* we have $\pi_i = \pi_j$. By Equation (4.20)

$$\pi_j = \pi_i = \sigma_u \sigma_{vi} J_* \sqrt{2}\, F(J_*, M)\, H(J_*, M)$$

and by (4.21) we have

$$J_* \pi_j + (M - J_*)\pi_i = M\pi_i = \frac{\sigma_u^2}{J_* \delta}.$$

Combining these yields

$$\frac{\sigma_u^2}{J_* \delta M} = \sigma_u \sigma_{vi} J_* \sqrt{2}\, F(J_*, M)\, H(J_*, M).$$

Making use of (4.16) to substitute for δ leaves

$$\mathcal{G}(J_*, M) \equiv 2M J_*^2\, F(J_*, M)^2\, H(J_*, M) - 1 = 0 \tag{5.1}$$

as the equation that identifies the equilibrium number of dealers, J_*, when liquidity is displayed. Since $\mathcal{G}$ is independent of all model parameters except J and M, the equilibrium number of dealers J_*, depends only on M.

Consider a solution to (5.1) of the form

$$J_*(M) = M(1 - \epsilon(M)) \tag{5.2}$$

where the function $\epsilon(\cdot)$ is non-negative and converges to zero as $M \to \infty$. Write (4.12) as

$$\mathcal{Z} = (2J - 1)\left(\frac{J}{J-1}\right)\left(\frac{M-J}{M-J+1}\right) - (J - 2).$$

Since both J and $M - J$ increase as M increases, we can use the asymptotic approximations $\frac{J}{J-1} \sim 1 + \frac{1}{J}$ and $\frac{M-J}{M-J+1} \sim 1 - \frac{1}{M-J}$ for the second and third factors of the first term. Substituting from (5.2) for J yields

$$\mathcal{Z} \sim [2M(1 - \epsilon(M)) - 1]\left(1 + \frac{1}{M(1 - \epsilon(M))}\right)\left(1 - \frac{1}{M\epsilon(M)}\right) - M(1 - \epsilon(M)) + 2$$

$$= M(1 - \epsilon(M)) - \frac{2}{\epsilon(M)} + 3 - \frac{1}{M\epsilon(M)} - \frac{1}{M(1 - \epsilon(M))} + \frac{1}{M^2\epsilon(M)(1 - \epsilon(M))}.$$

Ignoring the constant and lower-order terms we have

$$\mathcal{Z} \sim M(1 - \epsilon(M)) - \frac{2}{\epsilon(M)} \tag{5.3}$$

at J_* when M is large.

We repeat the same type of calculation four additional times. First, upon substituting (5.2) into (4.11) and dropping lower-order terms implies that $\psi_* \sim \mathcal{Z}$, so

$$\psi_* \sim M(1 - \epsilon(M)) - \frac{2}{\epsilon(M)}. \tag{5.4}$$

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Second, substituting (5.2) and (5.4) into (4.18), simplifying and ignoring lower-order terms yields

$$G(J_*, M) \sim \frac{-1}{M\epsilon(M)}. \tag{5.5}$$

Third, (5.2), (5.4) and (5.5) into (4.19), and again simplifying and dropping lower-order terms yields

$$H(J_*, M) \sim \frac{M}{(M\epsilon(M))^2}. \tag{5.6}$$

Finally, squaring the definition of $F(\cdot)$ yields

$$F(J_*, M)^2 = \frac{1}{(J_* - 1)^2} \left\{ (J_* - 1 + \psi_*)^2 \left[\frac{1}{J_* - \psi_* - 1} - \frac{1}{2} \left(\frac{1}{J_* - 1} \right) \right] \right.$$
$$\left. + \frac{J_*(M - J_*)}{J_* - 1} \left[\frac{J_* - 1 + \psi_*}{J_* - \psi_* - 1} - \frac{1}{2} \left(\frac{J_*}{J_* - 1} \right) \right] \right\}^{-1}.$$

Substituting from (5.2) and (5.4), noting that $J_* - \psi_* \sim \frac{2}{\epsilon(M)}$, simplifying and dropping lower-order terms yields

$$F(J_*, M)^2 \sim \frac{1}{2M^4\epsilon(M)} \tag{5.7}$$

when M is large.

Substituting (5.2), (5.6) and (5.7) into the expression for $\mathcal{G}$ in (5.1) yields the asymptotic approximation

$$\mathcal{G}(J_*, M) \sim 2M \left[M(1 - \epsilon(M)) \right]^2 \left(\frac{1}{4M^4\epsilon(M)} \right) \frac{M}{(M\epsilon(M))^2} - 1$$

when M is large. Setting this equal to zero and solving for $\epsilon(M)$ shows that (5.1) is satisfied when M is large if and only if

$$\epsilon(M) = \frac{1}{M^{\frac{2}{3}}}. \tag{5.8}$$

This implies that a solution of the form

$$J_*(M) = M - M^{\frac{1}{3}} \tag{5.9}$$

satisfies (5.1) when M is large. Note that this is the only asymptotic solution because candidates for J_* that possess other orders of M (i.e., that differ from M and $M^{\frac{1}{3}}$), or whose coefficients differ from 1 and -1, respectively, will violate (5.1) as $M \to \infty$. $\|$

Proof of Proposition 6: By Proposition 1, the loss of the uninformed in a market with hidden liquidity and J dealers is given by

$$\sigma_u^2 (J\delta)^{-1} = \sigma_u^2 \left\{ \frac{2\sigma_u}{\sigma_{vi}} \sqrt{\frac{J - 2}{J(M - 1)}} \right\}^{-1}$$

for arbitrary J. Since $J_{*,hidden} = M$ we have

$$\sqrt{M} \, \sigma_u^2 (J_*\delta)^{-1} = \sigma_u^2 \left\{ \frac{2\sigma_u}{\sigma_{vi}} \sqrt{\frac{M(M - 2)}{M(M - 1)}} \right\}^{-1}.$$

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As the market grows large,

$$\lim_{M \to \infty} \sqrt{M} \, \sigma_u^2 (J_* \delta)^{-1} = \frac{1}{2} \sigma_u \sigma_{vi}.$$

Thus,

$$\sigma_u^2 (J_{*,hidden} \delta)^{-1} \sim \frac{\sqrt{M}}{2} \sigma_u \sigma_{vi} \qquad (6.1)$$

in a large market.

By Proposition 4, the loss of the uninformed in a market with displayed liquidity and J dealers is given by

$$\sigma_u^2 (J \delta)^{-1} = \sigma_u^2 \left\{ \frac{\sigma_u}{\sigma_{vi}} \sqrt{2} \, J \, F(J, M) \right\}^{-1}.$$

By (5.7), $F \sim \frac{1}{M^2 \sqrt{2\epsilon(M)}}$ when evaluated at J_*, so

$$J_* \delta \sim \left(\frac{\sigma_u}{\sigma_{vi}} \right) \sqrt{2} \left[M(1 - \epsilon(M)) \right] \frac{1}{M^2 \sqrt{2\epsilon(M)}} \sim \left(\frac{\sigma_u}{\sigma_{vi}} \right) \frac{1}{\sqrt{M} \sqrt{M\epsilon(M)}}.$$

By (5.8), $M\epsilon(M) = M^{\frac{1}{3}}$, so

$$J_* \delta \sim \left(\frac{\sigma_u}{\sigma_{vi}} \right) \left(\frac{1}{M^{\frac{2}{3}}} \right)$$

and

$$\sigma_u^2 (J_{*,displayed} \delta)^{-1} \sim M^{\frac{2}{3}} \sigma_u \sigma_{vi}. \qquad (6.2)$$

The ratio of (6.2) to (6.1) indicates that the the expected loss of the uninformed is greater in a large displayed market than a large hidden market by a factor of $2M^{\frac{1}{6}}$. ‖

Proof of Proposition 7: Consider first a market with hidden liquidity. Equations for the price and market-order flow for arbitrary J are

$$\tilde{P} = \frac{1}{2} \sum_{j=1}^{J} \tilde{v}_j + \frac{1}{J\delta} \tilde{\omega} \qquad \text{and} \qquad \tilde{\omega} = \beta \sum_{i=J+1}^{M} \tilde{v}_i + \tilde{u}.$$

Because $J_{*,hidden} = M$, equation (6.1) is equivalent to $J_* \delta \sim \frac{2\sigma_u}{\sigma_{vi}} \frac{1}{\sqrt{M}}$. This, and the fact that $\beta = J\delta/2$, imply that $\beta \sim \frac{\sigma_u}{\sigma_{vi}} \frac{1}{\sqrt{M}}$. These relations imply the following asymptotics

$$\text{Cov}\left[\tilde{P}, \tilde{\omega} \right] = \frac{1}{J_* \delta} \text{Var}\left[\tilde{\omega} \right] \sim \frac{\sqrt{M}}{2} \sigma_u \sigma_{vi}$$

$$\text{Var}\left[\tilde{P} \right] = \frac{1}{4} J_* \sigma_{vi}^2 + \left(\frac{1}{J_* \delta} \right)^2 \text{Var}\left[\tilde{\omega} \right] \sim \frac{M}{2} \sigma_{vi}^2$$

$$\text{Var}\left[\tilde{\omega} \right] = \beta^2 (M - J) \sigma_{vi}^2 + \sigma_u^2 \sim \sigma_u^2.$$

Therefore,

$$R^2 \sim \frac{\frac{M}{4} \sigma_u^2 \sigma_{vi}^2}{\frac{M}{2} \sigma_u^2 \sigma_{vi}^2} = \frac{1}{2}$$

in a market with hidden liquidity.

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Now consider a market with displayed liquidity. Equations for the price and market-order flow for arbitrary J are

$$\tilde{P} = \frac{-\gamma}{J\delta} \sum_{j=1}^{J} \tilde{v}_j + \frac{1}{J\delta} \tilde{\omega} \qquad \text{and} \qquad \tilde{\omega} = \beta_1 \sum_{i=J+1}^{M} \tilde{v}_i + (M - J)\beta_2 \left(\frac{-\gamma}{J\delta} \sum_{j=1}^{J} \tilde{v}_j \right) + \tilde{u}.$$

where

$$J_*\delta = \frac{\sigma_u}{\sigma_{vi}} \sqrt{2} J_* F(M, J_*) \sim \left(\frac{\sigma_u}{\sigma_{vi}} \right) \left(\frac{1}{M^{\frac{2}{3}}} \right)$$

$$\frac{-\gamma}{J_*\delta} = \frac{1}{J_*} \left(\frac{J_* - 1 + \psi_*}{2\psi_*} \right) \sim \frac{2M(1 - \epsilon(M)) - 1}{4M(1 - \epsilon(M))} \sim \frac{1}{2}$$

$$\beta_1 = \frac{J_*\delta}{2} \sim \left(\frac{\sigma_u}{\sigma_{vi}} \right) \left(\frac{1}{2} \right) \left(\frac{1}{M^{\frac{2}{3}}} \right)$$

$$\beta_2 = \frac{-J_*\delta}{M - J_* + 1} \left(1 + \frac{J_*\delta}{\gamma} \right) \sim \frac{\sigma_u}{\sigma_{vi}} \left(\frac{M^{-\frac{2}{3}}}{M\epsilon(M) + 1} \right) = \frac{\sigma_u}{\sigma_{vi}} \left(\frac{1}{M + M^{\frac{2}{3}}} \right) \sim \left(\frac{\sigma_u}{\sigma_{vi}} \right) \frac{1}{M}.$$

These relations imply the following asymptotics

$$\text{Cov} \left[\tilde{P}, \tilde{\omega} \right] \sim M^{\frac{2}{3}} \sigma_{vi} \sigma_u$$

$$\text{Var} \left[\tilde{P} \right] \sim M^{\frac{4}{3}} \sigma_{vi}^2$$

$$\text{Var} \left[\tilde{\omega} \right] \sim \sigma_u^2.$$

Therefore,

$$R^2 \sim \frac{\left(M^{\frac{2}{3}} \sigma_{vi} \sigma_u \right)^2}{M^{\frac{4}{3}} \sigma_{vi}^2 \sigma_u^2} = 1$$

in a market with displayed liquidity.‖

Proof of Proposition 8: Consider non-dealers first. Suppose non-dealer i believes that (i) dealers $j \in \{1, \ldots, J\}$ follow symmetric strategies of the form $y_j = \gamma v_1 + \delta P$, and (ii) other non-dealers $k \neq i$ follow symmetric strategies of the form $x_k = \beta v_2$. Then non-dealer i perceives the market clearing condition as

$$J\gamma \tilde{v}_1 + J\delta \tilde{P} = \tilde{\omega} \qquad \text{where} \qquad \tilde{\omega} = x_i + (N - 1)\beta v_2 + \tilde{u},$$

or equivalently that the market clearing price is the realization of the random variable

$$\tilde{P} = \tilde{P}(\tilde{\omega}) = \frac{-\gamma}{\delta} \tilde{v}_1 + \frac{1}{J\delta} \tilde{\omega}.$$

Non-dealer i then chooses x_i to maximize expected profit conditional on his information $s_i = v_2$:

$$\max_{x_i} E \left[\left(\tilde{v}_1 + v_2 + \frac{\gamma}{\delta} \tilde{v}_1 - \frac{1}{J\delta} \left[x_i + (N - 1)\beta v_2 \right] + \tilde{u} \right) x_i \Big| v_2 \right].$$

Since $E \left[\tilde{v}_1 | v_2 \right] = 0$, the first-order condition implies

$$x_i = \frac{1}{2} \left[J\delta - (N - 1)\beta \right] v_2.$$

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The second-order condition is $\delta > 0$, which we later show is satisfied by choosing the positive root of the quadratic that defines δ. This verifies that if any non-dealer i believes the others follow symmetric linear strategies, he also follows a strategy that is linear in $s_i = v_2$. Symmetry among the non-dealers' strategies implies that any non-dealer i's strategy coefficient must be equal to the coefficient he believes defines the strategies of the other non-dealers:

$$\beta = \frac{1}{2}\left[J\delta - (N-1)\beta\right] \qquad \Longleftrightarrow \qquad \beta = \frac{J\delta}{N+1}. \tag{8.1}$$

Now consider dealers. Suppose dealer j believes that (i) other dealers $k \neq j$ follow symmetric strategies of the form $y_k = \gamma v_1 + \delta P$ and (ii) non-dealers $i \in \{1, \ldots, N\}$ follow symmetric strategies of the form $x_i = \beta v_2$. Then dealer j perceives the market clearing condition as

$$(J-1)\left(\gamma v_1 + \delta \tilde{P}\right) + y_j = \tilde{\omega} \qquad \text{where} \qquad \tilde{\omega} = N\beta\tilde{v}_2 + \tilde{u},$$

or equivalently that the market clearing price is a realization of the random variable

$$\tilde{P} = \tilde{P}(\tilde{\omega}) = \frac{-\gamma}{\delta}v_1 + \frac{1}{(J-1)\delta}\left(\tilde{\omega} - y_j\right). \tag{8.2}$$

Dealer j then chooses a price-contingent supply schedule $y_j(s_j, p)$ to maximize expected profit for each realization of the market clearing price $\tilde{P} = P$ and signal $\tilde{s}_j = v_1$

$$\max_{y_j(\cdot)} E\left[\left(\frac{-\gamma}{\delta}v_1 + \frac{1}{(J-1)\delta}\left(N\beta\tilde{v}_2 + \tilde{u} - y_j\right) - \tilde{v}\right)y_j\right].$$

Pointwise optimization yields a collection of first-order conditions, each relating to a particular realization of the signal and price:

$$E\left[\left(\frac{-\gamma}{\delta}v_1 + \frac{1}{(J-1)\delta}\left(N\beta\tilde{v}_2 + \tilde{u} - 2y_j\right) - \tilde{v}\right)\Big|v_1, \tilde{P} = P\right] = 0.$$

The second-order condition is satisfied if $\delta > 0$, which is verified below. Substituting for ω in (8.2) yields, $\tilde{P} = \frac{-\gamma}{\delta}v_1 + \frac{1}{(J-1)\delta}\left(N\beta\tilde{v}_2 + \tilde{u} - y_j\right)$ so the first-order condition is equivalent to

$$E\left[\tilde{P} - \tilde{v} - \frac{1}{(J-1)\delta}y_j\Big|v_1, \tilde{P} = P\right] = 0$$

or

$$y_j = (J-1)\delta\left(P - E\left[\tilde{v}|v_1, \tilde{P} = P\right]\right). \tag{8.3}$$

From the perspective of dealer j, all variates are jointly normal, so

$$E\left[\tilde{v}|v_1, \tilde{P} = P\right] = v_1 + E\left[\tilde{v}_2|v_1, \tilde{P} = P\right]$$
$$= v_1 + \frac{\mathcal{C}}{\mathcal{V}}\left(P - E\left[\tilde{P}|v_1\right]\right)$$
$$= v_1 + \frac{\mathcal{C}}{\mathcal{V}}\left(P + \frac{\gamma}{\delta}v_1 + \frac{1}{(J-1)\delta}y_j\right) \tag{8.4}$$

where

$$\mathcal{C} \equiv \text{Cov}\left[\tilde{v}_2, \tilde{P}|v_1\right] = \frac{N\beta}{(J-1)\delta}\sigma_{v_2}^2 \tag{8.5}$$

$$\mathcal{V} \equiv \text{Var}\left[\tilde{P}|v_1\right] = \frac{(N\beta)^2\sigma_{v_2}^2 + \sigma_u^2}{(J-1)^2\delta^2}. \tag{8.6}$$

Substituting (8.4) into (8.3) and solving for y_j yields

$$y_j(v_1, P) = (J-1)\delta\left\{\left(\frac{1 - \frac{\mathcal{C}}{\mathcal{V}}}{1 + \frac{\mathcal{C}}{\mathcal{V}}}\right)P - \left(\frac{1 + \frac{\mathcal{C}}{\mathcal{V}}\frac{\gamma}{\delta}}{1 + \frac{\mathcal{C}}{\mathcal{V}}}\right)v_1\right\}.$$

This verifies that if any dealer j believes the others follow symmetric linear strategies, he also follows a strategy that is linear in $s_j = v_1$ and the realized price P. Symmetry among the dealers' strategies implies that any dealer j's strategy coefficients must be equal to the coefficients he believes define other dealers' strategies:

$$\delta = (J-1)\delta\left(\frac{1 - \frac{\mathcal{C}}{\mathcal{V}}}{1 + \frac{\mathcal{C}}{\mathcal{V}}}\right) \qquad\Longleftrightarrow\qquad \frac{\mathcal{C}}{\mathcal{V}} = \frac{J-2}{J} \tag{8.7}$$

and

$$\gamma = -(J-1)\delta\left(\frac{1 + \frac{\mathcal{C}}{\mathcal{V}}\frac{\gamma}{\delta}}{1 + \frac{\mathcal{C}}{\mathcal{V}}}\right) \qquad \text{using (8.7)}$$

$$= -(J-1)\left(\frac{1 + \frac{J-2}{J}\frac{\gamma}{\delta}}{1 + \frac{J-2}{J}}\right) \qquad\Longleftrightarrow\qquad \frac{\gamma}{\delta} = -1. \tag{8.8}$$

To identify separately γ and δ, substitute (8.5) and (8.6) into (8.7)

$$\frac{N\beta}{(J-1)\delta}\sigma_{v_2}^2 = \frac{J-2}{J}\left\{\frac{(N\beta)^2\sigma_{v_2}^2 + \sigma_u^2}{(J-1)^2\delta^2}\right\}.$$

Substituting for β from (8.1) gives an expression in terms of δ alone. Solving that expression for δ and selecting the positive root so second-order conditions are satisfied yields

$$\delta = \frac{N+1}{J}\frac{\sigma_u}{\sigma_{v_2}}\sqrt{\frac{J-2}{N(J+N-1)}}. \tag{8.9}$$

Equations (8.1), (8.8) and (8.9) define the parameter values that characterize a symmetric equilibrium that is unique in the linear class. Note that the assumption in the proposition that $J > 2$ ensures that δ is real valued.

The unconditional expected profit of dealer j in equilibrium is

$$\pi_j \equiv E\left[\left(\tilde{P}_* - \tilde{v}\right)\tilde{y}_{j*}\right]$$

where y_{j*} is the dealer's optimal strategy and P_* is the equilibrium price schedule (i.e., evaluated at dealers' and non-dealers' optimal strategies). Thus,

$$\pi_j = E\left[\left(\tilde{v}_1 + \frac{\tilde{\omega}_*}{J\delta} - \tilde{v}_1 - \tilde{v}_2\right)\delta(\tilde{P}_* - \tilde{v}_1)\right]$$

$$= \delta E\left[\left(\frac{\tilde{\omega}_*}{J\delta} - \tilde{v}_2\right)\frac{\tilde{\omega}_*}{J\delta}\right] \qquad \text{where} \qquad \tilde{\omega}_* = \frac{NJ\delta}{N+1}\tilde{v}_2 + \tilde{u}.$$

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Substituting for $\tilde{\omega}_*$ and taking expectations yields

$$\pi_j = \delta \left\{ \frac{N}{N+1} \left(\frac{N}{N+1} - 1 \right) \sigma_{v_2}^2 + \left(\frac{1}{J\delta} \right)^2 \sigma_u^2 \right\}. \tag{8.10}$$

Substituting from (8.9) for δ and simplifying yields

$$\pi_j = \frac{\sigma_u \sigma_{v_2}}{J} \sqrt{\frac{N}{(J-2)(J+N-1)}}. \tag{8.11}$$

From this, it is easy to show that

$$\frac{\partial \pi_j}{\partial \sigma_{v_2}} > 0 \qquad \frac{\partial \pi_j}{\partial \sigma_u} > 0 \qquad \frac{\partial \pi_j}{\partial J} < 0 \qquad \text{and} \qquad \frac{\partial \pi_j}{\partial N} > 0.$$

Unconditional expected profit of non-dealer i in equilibrium is

$$\begin{aligned}
\pi_i &= E\left[\left(\tilde{v} - \tilde{P}_* \right) \tilde{x}_{i*} \right] \\
&= E\left[\left(\tilde{v}_1 + \tilde{v}_2 - \tilde{v}_1 - \frac{\tilde{\omega}_*}{J\delta} \right) \beta \tilde{v}_2 \right] \\
&= \beta E\left[\tilde{v}_2^2 - \frac{1}{J\delta}(N\beta \tilde{v}_2 + \tilde{u})\tilde{v}_2 \right] \\
&= \beta \left(1 - \frac{N\beta}{J\delta} \right) \sigma_{v_2}^2.
\end{aligned}$$

Substituting for β from (8.1) and simplifying yields

$$\pi_i = \frac{J\delta}{(N+1)^2} \sigma_{v_2}^2. \tag{8.12}$$

Substituting for δ from (8.9) yields

$$\pi_i = \frac{\sigma_{v_2} \sigma_u}{N+1} \sqrt{\frac{J-2}{N(J+N-1)}}. \tag{8.13}$$

From this, it is easy to show that

$$\frac{\partial \pi_i}{\partial \sigma_{v_2}} > 0 \qquad \frac{\partial \pi_i}{\partial \sigma_u} > 0 \qquad \frac{\partial \pi_i}{\partial N} < 0 \qquad \text{and} \qquad \frac{\partial \pi_i}{\partial J} > 0.$$

Finally, note that (8.10) is equivalent to

$$\pi_j = -N\delta \frac{\sigma_{v_2}^2}{(N+1)^2} + \frac{1}{J^2\delta} \sigma_u^2.$$

Using (8.12) to substitute for the first term

$$\pi_j = -\frac{N}{J} \pi_i + \frac{\sigma_u^2}{J^2\delta}$$

or

$$J\pi_j + N\pi_i = \frac{\sigma_u^2}{J\delta}.$$

This completes the proof of Proposition 8. $\|$

Figure 1
Dealer Strategies in Markets with Displayed Liquidity

The parameter $\psi(J,M)$ is plotted as a function of J for the case $M = 40$ over the feasible range where $J > 1 + \psi(J,M)$. M is the number of informed traders and J is the number of informed who trade as liquidity providers. The extent to which $\psi > 1$ indicates how much less aggressive are liquidity providers' orders when liquidity is displayed versus hidden. When $J = 39$, $\psi \approx 5$; liquidity providers trade about one-fifth as aggressively in a displayed market than an otherwise identical hidden market. When $J = 40$, $\psi = 1$; liquidity providers trade as aggressively in a displayed as in a hidden market.
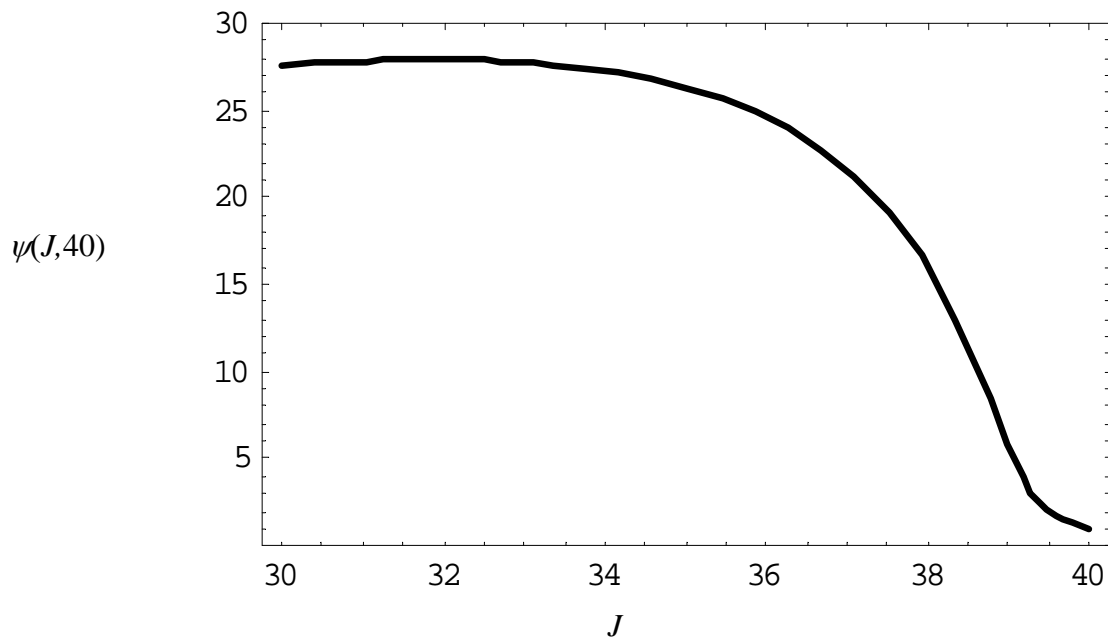
Figure 2
Equilibrium Profit Functions in a Market with Displayed Liquidity

The heavy (light) line is a graph of the profit of an individual dealer (non-dealer) as a function of the number of informed agents who trade as dealers, J, when there are a total of $M = 40$ informed agents. The dealer's profit decreases, and the non-dealer's profit increases in J. With $M = 40$, dealer and non-dealer profits are equal when approximately 37 informed agents trade as dealers.
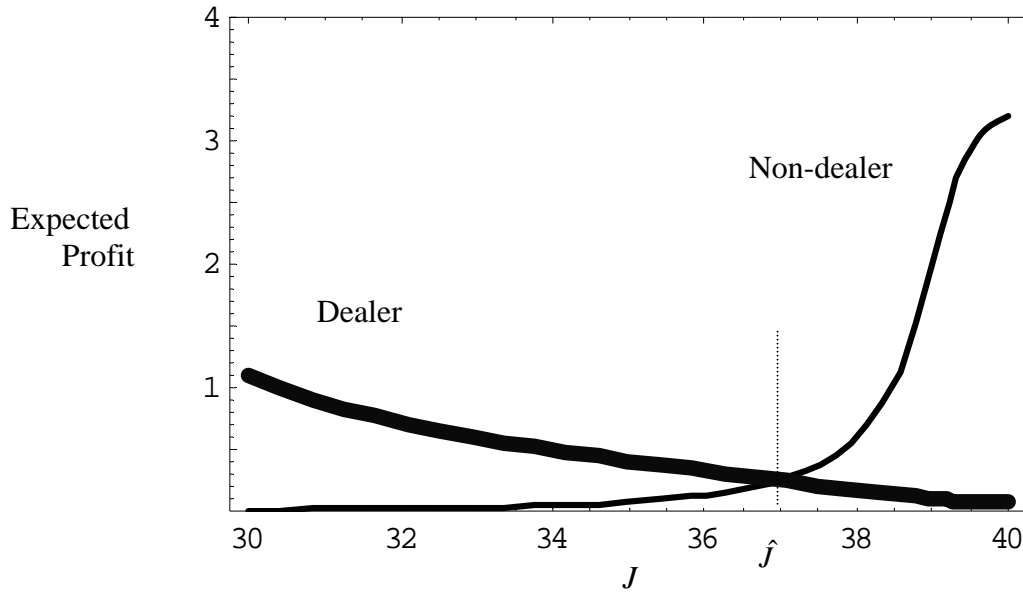
Figure 3
Equilibrium Number of Liquidity Providers when Liquidity is Displayed

The solid line plots the number of informed agents who trade as dealers, $J_*(M)$, as a function of the total number of informed agents M. The dashed line is the boundary $1+ \psi(J,M)$ below which the an equilibrium does not exist. The fact that $J_*(M)$ plots above the dashed line for all $M > 2$ implies that the equilibrium in Proposition 4 exists for all $M > 2$.
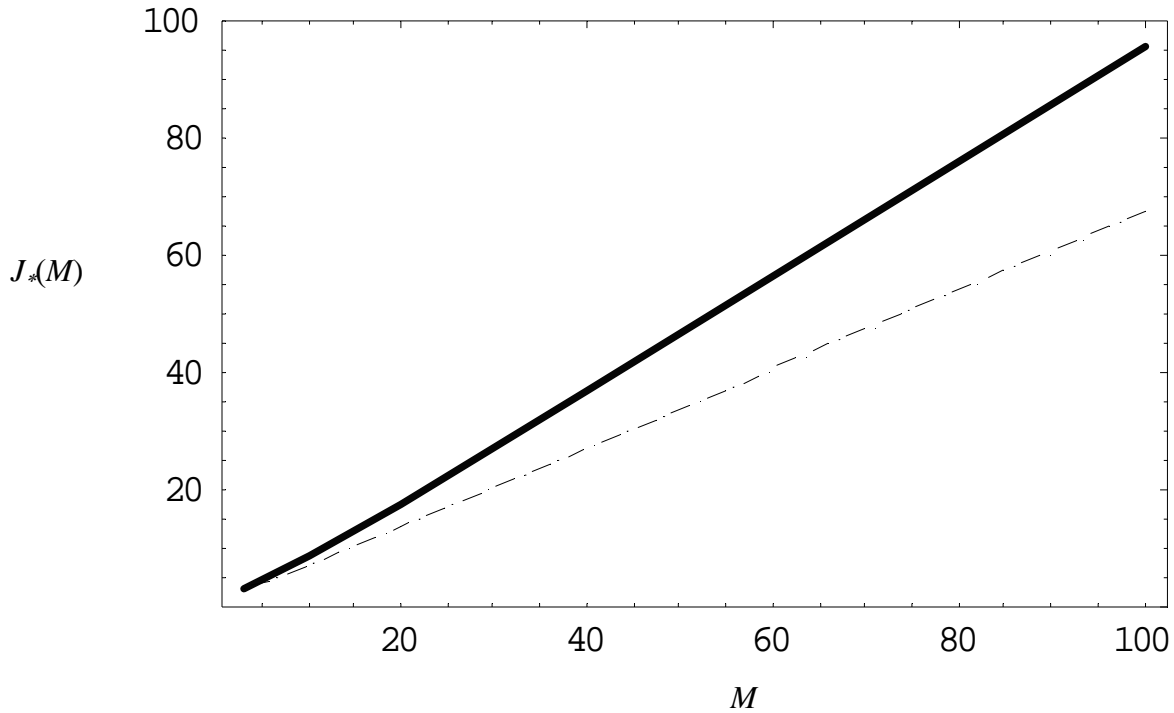
Figure 4
Uninformed Trader Expected Loss in a Market with Displayed Liquidity

The expected loss of the uninformed is plotted as a function of J for the case $M = 40$ over the feasible range where $J > 1 + \psi(J,M)$. The loss of the uninformed decreases monotonically as the number of dealers increases and are minimized when all the informed agents choose to provide liquidity, i.e., when $J = M$.
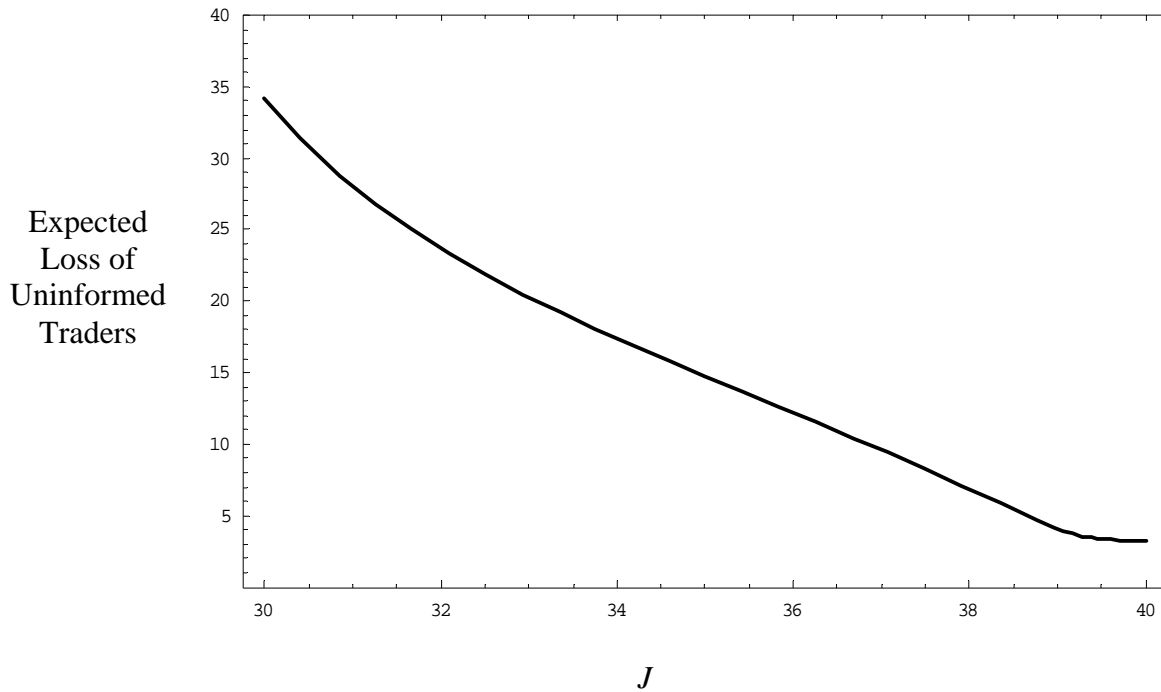
Expected
Loss of
Uninformed
Traders

J

Figure 5
Convergence to Large Market Limit

The line graphs the number of informed who trade as liquidity demanders (non-dealers), $M - J_*$, divided by $M^{1/3}$, as a function of the total number of informed agents M. As shown in Proposition 5, this ratio converges to one in the large-market limit as $M \to \infty$. The graph indicates that convergence occurs rather quickly, being close to one even for $M = 40$.

M
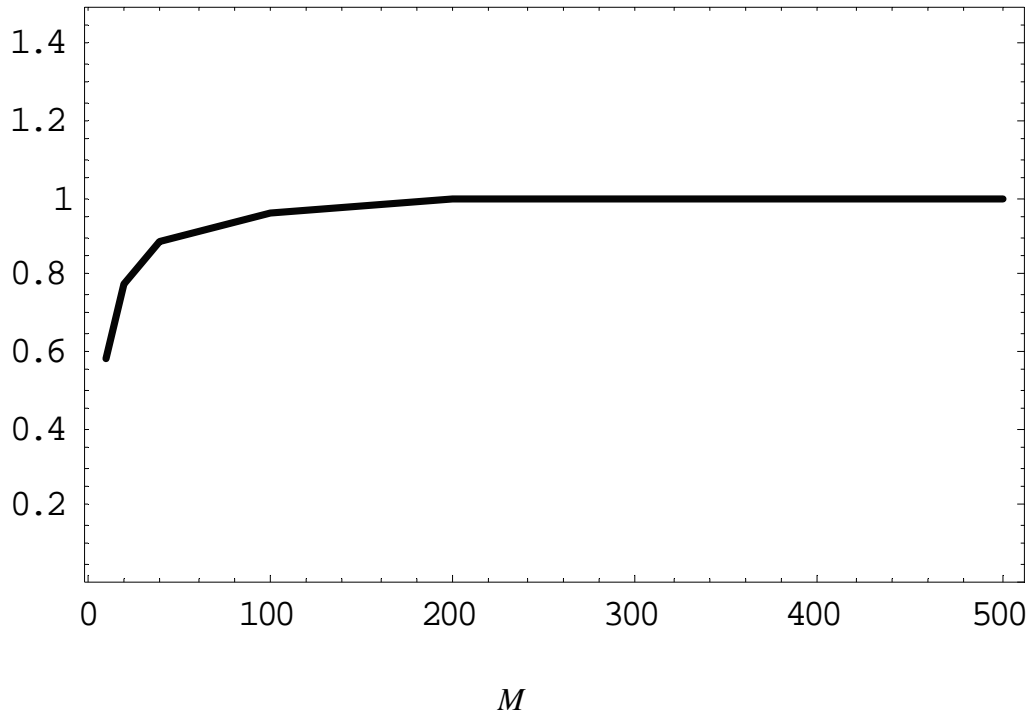
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