Financial Engineering

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Levy Processes

- Geometric Brownian Motion is very tractible, and captures some salient features of speculative price dynamics, but it is somewhat limiting.

- The continuity of the Brownian Motion is very convenient–it makes the hedging derivations go–but it is not necessarily realistic. Real world prices "jump" or "gap."

- A Levy Process is a more general way of characterizing price dynamics. It is more more general and flexible–a Brownian motion is a Levy Process, but not all Levy Processes are Brownian Motions.

A Definition of the Levy Process

- A Levy Process is (a) CADLAG, (b) has independent random increments, i.e., $X_1 - X_0, X_2 - X_1, \ldots, X_n - X_{n-1}$ are independent, (c) stationary, i.e., the probability law of $X_{t+h} - X_t$ does not depend on t, and (d) is stochastically continuous, i.e., $\lim_{h \to 0} \mathcal{P}(|X_{t+h} - X_t| \geq \varepsilon) = 0 \ \forall \varepsilon > 0$.

- Condition (d) does not mean that sample paths are necessarily continuous. It just excludes discontinuities at fixed (non-random) times.

- Moreover, if X_t is a Levy process, its distribution is infinitely divisible. This means that if the distribution of any increment has a given distribution F, then there exists a partition of the increment (i.e., the increment is a sum of sub-increments) where the elements of the partition (i.e., the sub-increments) have the same distribution.

- For instance, the normal distribution is infinitely divisible. Any normal variable can be expressed as the sum of other normals.

- There is a Levy Process associated with every infinitely divisible distribution. Similarly, there is an infinitely divisible distribution associated with every Levy Process.

- This means that any infinitely divisible distribution can be used as the law characterizing a Levy Process that describes the dynamics of some speculative price.

- Since a probability distribution is associated with a characteristic function, valuation using Levy Processes frequently utilizes the characteristic function associated with the relevant infinitely divisible distribution.

A Decomposition

- The Levy-Ito decomposition implies that every Levy Process is a sum of (a) a Brownian Motion with drift, (b) a finite activity jump process, and (c) an infinite activity jump process.

- The jump processes in the LP mean that it is not necessarily continuous.

- The jumps are represented as compound Poisson processes.

- The finite activity jump process means that there is a finite number of jumps with absolute value larger than 1.

- The infinite activity jump component can have infinitely many small jumps.

Model Building

- There are two basic types of jump models.

- In jump-diffusion models, the normal evolution of prices is characterized as a diffusion, but at random intervals there are periodic jumps (perhaps of random size).

- Infinite activity models have no diffusion part–instead, infinite numbers of jumps in every interval generate interesting small time behavior.

- In jump diffusion models, there is a Brownian component, jumps are rare, and the distribution of jump sizes is known. These models can readily characterize the volatility smile, and are easy to simulate, but their densities are not available in closed form.

- Infinite activity models needn't have a Brownian part; the process moves by jumping around a lot. There is is no distribution of jump sizes because they arrive infinitely often, but sometimes closed form densities of the process are available. These models can accurately capture historical price processes.

- Sometimes infinite activity models can be created by "subordination" of a Brownian process.

Subordination

- The basic idea behind subordination is that prices are represented as a “time changed” Brownian motion. That is, there is some increasing random process that depends on calendar time. This increasing random process measures “business time.” The price process depends on business time.

- Perhaps a better way to think about this is to view things in terms of information flow. Sometimes the rate of information flow is large. Sometimes the rate of information flow is small. Subordination essentially allows the rate of information flow to vary randomly over time.

- Let $X(t)$ be a Levy Process, and let T_t be a subordinator, i.e., a Levy Process with almost surely non-decreasing sample paths. Then $X(T_t)$ is a subordinated process.

- As an example, let T_t be a Gamma process. This is a stochastic process with increments that obey a Gamma distribution. (The Gamma distribution is a generalization of the factorial function). Draws from the Gamma distribution are always positive, so the sum of Gamma distributed variates is increasing. The Gamma distibution has two parameters, the mean and variance.

- One constructs a Variance Gamma Process by (a) for each time, take a draw of a Gamma increment, (b) add this Gamma increment to the sum of previous draws, (c) use this sum to measure the “trading” or “business time,” and (d) measure a GBM at this trading time.

- VG is an infinite activity process.

- There are other Levy Processes that can be constructed through subordination. In essence, any a.s. non-decreasing stochastic process can be used as the subordinator. Another example is an Inverse Gaussian process.

- These things are very easy to simulate, and sometimes have closed form distributions (or characteristic functions).

An Example: Variance Gamma

- The VG is an infinite variation process. It consists of a very larg number of small jumps.

- Usually the mean parameter for the Gamma distribution is set equal to 1. That is, on average, business time is the same as clock time. Put differently, the rate of information flow is on average σ, but sometimes it is faster and some times it is slower.

- Choose a variance parameter ν. This measures the variability in the rate of information flow.

- In the VG model, returns are normal *conditional* on the draw of the Gamma business time variable. Moreover, the draw of the conditionally Gaussian return is independent of the draw of business time. The business time/total information flow at t is T_t, which is the Gamma variate.

- In the true measure, the log price at clock time t is:

$$X_t = \theta T_t + \sigma\sqrt{T_t}Z$$

- Moving to an equivalent measure, we will change the drift of the price process to:

$$X_t = \theta T_t + \omega t + \sigma\sqrt{T_t}Z$$

  where ω is an adjustment to the drift so that the discounted stock price is a martingale.

- To price vanilla European options, exploiting the independence of the gamma and normal variates, we just integrate twice:

  $$V = e^{-r\tau} \int \gamma(T_t) \int f(S_0 e^{\theta T_t + \omega t + \sigma \sqrt{T_t} Z}) n(Z) dZ dT_t$$

  where $f(.)$ is the payoff function.

- Note that the market is incomplete in the VG case. We have two sources of risk (the business time/information flow and the Gaussian draw) but only one hedging instrument–the underlying.

- Indeed, we have an embarrasment of riches. We have three parameters that we can adjust to turn the process into a Martingale. This shouldn't be surprising. Recall if the market is not complete, that the EMM is not unique.

- Specifically, to make the process a Martingale,

$$\omega = \frac{1}{\nu} \ln(1 - \theta\nu - .5\sigma^2\nu)$$

- Remember what von Neumann said: Give me four parameters and I can fit an elephant! Give me 5, and I can make it swing its trunk.

Pros and Cons of LP

- A Levy Process can capture certain features of empirical return distributions that the Gaussian cannot. For instance, non-Gaussian LPs cn lead to heavy tails in the return distribution. Similarly, they can allow skewness. (For instance, in the VG the sign of θ determines the skew.)

- The ability to generate distributions exhibiting skewness and heavy tails allows LPs to result in volatility smiles and skews of various shapes.

- However, LPs cannot capture other features of price dynamics. These include, volatility clustering, positive autocorrelations in absolute returns (remember, LP increments are independent), and leverage effects (the fact that absolute/squared returns tend to be negatively correlated with returns).

- More complicated LP models can address some of these issues. For instance, adding jumps in the price process and stochastic volatility can generate volatility clustering and realistic smile behavior. An autocorrelated volatility process with a negative correlation between the price process and the vol process generates volatlity clustering and leverage effects. The jumps generate smiles in short date options. The stochastic volatility can be tuned to match longer dated smiles.

- Again–remember von Neumann.