The Pricing Implications of Counterparty Risk
For Non Linear Credit Products

September 6, 2005

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The comments of the two referees, Tobias Adrian, Roy Mashal, Marco Naldi, Michael Ong, Michael Pykhtin, Joshna Rosenberg, Til Schuermann, Wulin Suo, Jun Yang and seminar participants at the Bank of Canada, New York Federal Reserve Bank, Queen's University, Southern Methodist University and the University of Calgary are gratefully acknowledged. The standard disclaimer applies.

Abstract

In this paper we describe a methodology for deriving the upper and lower profit and loss (P&L) bounds in the presence of counterparty risk that does not rely on either structural or reduced form credit models. The methodology provides practitioners and regulators with a practical tool to estimate the impact on P&L of the two facets of counterparty risk: failure to perform and mark-to-market exposure. We show that for many applications, the bounds are tight and the credit worthiness of counterparties can have a major impact on the P&L.

Introduction

The recent decline in credit quality over the last few years, the demise of such firms as Enron and Parmalat, and the concentration of credit exposure among a few major financial institutions are salient reminders about the importance of counterparty risk exposure[1]. The rapid increase in the size of credit derivative markets has raised concerns about financial stability arising from credit risk transfer[2]. One of the many facets of counterparty risk is its effects on the pricing of contracts and the recording of profit and loss (P&L) and associated reserves.

Counterparty risk is the risk that a party to a contract might default over the life of the contract. There are two main facets to counterparty risk: (a) a mark-to-market risk and (b) failure to honor a contract when required to perform. For example, consider a credit default swap where there is the risk that the protection seller might default and for simplicity, we assume that there is no risk that the protection buyer will default. If the protection seller defaults, to restore the protection buyer to the position prior to default, necessitates pricing a swap with the same premium in the absence of counterparty risk[3]. If the credit quality of the reference entity has deteriorated (improved) then the protection buyer suffers an economic loss (gain) in entering into a new contract with the same premium. If the reference entity defaults, and the protection seller subsequently defaults prior to settlement, the protection buyer is exposed to the full loss from the reference entity.

Steps to mitigate counterparty risk span a wide spectrum: from limiting total exposure to individual counterparties, exposure to particular sectors, master contract agreements that facilitate netting, "haircuts" in pricing, posting of collateral[4], to payment in advance. Counterparty risk affects the value of a contract – real or financial - and failure to recognize this means that P&L is inaccurate and the incentive structure for traders possibly inappropriate. For many types of contracts it is possible to incorporate the effects of counterparty risk directly into the pricing and the P&L. For simple linear products, Johnson and Stulz (1986) consider counterparty risk using a structural model. Observing that structural models are difficult to apply to real world

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[1]See – Fitzpatrick (2002). Fitch Ratings (2004) reports 69 % of total counterparty exposure is held by the top 10 institutions.
[2] See the recent Basel Committee on Banking Supervision report "Credit Risk Transfer", (October, 2004).
[3]In calculating the mark-to-market risk the usual market convention is to ignore future counterparty risk. We follow this convention.
[4]The collateral may change over the life of the contract depending on the credit worthiness of the counterparty and the exposure.

applications, Jarrow and Turnbull (1992, 1995) address the pricing of counterparty risk using their reduced form methodology. Duffie and Huang (1996) and Jarrow and Turnbull (1997) use this reduced form methodology for the pricing of biltateral counterparty risk for interest rate and foreign exchange swaps[5] and a general review of pricing counterparty risk for standard linear financial products is given in Arvanitis and Gregory (2001, chapter 6)[6]. However, for non-linear products there is a major complication, that has not been addressed in the academic literature[7].

For non-linear products such as collateralized debt obligations, simulation is used for valuation, as closed form solutions do not in general exist because of the complexity of the contracts. Consequently, incorporating the mark-to-market facet of counterparty risk introduces a major complication. Default by, say, the protection seller can occur at any time over the life of the contract. When default occurs, it is necessary to perform a Monte Carlo simulation to determine the mark-to-market value. This implies that it is necessary to perform a simulation at each possible time, but default can occur at any point and there are an infinite number of points on the real line. Consequently brute force approaches are infeasible. A possible solution might be to assume that default can occur only at a finite number of times. Such an approach introduces its own set of problems and is of limited use. Given the dependence structure between the risky counterparty and the reference entities, it is necessary to determine the intensity functions of the surviving obligors, conditional on the information set at the chosen time – see Schönbucher and Schubert (2001). While this can be done for one reference entity, it quickly becomes infeasible for more than one reference entity – see Section 4.

Recently, Mashal and Naldi (2005) have introduced a way to avoid this computational limitation. They derive upper and lower bounds for the premium of a contract in the presence of counterparty risk. While it is easy to derive upper and lower bounds, the major challenge is to derive bounds that can be computed and thus avoid the computational barrier. In Mashal and Naldi the bounds can be computed and they demonstrate that the bounds are quite tight, at least for 'low' levels of risk. There are two advantages to the MN methodology. First, it can be applied to many types of credit structures, such as credit default swaps, synthetic CDO tranches,

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[5]The recent paper by Canabarro *et al* (2003) considers counterparty risk for a portfolio of securities. They do not address the questions of how counterparty risk affects the pricing of a security or the misstatement in the P&L.
[6]See Hughston and Turnbull (2001) for a treatment of collateral.
[7]Jarrow and Yu (2001) consider the pricing of the debt obligations of a firm, when its credit worthiness is affected by the default of another firm. They do not address the facets of counterparty risk that are the focus of this paper.

and first to default swaps. Second, the methodology does not explicitly rely on either the structural or reduced form approaches to pricing credit risky assets and allows interest rates and any relevant state variables to be stochastic.

Mashal and Naldi do not address how the recognition of counterparty risk affects the P&L. Yet senior management and regulators are concerned about how counterparty risk affects the value of a position. A trade may be recorded as profitable if counterparty risk is ignored, but possibly unprofitable if counterparty risk is recognized. We extend their analysis to derive upper and lower bounds for the misspecification in the P&L and examine the tightness of the bounds. Note that we are using the term profit and loss (P&L) in a broad sense. For most banks, P&L is not affected by changes in the counterparty credit quality. It does affect earnings through the reserving process.

In Section 2, we derive for a structured product expressions for the P&L when only one counterparty can default, or both counterparties might default. Except for simple products such as credit default swaps, these expressions cannot be evaluated. We derive in Section 3 upper and lower bounds for the P&L. All bounds can be computed. In Section 4, we present the simulation results. We first consider the pricing of a credit default swap in the absence of counterparty risk. A closed form solution exists for this case and consequently we have a bench mark to judge the accuracy of our simulation results. For the P&L bounds, we introduce a control variate technique. For a credit default swap, we compute the upper and lower bounds, first for the case of a risky protection seller, then a risky protection buyer, and finally when both counterparties are risky. We also examine the impact of settlement risk on the bounds for a risky protection seller. We next consider an example where a trader buys protection at the premium S from one counterparty and sells protection on the same reference entity at the price S + b to another counterparty. We show that the effects on the P&L due to counterparty risk can be substantial, especially if the counterparties have "high yield" ratings. Finally, we consider a synthetic CDO with a collateral portfolio of 100 names and examine the impact of counterparty risk for the different tranches. The sensitivity of the bounds to correlation is also examined. The practical usefulness of the methodology is evaluated in Section 5.

2 P&L In the Presence of Counterparty Risk

In this section we draw on the analysis first given by Mashal and Naldi (2005) (MN). We start by introducing some standard notation. Consider a probability space (Ω, F, Q), equipped with a filtration F_t and time zero representing the present time. We assume the complete information set can be expressed as a union of the form $F_t = G_t \cup H_t$, where G_t is the filtration generated by the trajectory of a multi-dimensional Brownian motion and H_t is the filtration generated by the default processes for the reference entities, $N_j^{RE}(t)$, which equals one if default has occurred, zero otherwise, j = 1, …, n, where n represents the number of reference entities. The default process for the protection seller (protection buyer) $N^{PS}(t)$, ($N^{PB}(t)$) equals one if default has occurred, zero otherwise. To simplify our expression, we will abuse this notation in cases where there is no risk of ambiguity and write $N(\tau)$ to denote the state of the default process, where the symbol τ is used to denote the time to default for a particular party.

We start by considering the profit and loss (P&L) for a general structure, ignoring counterparty risk. We then consider the effects on the P&L if there is the risk that the protection seller might default. In the third section, we consider the case of a risky protection buyer and in the last section we consider the case where both counterparties are risky.

2.1 Credit Structured Products

We start by considering a structured product tranche written on a pool of reference entities. The tranche buyer is effectively purchasing protection on the collateral portfolio over a defined range of losses. Let M denote the dollar exposure of a particular tranche, L(t) the cumulative dollar loss on the tranche up to time t, S the premium in the absence of counterparty risk[8]. The protection buyer promises to make payments at times $T_1, \ldots, T_n$. The time between payments dates is denoted by $\Delta_j = T_j - T_{j-1}$, j = 1, …, n. The premium is paid at the end of each period and is given by $S\Delta_j [M - L(T_{j-1})]^+$, j = 1, … , n, and protection is paid when losses occur[9]. The value of the *premium leg* is given by

[8] In the formal derivation, we ignore the complication of accrued interest. Accrued interest is incorporated in the simulations. See O'Kane and Turnbull (2003) for a treatment of accrued interest.
[9] The symbol X^+ is defined as $X^+ = \max(X, 0)$.

$$S\,E[\sum_{j=1}^{n} B(0,T_j)\Delta_j\,[M - L(T_{j-1})]^+ \,|\, F_0\,]$$

and the value of the *protection leg*

$$E[\int_0^{T_n} B(0,u)\,dL(u)\,I_{(M > L(u))}\,|F_0\,]$$

where $I_{(A)}$ is an indicator function that equals one if the event A is true, zero otherwise. In the above expression the indicator function equals one, provided losses over the life of the instrument do not exceed the dollar exposure of the tranche ($M > L(u)$). If losses do exceed M, the protection seller is not responsible for the excess losses and the indicator function is zero. It will prove convenient to define some general terms. Let

$$PB(k,K) \equiv \sum_{j=k}^{K} B(t,T_j)\Delta_j\,[M - L(T_{j-1})]^+$$

represent the discounted cash flows, where $t \le T_k$ and

$$PS(t,T) \equiv \int_t^{T} B(0,u)\;I_{(M > L(u))}\,dL(u)$$

the discounted cash flows made by the protection seller over the interval [t, T].

If we ignore counterparty risk, the booked profit to the protection buyer is

$$\pi^{NC} = E[PS(0,T_n)|F_0] - S\,E[PB(1,n)|F_0] \qquad (1)$$

By ignoring counterparty risk, we are incorrectly recording the P&L.

2.2 Default by the Protection Seller

We start by assuming that we have purchased credit protection and there is no risk that we will default. However, there is risk that the protection seller might default. Two events will impose costs on us if the protection seller defaults:

(a) no default by the reference entity, default by the protection seller;
(b) the reference entity defaults and before settlement, the protection seller defaults.

The first case imposes a mark-to market risk on the protection buyer and the second case a failure to perform risk. We analyze both cases.

Suppose that the protection seller defaults at time τ_{PS} over the life of the contract. Given the default by the protection seller, if the tranche buyer purchased new credit protection from a default free protection seller, the premium remaining unchanged, the value of this new tranche to the protection buyer is given by

$$V(\tau_{PS}) \equiv E[PS(\tau_{PS}, T_n)|F_{\tau_{PS}}] - SE[PB(z+1, n)|F_{\tau_{PS}}] \qquad (2)$$

where the index z is defined by $T_Z < \tau_{PS} \le T_{Z+1}$. The value of the tranche is the difference between the values of the protection leg and premium leg.

If the credit quality of the reference portfolio has improved, implying that the tranche had value to the original protection seller, $V(\tau_{PS}) < 0$, the protection buyer makes a payment of

$$-R_{PB}V(\tau_{PS}) \ge 0$$

to the protection seller, where $0 \le R_{PB} \le 1$ represents the fraction of the amount due. Note that we have a minus sign, given that $V(\tau_{PS})$ is negative.

The value at time zero of the payments by the protection buyer is given by

$$\begin{aligned}&SE[\sum_{j=1}^{n} B(0, T_j)\Delta_j\,[M - L(T_{j-1})]^{+}\,I_{(\tau_{PS} > T_j)}|F_0]\\ &- E[\,R_{PB}\,U(\tau_{PS})V(\tau_{PS})\,I_{(V(\tau_{PS}) \le 0)}\,|F_0]\\ &\equiv SE[PB_{PS}(1, n)|F_0] - E[\,R_{PB}\,U(\tau_{PS})V(\tau_{PS})I_{(V(\tau_{PS}) \le 0)}\,|F_0]\end{aligned}$$

where we redefine $PB_{PS}(1, n)$ as the payments are contingent on no default by the protection seller and

$$U(\tau_{PS}) \equiv B(0, \tau_{PS})\,I_{(T_n > \tau_{PS})}\,I_{(N(\tau_{PS})=1)} \quad \ge 0$$

which represents the discount factor given that default by the protection seller occurs before default of the reference entity during the life of the contract.

If $V(\tau_{PS}) \ge 0$, the protection seller makes a payment

$$R_{PS}^{D}V(\tau_{PS}) \ge 0)$$

to the protection buyer, where $0 \le R_{PS}^{D} \le 1$ represents the fraction of the amount due. We use the superscript D to indicate that default by the protection seller has occurred. If the protection seller defaults at time τ_{PS} during the life of the contract, then the losses within a settlement period Δ will not be covered. This means subtracting the amount $L(\tau_{PS}) - L(\tau_{PS} - \Delta)$ from the stated losses paid and the protection seller making a payment $R_{PS}^{D}[L(\tau_{PS}) - L(\tau_{PS} - \Delta)]$. The value at time zero of the payments by the protection seller is given by

$$E[\int_0^{T_n} B(0,u)\,dL(u)\,I_{(M>L(u))}\,I_{(\tau_{PS}>u)}|F_0]$$
$$-E[B(0,\tau_{PS})(1-R_{PS}^D)[L(\tau_{PS})-L(\tau_{PS}-\Delta)]I_{(T_n\geq\tau_{PS})}|F_0]$$
$$+\;E[R_{PS}^D\;U(\tau_{PS})V(\tau_{PS})I_{(V(\tau_{PS})>0)}|F_0]$$
$$\equiv E[PS_{PS}(0,\;T_n)|F_0]+\;E[R_{PS}^D\;U(\tau_{PS})V(\tau_{PS})I_{(V(\tau_{PS})>0)}|F_0]$$

where we redefine $PS_{PS}(0,T_n)$ to include the loss during the settlement period.

We assume that $E[R_{PB}]\geq E[R_{PS}^D]$. This is a reasonable assumption, given that only the protection seller has defaulted. Second, we assume that the random recovery rates are independent of the other stochastic factors. This is a simplifying assumption, as we know from the work of Acharya *et al* (2003) and Altman *et al* (2005) that default rates and recovery rates are negatively correlated. Given these assumptions, the P&L becomes

$$\begin{aligned}\pi_{PS}^C &= E[PS_{PS}(0,T_n)|F_0]-\;SE[PB_{PS}(1,n)|F_0]\\ &\quad+E[U(\tau_{PS})\min[E(R_{PS}^D)V(\tau_{PS}),E(R_{PB})\;V(\tau_{PS})]|F_0]\\ &\equiv FP_{PS}+E[U(\tau_{PS})\min[E(R_{PS}^D)V(\tau_{PS}),E(R_{PB})\;V(\tau_{PS})]|F_0]\end{aligned}\tag{3}$$

where FP_{PS} denotes the P&L incorporating the effects of failure to perform in the event of default by the risky protection seller. The second term on the right side reflects the mark-to-market facet of counterparty risk on the P&L.

2.3 Default By the Protection Buyer

Here we now assume that the counterparty risk is on the side of the protection buyer. Suppose that the protection buyer defaults at time τ_{PB}. If $V(\tau_{PB})<0$, the swap has value to the protection seller[10] and consequently the protection buyer makes a payment of

$$-R_{PB}^D V(\tau_{PB})\geq 0$$

to the protection seller, where R_{PB}^D denotes the recovery rate given default by the protection buyer. At time zero, the value of the payments by the protection buyer is given by

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[10] Recall that we are defining the value $V(\tau_{PB})$, as

$V(\tau_{PB})\equiv E[PS(\tau_{PB},T_n)|F_{\tau_{PB}}]-S\,E[PB(z+1,n)|F_{\tau_{PB}}]$

$$
\begin{aligned}
& S\, E[\sum_{j=1}^{n} B(0, T_j)\, \Delta_j\; [M - L(T_{j-1})]^{+}\, I_{(\tau_{PB} > T_j)}\, |F_0] \\
& - E[\, R_{PB}^{D}\, U(\tau_{PB})\, V(\tau_{PS})\; I_{(V(\tau_{PS}) \le 0)}\; |F_0] \\
& \equiv S\, E[PB_{PB}(1, n)|F_0] - E[\, R_{PB}^{D}\, U(\tau_{PB})\, V(\tau_{PB})\, I_{(V(\tau_{PB}) \le 0)}\; |F_0]
\end{aligned}
$$

where we redefine $PB_{PB}(1, n)$ as the payment stream is contingent on no default by the protection buyer and

$$
U(\tau_{PB}) \equiv B(0, \tau_{PB})\, I_{(N(\tau_{PB})=1)}\, I_{(\tau_{RE} > \tau_{PB})} \; \ge 0
$$

which represents the discount factor given that default by the protection buyer occurs before default of the reference entity during the life of the contract.

If $V(\tau_{PB}) \ge 0$, the protection seller makes a payment

$$
R_{PS} V(\tau_{PB}) \ge 0
$$

to the protection buyer. At time zero, the value of the payments by the protection seller is given by

$$
E[PS(0, T_n)|F_0] + E[R_{PS}\, U(\tau_{PB})\, V(\tau_{PB})\, I_{(V(\tau_{PB}) > 0)}\, |F_0]
$$

where

$$
PS_{PB}(0,\; T_n) \equiv \int_{0}^{T_n} \; B(0, u)\; I_{(M > L(u))}\, I_{(\tau_{PB} > u)}\, dL(u)
$$

Note that we do not consider the joint default of the reference entity and the protection buyer. If the protection buyer defaults, apart from any mark-to-market payment, accrued interest is also due. In this analysis we are ignoring accrued interest. We do include accrued interest in the simulations, where the protection buyer is assumed to make a payment of $S_C\, R_{PB}^{D}\, (\tau_{PB} - T_{j-1})\, I_{(T_{j-1} < \tau_{PB} \le T_j)}$. Default by the reference entity triggers payment by the protection seller.

If we had sold protection, the P&L becomes

$$
\begin{aligned}
\pi_{PB}^{C} = {} & S\, E[PB(1, n)|F_0] - E[PS(0, T_n)|F_0] \\
& - E[R_{PB}^{D}\, U(\tau_{PB})\, V(\tau_{PB})\, I_{(V(\tau_{PB}) \le 0)}\, |F_0] \\
& - E[R_{PS}\, U(\tau_{PB})\, V(\tau_{PB})\, I_{(V(\tau_{PB}) > 0)}\, |F_0]
\end{aligned}
$$

In this case, it is reasonable to assume that $E(R_{PB}^{D}) \le E(R_{PS})$, implying that the above expression can be written in the form

$$\begin{aligned}\pi_{PB}^{C} &= S\,E[PB_{PB}(1,n)|F_0] - E[PS_{PB}(0,T_n)|F_0] \\ &\quad - E[U(\tau_{PB})\max[E(R_{PS})V(\tau_{PB}),\ E(R_{PB}^{D})V(\tau_{PB})]|F_0] \\ &\equiv FP_{PB} - E[U(\tau_{PB})\max[E(R_{PS})V(\tau_{PB}),\ E(R_{PB}^{D})V(\tau_{PB})]|F_0]\end{aligned} \tag{4}$$

The term FP_{PB} denotes the P&L incorporating the effects of failure to perform in the event of default by the risky protection buyer. The second term on the right side reflects the mark-to-market facet of counterparty risk on the P&L.

Note that the last term on the right side of expressions (3) and (4) arises because of the mark-to-market risk. If Monte Carlo simulation is used for the valuation ignoring counterparty risk, then in both cases we face a computation problem. The risky counterpart can default any time over the life of the contract and when default occurs, it is necessary to undertake a new simulation to determine the value $V(\tau)$, where τ denotes the stopping time of the risky counterparty, implying that we need to undertake a simulation within a simulation.

2.4 Both Counterparties Risky

Up to the present point, we have only considered the case of either the protection seller or the protection buyer being risky. This is a common assumption, in the setting of reserves. In reality, the protection buyer and seller may be of similar credit quality and hence either is equally likely to default. In the following analysis, we allow either party to default. However, we ignore the possibility of both counterparties defaulting together, as this is such a rare event it adds only complication without additional benefit[11]. The default swap premium is denoted by S.

Suppose that the protection seller defaults at time τ_{PS} *before* default by the protection buyer. Given the default by the protection seller, if the protection buyer purchased new credit protection, the premium remaining unchanged, the value of this new contract to the protection buyer, ignoring all future counterparty risk, is given by

[11] Regulators are concerned by systemic risk – the failure of one institution causing the failure of other institutions. This analysis allows for systemic risk through the modeling the dependence of the stopping times. We are simply ruling out two counterparties defaulting at the same time.

$$V(\tau_{PS}) = E[PS(\tau_{PS}, T_n)|F_{\tau_{PS}}] - S E[PB(z+1, n)|F_{\tau_{PS}}]$$

where the index z is defined by $T_Z < \tau_{PS} \leq T_{Z+1}$.

If $V(\tau_{PS}) < 0$, the swap had value to the protection seller, and the protection buyer makes a payment of

$$-R_{PB} V(\tau_{PS}) < 0$$

to the protection seller. Note that we have a minus sign given that V(.) is negative. If $V(\tau_{PS}) \geq 0$, the protection seller makes a payment $R^D_{PS} V(\tau_{PS}) \geq 0$ to the protection buyer.

Suppose now that the protection buyer defaults at time τ_{PB} *before* the protection seller. Given the default by the protection buyer, the value of a new credit contract on the same reference portfolio to the protection buyer, is given by

$$V(\tau_{PB}) \equiv E[PS(\tau_{PB}, T_n)|F_{\tau_{PB}}] - S E[PB(z+1, n)|F_{\tau_{PB}}]$$

where the index z is defined by $T_Z < \tau_{PB} \leq T_{Z+1}$.

If $V(\tau_{PB}) < 0$, the swap had value to the protection seller, and the protection buyer makes a payment of

$$-R^D_{PB} V(\tau_{PB}) \geq 0$$

to the protection seller. If $V(\tau_{PB}) \geq 0$, the protection seller makes a payment $R_{PS} V(\tau_{PB}) \geq 0$ to the protection buyer.

For the two cases, the value of the payments by the protection buyer is given by

$$S E[PB(1, n)|F_0] - E[R_{PB} U(\tau_{PS}) V(\tau_{PS}) I_{(V(\tau_{PS}) \leq 0)} |F_0] \\ - E[R^D_{PB} U(\tau_{PB}) V(\tau_{PB}) I_{(V(\tau_{PB}) \leq 0)} |F_0]$$

where

$$U(\tau_{PS}) \equiv B(0, \tau_{PS}) I_{(\tau_{RE} > \tau_{PS})} I_{(N(\tau_{PS})=1)} \geq 0$$

and

$$U(\tau_{PB}) \equiv B(0, \tau_{PB}) I_{(\tau_{RE} > \tau_{PB})} I_{(N(\tau_{PB})=1)} \geq 0$$

The value of the payments by the protection seller is given by

$$E[PS(0,\ T_n)|F_0] + E[\ R^D_{PS}\ U(\tau_{PS})V(\tau_{PS})I_{(V(\tau_{PS})>0)}\ |F_0]$$
$$+\ E[\ R_{PS}\ U(\tau_{PB})V(\tau_{PB})I_{(V(\tau_{PB})>0)}\ |F_0]$$

where we redefine $PS(0, T_n)$ to include the loss during the settlement period.

The P&L from buying protection is

$$\begin{aligned}\pi^C_{PBS} = \ & E[PS(0, T_n)|F_0] - S\,E[PB(1, n)|F_0] \\ & + E[U(\tau_{PS})\ \min[E(R^D_{PS})V(\tau_{PS}),\ E(R_{PB})V(\tau_{PS})]|F_0] \\ & + E[U(\tau_{PB})\ \max[E(R_{PS})V(\tau_{PB}),\ E(R^D_{PB})V(\tau_{PB})]|F_0]\end{aligned} \qquad (5a)$$

Again, we rewrite this expression in the form

$$\begin{aligned}\pi^C_{PBS} = \ & FP_{PBS} \\ & + E[U(\tau_{PS})\ \min[E(R^D_{PS})V(\tau_{PS}),\ E(R_{PB})V(\tau_{PS})]|F_0] \\ & + E[U(\tau_{PB})\ \max[E(R_{PS})V(\tau_{PB}),\ E(R^D_{PB})V(\tau_{PB})]|F_0]\end{aligned} \qquad (5b)$$

The first term on the right side, FP_{PBS} is the P&L incorporating the failure to perform in the event of default by either counterparty. The second and third terms reflects the mark-to-market facet of counterparty risk on the P&L if either the protection seller or protection buyer defaults.

3 Upper and Lower Bounds

In this section, we establish upper and lower bounds for the P&L in the presence of counterparty risk. We first consider the case that the protection seller might default, second, the case that the protection buyer might default and finally the case when either the protection buyer or seller might default.

3.1 Credit Risky Protection Seller

If we purchase protection, paying a premium S on a tranche of a structured product, from expression (3) the P&L can be written in the form

$$\pi_{PS}^{C} \equiv FP_{PS} + E[\ \min[E(R_{PS}^{D})\ U(\tau_{PS})V(\tau_{PS}), E(R_{PB})U(\tau_{PS})V(\tau_{PS})]\,|\,F_0] \qquad (6)$$

The above expression cannot be evaluated, in general. To understand why, consider the type of algorithm we would use if we attempted to evaluate the above expression. Suppose that we perform a simulation involving, say, 100,000 runs.

Step 1

For each run, generate the default times for all the reference entities and the protection seller over the life of the tranche. We refer to the history of defaults over the life of the tranche as a path. Calculate the discounted cash flows to the different parties[12].

Step 2

If the protection seller defaults over the life of the contract, it is necessary to determine the value of the tranche at the time of the default, $V(\tau_{PS})$, in order to evaluate whether the protection seller makes a payment to the protection buyer if the value of the replacement tranche is positive or vice versa. To determine the value of the tranche, $V(\tau_{PS})$, we must undertake a separate simulation.

Step 3

Go back to Step 1 and repeat steps 1 and 2 for the total number of runs.

It is the requirement to perform a separate simulation in Step 2 that makes this approach infeasible. Default by the protection seller can occur at any point over the life of the contract, and

[12] The cash flows are discounted at the risk free rate of interest, given that we are using the pricing measure.

there are an infinite number of points. At each point, it is necessary to undertake a separate simulation.

While we cannot evaluate expression (6), we can however derive upper and lower bounds. We first derive the upper bound.

Proposition Upper Profit Bound

An upper profit bound is given by

$$\pi_{PS}^{UC} = FP_{PS} + \min\{E(R_{PS}^{D})E[U(\tau_{PS})V(\tau_{PS})|F_0], E(R_{PB})E[U(\tau_{PS})V(\tau_{PS})|F_0]\} \qquad (7)$$

Proof

Given that min(. ,.)is a concave function, then applying Jensen's inequality to expression (6) gives

$$\begin{aligned}\pi_{PS}^{C} &\leq FP_{PS} + \min\{E(R_{PS}^{D})E[U(\tau_{PS})V(\tau_{PS})|F_0], E(R_{PB})E[U(\tau_{PS})V(\tau_{PS})|F_0]\} \\ &= \pi_{PS}^{UC}\end{aligned}$$

We can evaluate expression (7). The expression involves the present value of replacement tranche, $E[U(\tau_{PS})V(\tau_{PS})|F_0]$, which we can estimate. The pricing algorithm now becomes:

Step 1

Generate the default times for all the reference entities and the protection seller over the life of the tranche. Calculate the discounted cash flows to the different parties.

Step 2

If the protection seller defaults over the life of the contract, we need to calculate the cash flows to the protection buyer and the new protection seller. But along the path, we know the history of remaining defaults and we can determine the discounted value of the payments made by the protection buyer and the new protection seller. We store these values.

Step 3

Go back to Step 1 and repeat steps 1 and 2 for the total number of runs.

Step 4

We can now estimate the present value of the replacement tranche $E[U(\tau_{PS})V(\tau_{PS})|F_0]$ and evaluate expression (7).

Lower Profit Bound

To derive a lower bound requires more work. The trick that facilitated ease of computation in expression (7) was avoiding computation of the value of the contract at the time of the default by the protection seller. At the time of default by the protection seller and assuming that default by the reference entity had not occurred, condition on a particular path over the remaining life of the contract; that is, conditional on the information set F_{T_n}, the net cash flow to the protection buyer is

$$C(\tau_{PS}) \equiv PS(\tau_{PS}, T_n) - S\,PB(z+1, n) \tag{8}$$

We know if and when the reference entity defaults over the remaining interval (τ_{PS}, T_n), given that we have conditioned on the information set F_{T_n}. Unlike expression (6), which is in terms of the market value of the future cash flows, expression (8) simply considers a single conditional cash flow.

Proposition

A lower profit bound is given by

$$\pi_{PS}^{LC} \equiv FP_{PS} + E\{U(\tau_{PS})\min[E(R_{PS}^{D})C(\tau_{PS}), E(R_{PB})C(\tau_{PS})]|F_0\} \tag{9}$$

Proof

To prove the lower profit bound, we need to prove

$$\pi_{PS}^{C} - \pi_{PS}^{LC} \geq 0$$

Now

$$\begin{aligned}\pi_{PS}^{C} - \pi_{PS}^{LC} = {} & E\{U(\tau_{PS})E[\min[E(R_{PS}^{D})V(\tau_{PS}), E(R_{PB})V(\tau_{PS})]|F_{\tau_{PS}}]|F_0\} \\ & - E\{U(\tau_{PS})E[\min[E(R_{PS}^{D})C(\tau_{PS}), E(R_{PB})C(\tau_{PS})]|F_{\tau_{PS}}]|F_0\}\end{aligned}$$

Consider a particular state in $F_{\tau_{PS}}$ and define

$$\begin{aligned}L \equiv {} & \min[E(R_{PB})V(\tau_{PS}), E(R_{PS}^{D})V(\tau_{PS})] \\ & - E[\min[E(R_{PB})C(\tau_{PS}), E(R_{PS}^{D})C(\tau_{PS})]|F_{\tau_{PS}}]\end{aligned}$$

We need to show that $L \geq 0$. Again, using Jensen's inequality, we have

$$
\begin{aligned}
& -E[\min[E(R_{PB})C(\tau_{PS}),\ E(R_{PS}^{D})C(\tau_{PS})]|F_{\tau_{PS}}] \\
& \geq -\min[E(R_{PB})V(\tau_{PS}),\ E(R_{PS}^{D})V(\tau_{PS})]
\end{aligned}
$$

which implies that $L \geq 0$, and hence

$$\pi_{PS}^{C} - \pi_{PS}^{LC} \geq 0$$

implying that (9) is a lower bound.

We can evaluate expression (9). The expression involves the net cash flow to the protection buyer along a path, $C(\tau_{PS})$. The pricing algorithm now becomes:

Step 1

Generate the default times for all the reference entities and the protection seller over the life of the tranche. Calculate the discounted cash flows to the different parties.

Step 2

If the protection seller defaults over the life of the contract, along the path we know the history of remaining defaults and we can determine the net cash flow to the protection buyer. We then compute the term $\min[E(R_{PS}^{D})C(\tau_{PS}), E(R_{PB})\, C(\tau_{PS})]$ and store the value.

Step 3

Go back to Step 1 and repeat steps 1 and 2 for the total number of runs.

Step 4

We can now evaluate expression (9).

We have now established upper and lower bounds for the profit

$$\pi_{PS}^{UC} \geq \pi_{PS}^{C} \geq \pi_{PS}^{LC} \qquad (10)$$

The difference

$$\Delta\pi_{PS}^{C} \equiv \pi_{PS}^{NC} - \pi_{PS}^{C}$$

is a measure of the misstatement in the profit of the trade from ignoring counterparty risk. We can bound this misspecification in the P&L:

$$\pi_{PS}^{NC} - \pi_{PS}^{LC} \geq \Delta\pi_{PS}^{C} \geq \pi_{PS}^{NC} - \pi_{PS}^{UC} \qquad (11)$$

3.2 Credit Risky Protection Buyer

If we had sold protection to a credit risky protection buyer, rewriting expression (4), the P&L is

$$\pi_{PB}^{C} = FP_{PB} - E\{\max[E(R_{PS})U(\tau_{PB})V(\tau_{PB}),\ E(R_{PB}^{D})U(\tau_{PB})V(\tau_{PB})] \mid F_0\} \quad (12)$$

Proposition Upper Profit Bound

An upper bound is given by

$$\pi_{PB}^{UC} = FP_{PB} - \max\{E(R_{PS})E[U(\tau_{PB})V(\tau_{PB}) \mid F_0],\ E(R_{PB}^{D})E[U(\tau_{PB})V(\tau_{PB}) \mid F_0]\} \quad (13)$$

Proof

Given that max(.,.) is a convex function, then applying Jensen's inequality to expression (12) gives

$$\begin{aligned}\pi_{PB}^{C} &\le FP_{PB} - \max\{E(R_{PS})E[U(\tau_{PB})V(\tau_{PB}) \mid F_0],\ E(R_{PB}^{D})E[U(\tau_{PB})V(\tau_{PB}) \mid F_0]\} \\ &= \pi_{PB}^{UC}\end{aligned}$$

Proposition Lower Profit Bound

A lower profit bound is given by

$$\pi_{PB}^{LC} = FP_{PB} - E[U(\tau_{PB})E[\max[E(R_{PS})C(\tau_{PB}),\ E(R_{PB}^{D})C(\tau_{PB})] \mid F_{\Gamma_{PB}}] \mid F_0] \quad (14)$$

The proof is similar to the proof for the lower profit bound in the case of a risky protection seller, so details are omitted[13].

We have now established upper and lower bounds for the profit

$$\pi_{PB}^{UC} \ge \pi_{PB}^{C} \ge \pi_{PB}^{LC} \quad (15)$$

[13] Let

$$\begin{aligned}L = &E[\max[E(R_{PB})C(\tau_{PB}),\ E(R_{PS}^{D})C(\tau_{PB})] \mid F_{\Gamma_{PB}}] \\ &- \max[E(R_{PB})V(\tau_{PB}),\ E(R_{PS}^{D})V(\tau_{PB})]\end{aligned}$$

and applying Jensen's gives the result.

The difference

$$\Delta\pi_{PB}^{C} \equiv \pi_{PB}^{NC} - \pi_{PB}^{C}$$

is a measure of the misstatement in the P&L of the trade.

3.3 Both Counterparties Risky

If we had purchased protection, rewriting expression (5), the P&L is given by

$$\begin{aligned}\pi_{PBS}^{C} = \; & FP_{PBS} + E[\,U(\tau_{PS})\,\min[\,E(R_{PS}^{D})V(\tau_{PS}),\; E(R_{PB})V(\tau_{PS})\,]|F_0\,] \\ & + E[\,U(\tau_{PB})\,\max[\,E(R_{PS})V(\tau_{PB}),\; E(R_{PB}^{D})V(\tau_{PB})\,]|F_0\,]\end{aligned} \tag{16}$$

Proposition Upper Profit Bound

An upper profit bound is given by

$$\begin{aligned}\pi_{PBS}^{UC} = \; & FP_{PBS} + \min\{E(R_{PB})E[\,U(\tau_{PS})V(\tau_{PS})\;F_0\,],\; E(R_{PS}^{D})E[\,U(\tau_{PS})V(\tau_{PS})\,|F_0\,]\} \\ & + E[\;U(\tau_{PB})\,E[\max[E(R_{PS})C(\tau_{PB}),\; E(R_{PB}^{D})C(\tau_{PB})]\;|\;F_{\Gamma_{PB}}\,]\,|F_0\,]\end{aligned} \tag{17}$$

The proof follows by combining the proofs for the upper bound for a risky protection seller and the lower bound for a risky protection buyer, so details are omitted.

Proposition Lower Profit Bound

A lower profit bound is given by

$$\begin{aligned}\pi_{PBS}^{LC} = \; & FP_{PBS} + E\{U(\tau_{PS})\min[\;E(R_{PB})C(\tau_{PS}),\; E(R_{PS}^{D})C(\tau_{PS})]\,|F_0\;\} \\ & +\; \max\{E(R_{PS})E[\,U(\tau_{PB})\,V(\tau_{PB})|F_0\,],\; E(R_{PB}^{D})E[\,U(\tau_{PB})\,V(\tau_{PB})\,|F_0\,]\}\end{aligned} \tag{18}$$

The proof follows by combining the proofs for the lower bound for a risky protection seller and the upper bound for a risky protection buyer.

4 Results

In this section we investigate the practical implications of counterparty risk. We consider two different types of instruments: a credit default swap and a synthetic collateralized debt obligation (CDO). For a credit default swap we have a closed form solution and this provides us with a benchmark to check the accuracy of our Monte Carlo simulations. In cases where a closed form solution exists, we can generate an exact solution incorporating counterparty risk and consequently there is no need for upper and lower bounds. Such cases provide a bench mark for the upper and lower bounds. For our second case, a synthetic CDO, closed form solutions do not exist.

We start by first documenting the accuracy of the Monte Carlo simulation. In the second part, we compute the P&L bounds for three cases: (a) the protection seller; (b) the protection buyer; and (c) both counterparties are subject to default risk. Often a trader will buy protection at the price S and off-set the position by selling protection on the same reference entity at the price (S + b), where b is the spread. The normal procedure is to record the P&L as the present value of the spread. We estimate the misstatement in the P&L due to neglecting counterparty risk. In the last part, we consider a synthetic CDO.

4.1 Modeling Default Dependence

To model default dependence we use a normal copula. A copula function binds together individual marginal distributions to form the multivariate distribution[14]. The normal copula, first introduced by Li (2000), has become the industry standard[15].

Parameter Assumptions

We list the assumptions used in the simulations. These assumptions are similar to those used in Mashal and Naldi (2005).

1 We assume a flat term structure of interest rates, at 2%, assuming continuous compounding[16].

2 The reference entity has a deterministic recovery rate of 35%.

[14] For a short introduction to copula functions see Schonbucher (2003, section 10.7) and for a more detailed discussion see Nelsen (1999).

[15] See Bank of America (2004).

[16] In practice the current swap curve would be used.

3 If the protection seller defaults, the recovery rates for the protection seller and for the protection buyer are assumed to be independent of other state variables and $E(R_{PS}^{D}) = 35\%$ and $E(R_{PB}) = 100\%$.

4 If the protection buyer defaults, the recovery rates for the protection seller and for the protection buyer are assumed to be independent of other state variables and $E(R_{PB}^{D}) = 35\%$ and $E(R_{PS}) = 100\%$.

5 Asset correlations between any two entities are all equal to 25%.

6 The intensities for all reference entities are assumed equal, and state independent.

7 Notional value is \$1mm.

8 The number of simulations is fixed at 100,000.

9 The settlement period is assumed to be zero, except in Table 2, Part (B).

Many of theses assumptions can be relaxed without affecting the bounds, though the simulations may become more involved. For example, if the reference term structure of interest rates is stochastic, then this is also simulated.

4.2 Pricing a Simple Credit Default Swap

For a credit default swap, we have a closed form solution that allows us to compute the present value of the cash flows to the protection buyer, seller and the theoretical premium, p, given our assumptions and ignoring counterparty risk. This provides us with a benchmark, with which to compare the accuracy of our simulations.

We estimate the present value of the payment by the protection seller, $E[PS(0, \tau_{RE}) | F_0]$, the present value of the payments by the protection buyer for a unit premium, $E[PB(1, n) | F_0]$, and the estimated premium, $\hat{p}$, using expression (1). We also calculate the estimated value of the swap, defined as

$$\hat{\pi} \equiv E[PS(0, \tau_{RE}) | F_0] - p\, E[PB(1, n) | F_0]$$

where p is the theoretical premium. If the estimated probability of default is less (greater) than the theoretical value, then the present value of the payment by the protection seller will be under (over) estimated and the present value of the payments by the protection buyer over (under) estimated, implying that the estimated value of the swap will be negative (positive). The estimated value of the swap will tend to zero, as the estimated premium tends to its theoretical value. We repeat the estimation exercise 30 times and report in Table 1 the mean value and the standard deviation. The mean values are compared to their theoretical values. For all variables, the mean value is within two standard deviations of its theoretical value.

For the reference entity, we consider two intensities of 1 percent and 4 percent, which are representative of investment grade and high-yield names. When the intensity of the reference entity is 1 percent, the mean of the estimated probability of default over the five year period is slightly greater than its theoretical value. Consequently, the mean of the payment to the protection buyer (PV to PB) is slightly greater than its theoretical value and the mean of the payment to the protection seller (PV to PS) slightly less than its theoretical value. The estimated premium is greater than its theoretical value. When the intensity of the reference entity is 4 percent, the mean of the estimated probability of default over the five year period is slightly less than its theoretical value. Consequently, the estimated premium is less than its theoretical value and the mean value of the swap is slightly negative.

The high degree of accuracy more than justifies the number of simulations being fixed at 100,000. However, we did a number of additional checks using one million runs. These results are not reported.

Exact Valuation

With a simple default swap it is possible to value a replacement instrument just after a counterparty defaults using a closed form expression. Schönbucher and Schubert (2001) derive general expressions for the intensity function and survival probabilities of the reference entity after the protection seller defaults. Given our assumptions, the intensity function for the reference entity, after default by a counterparty, is now time dependent and consequently numerical integration is used to estimate the value of the replacement swap.

In all the simulation results, we find that the value of the replacement swap, $V(\tau_{PS})$, is positive . This greatly simplifies the explanation of the results. For the case of a risky protection seller, the value of the swap with counterparty risk, as given by expression (3), simplifies as

$$\min[E(R_{PS}^{D})V(\tau_{PS}), E(R_{PB})V(\tau_{PS})] = E(R_{PS}^{D})V(\tau_{PS})$$

given that $V(\tau_{PS})$ is positive. Therefore expression (3) can be written

$$\pi^{C}_{PS} = FP_{PS} + E[\, U(\tau_{PS})\, E(R^{D}_{PS})\, V(\tau_{PS}) \,|\, F_0 \,] \qquad (19)$$

The upper P&L bound – see expression (7) - becomes

$$\pi^{UC}_{PS} = FP_{PS} + E[\, U(\tau_{PS})\, E(R^{D}_{PS})\, V(\tau_{PS}) \,|\, F_0 \,]$$

implying that the upper premium bound becomes the exact:

$$\pi^{C}_{PS} = \pi^{UC}_{PS} \qquad (20)$$

For the case of a risky protection buyer, see expression (4), the value of the swap simplifies, as

$$\max[\, E(R_{PS})\, V(\tau_{PB}),\, E(R^{D}_{PB})\, V(\tau_{PB}) \,] = E(R_{PS})\, V(\tau_{PB})$$

Therefore the expression (4) for the value of a swap with counterparty risk becomes

$$\pi^{C}_{PB} = FP_{PB} - E[\, U(\tau_{PB})\, E(R_{PB})\, V(\tau_{PB}) \,|\, F_0 \,] \qquad (21)$$

The P&L upper bound becomes

$$\pi^{UC}_{PB} = FP_{PB} - E[\, U(\tau_{PB})\, E(R_{PB})\, V(\tau_{PB}) \,|\, F_0 \,]$$

Hence, from expression (21), the upper P&L bound becomes exact:

$$\pi^{C}_{PB} = \pi^{UC}_{PB} \qquad (22)$$

Both valuation expressions (19) and (21) can be directly estimated in the simulation. While there is no need to undertake a separate valuation using numerical integration, in the results reported below we did both methods as a check.

4.3 Measurement of the P&L Bounds

We first compute the profit bounds for the case of a risky protection seller, using expression (10), and then for the case of a risky protection buyer using expression (15). Given the size of the volatility for the estimate of the swap value in Table 1, we use a control variate approach[17], to improve the accuracy of our estimates. The premium for a swap with no counterparty risk is set such that the value of the swap is zero, when initiated. We can estimate

[17] See Rubinstein (1981), chapter 4 for details.

the value of this swap using Monte Carlo and can use it as a control variate. The estimate of the profit bound, $Y(\beta)$, using the control variate is given by

$$Y(\beta) = Y - \beta(\pi_S - v)$$

where Y is the estimate of the profit bound; π_S is the estimate of the value of a simple swap without counterparty risk; v is the value of the simple swap; and β is set to minimize the variance of the estimate: $\beta = \mathrm{cov}(Y, \pi_S) / \mathrm{var}(\pi_S)$. The premium is set such that the value of the simple swap is zero (v = 0).

In Table 2, we examine how the width of the P&L bounds changes as the risk of a counterparty changes. The intensity of the counterparty is varied from 50 basis points, which is representative of an A credit, to 400 basis points, representative of a high-yield name. In Part A we consider only a risky protection seller. All the bounds are negative and relative tight. When the intensity of the reference entity is 1 percent, the upper bound varies from - 431 for an investment grade counterparty to -1,890 for a high-yield counterparty. Figure 1 shows the variation of the width between the upper and lower bounds with counterparty risk. It is seen that the variation is approximately linear. We know that for this special case the upper P&L bounds are exact. Consequently, for the case of a risky protection seller with an intensity of 50 basis points, we are over estimating the P&L by approximately \$431 for a swap on a reference entity with an intensity of 1 percent. The P&L is over estimated by \$804, when the intensity of the reference entity is 4 percent.

Failure to perform is one facet of counterparty risk. It occurs when the reference entity defaults and before settlement, the protection seller also defaults. Ex ante the effect on the price of a swap should be relatively small, given that the probability of this occurrence is small. It depends on the probability of the joint events of the reference entity defaulting and then default by the protection seller before settlement. Ex post, the consequences are however large. The protection buyer was expecting a payment of $N(1 - R_{RE})$ from the protection seller, given default by the reference entity, where N is the notional of the swap. The payment is reduced to $N(1 - R_{RE})R_{PS}^{D}$, if the protection seller defaults before settlement.

In Table 2, Part A, the settlement period is zero. In Part B, settlement period is one month. Comparing the results in the two tables, we see that settlement risk increases the absolute values of the bounds, though the effect is quite small if the intensity of the protection seller is small. The impact of settlement risk becomes more noticeable, as the risk of default by either the protection seller or the reference entity increases.

In Part C, we consider a risky protection buyer. In this case, counterparty risk is miniscule. If the protection buyer defaults, then assuming the value of the replacement swap is positive[18], the protection seller makes a payment of $V(\tau_{PB})$ to the protection buyer. There is no mark-to-market risk for the protection buyer. The protection seller is however exposed to the loss of accrued interest if the protection buyer defaults. In present value terms this is very small. In the reported results, the upper bound is practically zero, irrespective of the risk of the protection buyer. In Figure 2, the width of the bounds increases with the intensity of the protection buyer. The variation is approximately linear.

Both Counterparties Risky

In the setting of reserves, it is common practice to assume that it is only the other counterparty is risky, while in reality either party to the contract might default. In Table 3, we consider the upper and lower bounds when buying credit protection. In Part A, it is assumed that the intensity of the protection buyer and seller are equal. It is seen that the lower bound is decreasing, as the intensity of the counterparties increases. The upper bound is U-shaped. This is not surprising, given the results in Table 2. Recall the upper bound for buying protection, is the upper bound for the risky protection seller plus the upper bound for the risky protection buyer. Table 2, Part A, shows the upper bound for a risky protection seller, assuming we are *buying* protection. As the intensity of the counterparties increases, the upper bound decreases. Table 2, Part C, shows the lower bound for a risky protection buyer, assuming we are *selling* protection. The lower bound decreases, as the intensity of the counterparties increases. This implies that if we had *purchased* protection and there is the risk that we might default, the upper bound increases as the intensity of default increases. We can use the results in Table 2 to approximate the bounds. For example, if the intensities for the reference entity is 1 percent and for the counterparties 50 basis points, an estimate of the upper bound is -431 - (-204) = -227, which is close to reported value of -218 in Table 3.

[18] It should be stressed that this assumption is not made in the simulations.

In Part B, the intensity of the protection seller is fixed at 50 basis points and the intensity of the protection buyer varied. The lower bound is relatively insensitive to variations in the credit worthiness of the protection buyer, while the upper bound is U-shaped. When the upper bound is positive and the lower bound negative, this raises the question of whether the true P&L is positive or negative. In general, this question cannot be answered. The upper bounds in Table 3 depend in large part on the upper bound for the protection buyer. We know from expression (22) that for *selling* protection the upper bound for a risky protection buyer is exact, not the lower bound. This implies that it is perhaps imprudent to assume the true P&L is positive.

4.4 P&L Misstatement: Off-setting Position

Consider the case of a trader buying protection on a reference entity at the price S from one counterparty and immediately off-setting the risk by selling protection at the price (S + b) to another counterparty, where b denotes the bid/ask spread and is non-negative. If we neglect counterparty risk, the P&L simply reflects the present value of the spread, b, over the life of the instrument:

$$\begin{aligned}\pi_b^{NC} &= E[\,PS(0, T_n)\,|\,F_0\,] - S\,E[\,PB(1, n)\,|\,F_0\,] \\ &\quad + (S + b)\,E[\,PB(1, n)\,|\,F_0\,] - E[\,PS(0, T_n)\,|\,F_0\,] \\ &= b\,E[\,PB(1, n)\,|\,F_0\,]\end{aligned}$$

For the first part of the trade, the trader is exposed to counterparty risk of the protection seller and in the second part of the trade, to the counterparty risk of the protection buyer. Taking into account the risky counterparties, the P&L should be

$$\pi_b^C \equiv \pi_{PS}^C + \pi_{PB}^C$$

Using expressions (10) and (15) we can bound this P&L

$$(\pi_{PS}^{LC} + \pi_{SB}^{LC}) \le \pi_b^C \le (\pi_{PS}^{UC} + \pi_{PB}^{UC}) \tag{23}$$

The misstatement in the P&L of the swap is given by

$$\Delta\pi_b^C \equiv \pi_b^{NC} - \pi_b^C$$

and using expression (19) the bounds are given by

$$\pi_b^{NC} - (\pi_{PS}^{LC} + \pi_{PB}^{LC}) \ge \Delta\pi_b^C \ge \pi_b^{NC} - (\pi_{PS}^{UC} + \pi_{PB}^{UC}) \tag{24}$$

In Table 4, we consider two cases. In Part A, the intensity of the reference entity is 1 percent and the spread is 10 basis points, which is a spread often observed for liquid investment grade names. The P&L ignoring counterparty risk is

$$\pi_b^{NC} = 4{,}631$$

on a notional of $1mm. The first part of the trade involves buying protection from a risky protection seller. If the intensity of the protection seller is 50 basis points, the value of the P&L lies between

$$-641 \le \pi_{PS}^{C} \le -431$$

The second part of the trade involves selling protection to a risky protection buyer at a spread of 10 basis points. If the intensity for the protection buyer is 50 basis points, the upper and lower bounds are

$$4{,}376 \le \pi_{PB}^{C} \le 4{,}631$$

We know from Table 2, that upper bound is insensitive to the credit risk of the protection buyer. In this case, the upper bound is 4,631 irrespective of the intensity of the protection buyer. Combining these two expressions, we have

$$3{,}735 \le \pi_b^{C} \le 4{,}200$$

where $\pi_b^C = \pi_{PS}^C + \pi_{SB}^C$.

Given the assumptions, we know that upper P&L bounds are exact, implying that

$$\pi_{PS}^{C} = -431 \text{ and } \pi_{PB}^{C} = 4{,}631$$

and the correct P&L is

$$\pi_b^{C} = \pi_{PS}^{C} + \pi_{PB}^{C} = 4{,}200$$

The misstatement in the P&L is

$$4{,}631 - 4{,}200 = 431$$

or 9.3 % of the recorded amount. The misstatement in the P&L is sensitive to the risk of the protection seller, though insensitive to the risk of the protection buyer. If the intensity of the protection seller is 400 basis points, the misstatement in the P&L is 40.8 %.

In Table 4, Part B, the intensity of the reference entity is 4 percent and the spread is 30 basis points, which is a spread often observed for high yield names. The P&L ignoring counterparty risk is 12,927. The correct P&L, taking into account counterparty risk is

$$\pi_b^C = \pi_{PS}^C + \pi_{SB}^C = -804 + 12{,}924 = 12{,}120$$

and the misstatement in the P&L is

$$12{,}927 - 12{,}120 = 807$$

or 6.2 % of the recorded amount. Again, the misstatement in the P&L increases as the risk of the protection seller increases.

The results in Table 4 provide a quantification of the well known fact that the main risk is on the side of the protection seller. The usual rationale is that if you purchase protection and the reference entity defaults, failure to perform by the protection seller has a large cost. There is a small probability[19] of a large loss. Here, however, the rationale is different. The risk arises because of the mark-to-market loss if the protection seller defaults. Default increases the intensity of the reference entity and the cost of new protection increases, implying a mark-to-market loss for the protection buyer. The impact on the P&L can be significant. Even when the intensities for the protection seller and buyer are 50 basis points, the misstatement in the P&L is 9.3 % for an investment grade reference entity and 6.2 % for a high-yield reference entity.

4.5 Aggregation

Up to the present point we have considered a contract in isolation in order to determine the impact of counterparty risk on the P&L. This information is useful in its own right, as it provides information on how the contract will be priced in the market. However, any financial institution usually has a portfolio of contracts with the same counterparty. The desirability of an additional contract with the same counterparty will depend in part on its impact on the aggregate dollar cost of the counterparty risk exposure. In risk management, various ways are used to monitor aggregate counterparty risk: two common measures are the total nominal exposure and the exposure at default[20]. Here we are measuring the impact of counterparty risk on aggregate valuation. This provides information to senior management about the cost of counterparty risk and can affect the desirability of the contract to the institution.

Some of the contracts may be covered under a Master Agreement (MA) that facilitates netting. Some reference entities may be in contracts that are covered under a MA and in contracts

[19] The probability is small because it depends on the joint event of default by the reference entity followed by default of the protection seller within the settlement period.
[20] Both measures are usually adjusted for netting if a Master Agreement exists.

not covered. To determine the marginal impact of the contract it is necessary to simulate the default times of the counterparty and all the reference entities in the different contracts. For each simulation path, the cash flows are determined for all the contracts. If the counterparty defaults, the contracts in existence at the time of the default are split into two groups: those covered under a MA and the remaining contracts. For those covered under a MA the aggregate value is calculated and upper/lower bounds determined for the aggregate. For those not covered they are treated individually. By aggregating across the upper (lower) bounds, the institution can determine the marginal impact of the contract.

4.6 Upper and Lower P&L Bounds for a Synthetic CDO.

In Table 5, we consider a synthetic CDO. The collateral is composed of 100 names, each with the same notional amount of one million. We compute the spreads for three tranches of different seniority: an equity tranche exposing the investor (the protection seller) to the first 5% of the portfolio losses, a mezzanine tranche covering the (5 – 10) % of portfolio losses and a senior tranche for all remaining losses. The notional value for the equity and mezzanine tranches is 5 million and the senior tranche 90 million. All names are assumed to have an intensity of 2 percent and the same correlation. All other assumptions remain unchanged. We consider only counterparty risk on the side of the protection seller. We treat the CDO as a unity: there is only one protection seller for the whole instrument. Alternatively, we could treat each tranche as a separate instrument with different investors.

In all cases the bounds are negative, implying that the P&L is being over stated. For example, if the intensity for the protection seller is 50 basis points, the correct P&L for the equity tranche is between -9,209 and -10,394. The misstatement is not significant, given the notional is 5 million. For all tranches, the upper and lower bounds decrease, as the intensity of the protection seller increases. This is to be expected, given the results in Table 2. The bounds for the mezzanine tranche are substantially greater in absolute magnitude than the bounds for the equity tranche. A similar comment applies to the senior tranche, though the notional for this tranche is 90 million.

Table 6 shows the sensitivity of the bounds to correlation. The correlation between all names is assumed to be equal. For the equity tranche, the premium (not shown) and the upper bound, increase in absolute magnitude, as the correlation increases. The lower bound is relatively insensitive. A similar comment applies for the mezzanine tranche. For the senior tranche, increases in the correlation increase the risk to the tranche and the bounds increase in absolute magnitude. In all cases, the bounds are negative and tight.

5 Evaluation and Summary

We have described a methodology for deriving the upper and lower P&L bounds in the presence of counterparty risk. The methodology is quite general and can be applied to a wide range of contracts subject to counterparty risk. It does not rely on either structural or reduced form credit models. The methodology provides practitioners and regulators with a practical tool for the determination of reserves to incorporate the two facets of counterparty risk: failure to perform and mark-to-market exposure. To be of use, the bounds must be relative tight and to be of importance, the misspecification must be substantial. The usefulness and relevance of the methodology will depend on the particular application.

If the bounds have the same sign, then there is either over or under estimation of the P&L. Tight bounds allow the P&L to be pinned down. For many of the examples shown, the bounds are relatively tight and of the same sign. This was not the case when we allowed either counterparty to default. Depending on the risk (intensity) of the reference entity, the bounds could differ in sign. In Table 4, we considered a common form of trade: buying protection of a reference entity from one counterparty and selling protection on the same reference entity to another counterparty. It is shown that the misstatement in the P&L can be substantial. The sensitivity of the bounds to correlation depends on the type of contract. For equity and mezzanine tranches of a synthetic CDO, the bounds were relatively insensitive. This was not the case for the senior tranche. For all tranches the bounds were negative and relatively tight.

Table 1

Credit Default Swap: No Counterparty Risk

	Intensity 1%		**Intensity 4%**	
	Estimated Value	Theoretical Value	Estimated Value	Theoretical Value
*Probability of Default %	4.9380 (0.05)	4.8771	18.1165 (0.06)	18.1269
PV to PB	30,557 (336)	30,180	112,262 (371)	112,312
PV to PS	4,630,053 (1,272)	4,631,475	4,308,067 (1,907)	4,308,924
Value of Swap	386.21 (345)	0	-27.84 (421)	0
Estimated Premium (bps)	66.00 (0.75)	65.16	260.58 (0.98)	260.65

The figures in parenthesis are the standard deviations of the estimates.

The estimate values and their standard deviation are estimated by repeating the simulation 30 times.

PV to PB denotes the present value of the cash flow to the protection buyer.

PV to PS denotes the present value of the cash flows to the protection seller, given a unit premium.

The value of the swap is defined as:

PV of payment to protection buyer – Theoretical Premium * PV of unit payments to protection seller.

The maturity of the swap is 5 years.

*This is the probability of default over the life of the swap.

Table 2

P&L Bounds

Part A

Risky Protection Seller

	Intensity of Reference Entity			
	Intensity 1 %		Intensity 4 %	
Intensity of Protection Seller (bps)	Lower Bound	Upper Bound	Lower Bound	Upper Bound
50	-641	-431	-1,416	-804
100	-1,156	-735	-2,713	-1,466
200	-2,009	-1,172	-5,402	-2,884
300	-2,821	-1,572	-7,541	-3,729
400	-3,542	-1,890	-9,549	-4,452

Part B

Risky Protection Seller: Settlement Risk

	Intensity of Reference Entity			
	Intensity 1 %		Intensity 4 %	
Intensity of Protection Seller (bps)	Lower Bound	Upper Bound	Lower Bound	Upper Bound
50	-661	-451	-1,461	-849
100	-1,180	-759	-2,777	-1,529
200	-2,042	-1,205	-5,525	-3,006
300	-2,873	-1,625	-7,693	-3,881
400	-3,630	-1,978	-9,760	-4,664

Settlement period is one month.

Part C

Risky Protection Buyer

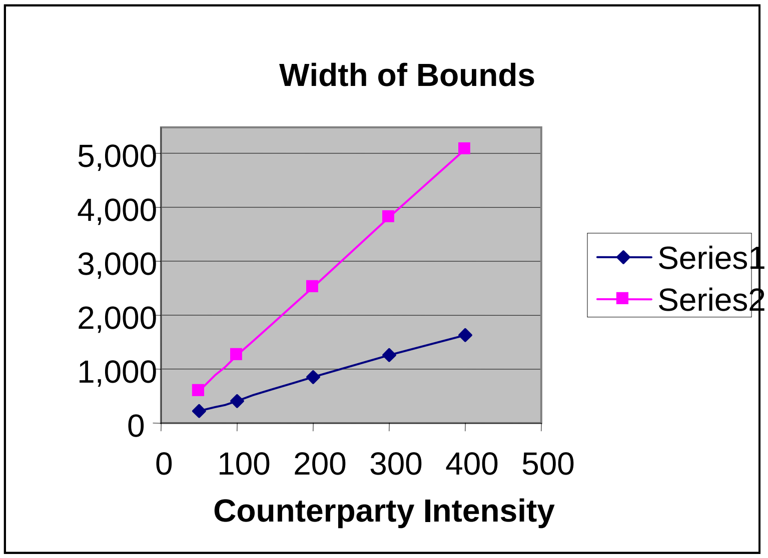
	Intensity of Reference Entity			
	Intensity 1 %		Intensity 4 %	
Intensity of Protection Buyer (bps)	Lower Bound	Upper Bound	Lower Bound	Upper Bound
50	-204	0.03	-580	0.10
100	-413	0.06	-1,205	0.21
200	-824	0.11	-2,476	0.40
300	-1,235	0.16	-3,759	0.61
400	-1,644	0.22	-5,057	0.80

Figure 1

Width of Bounds

Part A

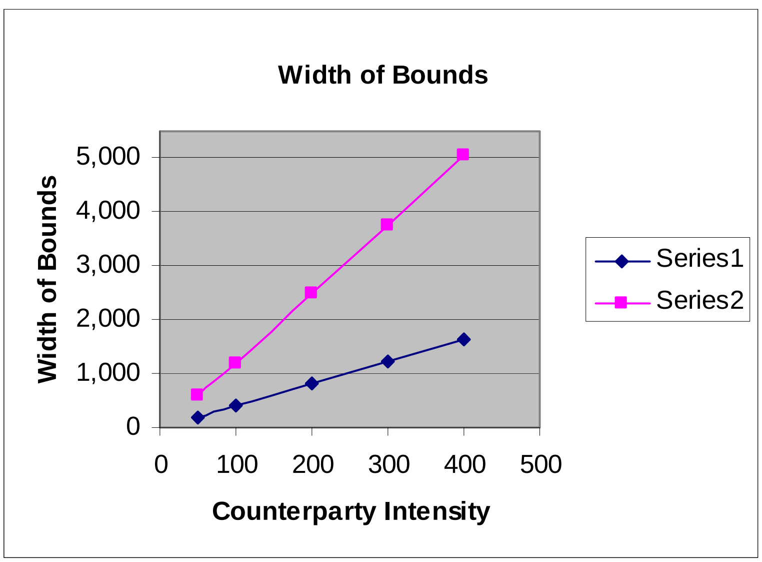
Risky Protection Seller



Series 1 refers to reference entity with an intensity of 1 percent.
Series 2 refers to reference entity with an intensity of 4 percent.

Figure 2

Risky Protection Buyer



Series 1 refers to reference entity with an intensity of 1 percent.
Series 2 refers to reference entity with an intensity of 4 percent.

Table 3

P&L Bounds: Both Counterparties Risky

Part A

	Intensity of Reference Entity			
	Intensity 1 %		Intensity 4 %	
Intensity of Counterparties (bps)	**Lower Bound**	**Upper Bound**	**Lower Bound**	**Upper Bound**
50	-622	-218	-1,363	-196
100	-1,082	-282	-2,585	-223
200	-1,862	-306	-4,963	-253
300	-2,518	-242	-6,701	301
400	-3,076	-114	-8,290	922

Part B

Sensitivity to Risky Protection Buyer

	Intensity of Reference Entity			
	Intensity 1 %		Intensity 4 %	
Intensity of Protection Buyer (bps)	**Lower Bound**	**Upper Bound**	**Lower Bound**	**Upper Bound**
50	-622	-218	-1,363	-196
100	-606	-2	-1,348	423
200	-356	447	-1,307	1,698
300	-528	873	-1,259	2,999
400	-513	1,286	-1,229	4,297

The intensity of the Protection Seller is 50 basis points.

Table 4
P&L Misstatement: Off-setting Position

Part A
Spread 10 basis points
Bounds for Each Leg of the Trade

Protection Seller			Protection Buyer		
Intensity (bps)	Lower Bound	Upper Bound	Intensity (bps)	Lower Bound	Upper Bound
50	-641	-431	50	4,376	4,631
100	-1156	-735	100	4,116	4,631
200	-2009	-1172	200	3,603	4,631
400	-3542	-1890	400	2584	4,631

Intensity of reference entity 1 percent.

Round Trip Bounds

Intensity Protection Seller (bps)	Intensity Protection Buyer (bps)	Lower Bound	Upper Bound	Misstatement	Percentage Error
50	50	3,735	4,200	431	9.3
100	100	2,960	3,896	735	15.9
200	200	1,594	3,459	1,172	25.3
400	400	-958	2,741	1,890	40.8

P&L of trade ignoring counterparty risk $4,631

Part B

Spread 30 basis points
Bounds for Each Leg of the Trade

Protection Seller			Protection Buyer		
Intensity (bps)	Lower Bound	Upper Bound	Intensity (bps)	Lower Bound	Upper Bound
50	-1,416	-804	50	12,223	12,924
100	-2,713	-1,466	100	11,471	12,924
200	-5,402	-2,884	200	9,946	12,925
400	-9,549	-4,452	400	6,852	12,925

Intensity of reference entity 4 percent. .

Round Trip Bounds

Intensity Protection Seller (bps)	Intensity Protection Buyer (bps)	Lower Bound	Upper Bound	Misstatement	Percentage Error
50	50	10,807	12,120	807	6.2
100	100	8,758	11,458	1,469	11.4
200	200	4,544	10,040	2,886	22.3
400	400	-2,697	8,472	4,454	34.5

P&L of trade ignoring counterparty risk $12,927

Table 5

Synthetic CDO P&L Bounds

Risky Protection Seller

	Equity (0 – 5) %		Mezzanine (5 – 10) %		Senior (10- 100) %	
Intensity (bps)	Lower Bound	Upper Bound	Lower Bound	Upper Bound	Lower Bound	Upper Bound
0	0	0	0	0	0	0
50	-10,394	-9,209	-21,917	-20,078	-53,646	-50,476
100	-22,706	-19,689	-43,823	-39,323	-96,009	-88,319
200	-46,073	-38,666	-81,109	-70,467	-157,579	-140,646
300	-69,460	-56,824	-116,500	-98,730	-210,897	-183,850
400	-92,702	-73,846	-152,112	-126,362	-263,303	-224,720

Portfolio of 100 reference entities.

Notional of each reference entity = $1 mm.

Intensity of reference entities = 2 %

. **Table 6**

Synthetic CDO P&L Bounds

Sensitivity to Correlation

	Equity (0 – 5) %		Mezzanine (5 – 10) %		Senior (10- 100) %	
Correlation	Lower Bound	Upper Bound	Lower Bound	Upper Bound	Lower Bound	Upper Bound
5	-21,201	-14,133	-35,062	-26,935	-11,550	-8,551
10	-22,469	-16,707	-40,330	-33,159	-28,710	-23,649
15	-22,929	-18,114	-42,702	-36,451	-48,433	-41,933
20	-22,975	-19,152	-43,571	-38,253	-71,105	-63,779
25	-22,706	-19,689	-43,823	-39,323	-96,009	-88,319
30	-21,976	-19,644	-43,331	-39,580	-123,832	-116,068
35	-21,091	-19,335	-42,455	-39,448	-152,569	-144,989

Portfolio of 100 reference entities.

Notional of each reference entity = $1 mm.

Intensity of reference entities = 2 %

Intensity of protection seller = 100 bps.

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