DERIVATIVES PRICING

SWAPS

Swaps

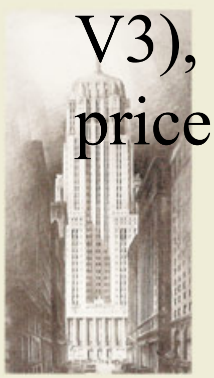
- Swaps are the most common OTC derivative.
- Most swaps are "vanilla" fixed price swaps. These are like bundles of forward contracts. (Though some swaps have only one pricing date.)
- Vanilla swaps are typically cash settled.
- The parties to a swap set: (a) the notional quantity; (b) the "tenor" or maturity of the swap (how long it lasts); (c) the payment dates; (d) the floating price index; and (e) the fixed price.

Swap Mechanics

- The mechanics of a swap are as follows: There is a fixed payer price (the "long") and a floating price payer (the "short"). The fixed price payer makes the same payment to the counterparty (the floating price payer) every payment date. The floating payer pays the counterparty (the fixed price payer) an amount that depends on the floating price index.
- The most common floating price index is currently the settlement price on the next to last or last trading day of the contract corresponding to the payment month (e.g., the January futures for the January payment date). In the past contracts based on the average of the last 3 days of the NYMEX NG futures settlement prices corresponding to the payment month were common.
- However, parties could agree to any price index, e.g., Chicago City Gate quoted by Platts.

Swap Mechanics (con't)

- The party whose payment obligation is larger pays the net amount to the counterparty. The payment equals the absolute value of the difference between fixed and floating prices times the notional quantity.
- An example works best. Consider a swap for 100,000 MMBTU per month (equivalent to 10 NYMEX contracts) for each month in April-October 2023. The fixed price in the swap is $4.00 per MMBTU. Payments are made 5 days after the last trading day of the NYMEX NG futures (each month from J3-V3), and the floating price is based on the LD NYMEX settle prices



Example (con't)

- April contract stops trading on 29 March 2023
- Assume the J3 price on this date is $5.00. Then the swap seller owes the buyer $1 x 100,000=$100,000.
- Assume the J3 price on this date is $3.75. Then the swap buyer owes the seller $.25 x 100,000=$25,000.

Determining the Fixed Price

- Fixed prices are determined at the time that the swap is negotiated based on the relevant forward prices at that time. For NYMEX LTD, the NYMEX futures prices prevailing when the swap is negotiated are used to determine the fixed price
- The fixed price is set so that a swap has zero value on the date that it is created.
- A hedging argument produces the fixed price of the swap.
- Buy a calendar 2024 NYMEX swap
- Sell a "strip" of 12 NYMEX futures

Determining the Fixed Price (con't)

- On each payment date, the cash flow from the swap is $F_{T,T} - P$

Where $F_{T,T}$ is the futures price at expiration and P is the fixed price.

- The payoff from the futures is $F_{t,T} - F_{T,T}$ where $F_{t,T}$ is the futures price on the day you initiate the hedge/enter the swap.
- The net payoff is therefore $F_{t,T} - P$ which is known when you initiate the hedge.
- This strategy therefore produces a set of 12 known (riskless) cash flows.
- Discount these back to the present and set equal to zero (the value of the swap at initiation).

Determining the Fixed Price (end)

- $0 = \sum_{i=1}^{N} D_i(F_{t,i} - P)$ where D_i is the discount factor from payment date *i* to the present.
- This then implies:

$$P = \frac{\sum_{i=1}^{N} D_i F_{t,i}}{\sum_{i=1}^{N} D_i}$$

- The swap fixed price therefore depends on the forward curve for the commodity, and is a weighted average of forward prices.

The Swap Value After Initiation

- After the swap is initiated, the forward/futures prices change, and hence the value of the swap can become either positive or negative. (Remember this is a zero sum deal, so if it is positive to one party it is negative to the other.)

$$V_{t'} = \sum_{i=1}^{N} D_i(F_{t',i} - P)$$

Uses of Swaps

- Like all derivatives, swaps can be used to speculate or hedge.
- Consider a gas producer that wants to lock in the first 12 cash flows on a new well: sell a 12 month “forward” starting swap where the cash flows of the swap start on expected first production date.
- Bearish on gas: sell a swap.
- European gas consumer wants to hedge the cost of natural gas for 2024: buy calendar year 2024 TTF swap.
- Bullish on gas: buy a swap.

Other Kinds of Energy Swaps

- Index swap: pay (receive) index (*e.g*., Waha monthly index), receive (pay) fixed.
- Gas daily swap: pay (receive) Gas Daily price (*e.g*., Waha GD), receive (pay) monthly index at same location.
- Basis swap: pay (receive) basis (monthly index at a location minus NYMEX), receive a fixed payment negotiated when the swap is initiated.

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OPTIONS

Options Basics

- Call option: right, but *not the obligation* to buy the "underlying" (e.g., a stock, a futures contract) at a fixed price (the "strike price").
- Put option: right, but *not the obligation* to sell the "underlying" (e.g., a stock, a futures contract) at a fixed price (the "strike price").
- Calls and puts have fixed expiration dates. These can range from a day to years into the future.
- Key feature: they give the owner a *right*. Conversely, the seller has granted the buyer the right to make the seller do something (*e.g.*, to buy the underlying at the strike price).

Options Payoffs

- A call option payoff is:

$$\max[F_T - K, 0]$$

- A put option payoff is:

$$\max[K - F_T, 0]$$

- Note that these payoffs are mirror images of one another. We will use this feature later.

Options "Geography"

- European: can exercise only at expiration date.
- American: can exercise any time up to and including the expiration date.
- Bermudan: any guesses?
- Asian: payoff based on the average price of the underlying over some period of time, e.g., the average of month end prices in the year prior to expiration.
- Russian: no expiration date. That's crazy!

Exotic Options

- Puts and calls are considered “vanilla” options: there are a wide variety of “exotic” options. The variety limited only by the perverse imaginations of traders.
- Asians are a kind of exotic option.
- Barrier options: knock-in (up-and-in, down-and-in), knock-out (up-and-out, down-and-out).
- Parisian: a kind of barrier option where the price has to remain above/below the barrier for an interval of time.
- Swing: common commodity option. In a swing call the buyer has the right to buy gas (say) on a finite number of days (*e.g*., 5) over a period of time (*e.g*., a month).
- I could go on. And on. And on.

Early Exercise

- American options have an "early exercise" feature. Since Americans give more options (choices) than Europeans, they have to be at least as valuable, and perhaps more valuable.
- What determines the value of early exercise?
- Early exercise always has a disadvantage: you give up the price protection (the "insurance") that the option gives.
- Consider a call on a future. With the future, you can profit from price increases but lose from price declines. With the call, you can profit from the price increases, but are protected against price declines below the strike.
- Therefore, for early exercise to be valuable, there must be some benefit.

Early Exercise Benefit

- The only benefit of exercising early is receiving a cash flow earlier. Some options provide this benefit, some do not.
- A call on a non-dividend paying stock provides no benefit: early exercise means that you have a cash outflow sooner, which is bad not good.
- Put on a non-dividend paying stock accelerates receipt of cash flow, so might exercise early.
- Call on a dividend paying stock: exercise right before *ex* dividend date, get the dividend. Don't get the dividend if you don't exercise.

Futures Option Early Exercise

- Most important options in commodities are options on futures.
- For a call option on futures, upon exercise you receive (a) a long futures contract marked at the current futures price (and thus having zero value), and a cash payment equal to the difference between the futures price and the strike price.
- For a put option on futures, upon exercise you receive (a) a short futures contract marked at the current futures price (and thus having zero value), and a cash payment equal to the difference between the strike price and the futures price.
- Since for both put and call there is a cash flow at exercise, might be beneficial to exercise early.

Determinants of Early Exercise

- Early exercise is a trade-off between lost “insurance” value and gained time value of money.
- Insurance value depends on “moneyness” of option, time to expiration, and volatility.
- Time value depends on interest rate.
- Early exercise more likely when: (a) option is deep in the money, (b) time to expiration is low, (c) volatility is low, and (d) interest rates are high. Factors a-c reduce insurance value: factor d increases time value.

OPTIONS PRICING

Arbitrage Restrictions

- Certain arbitrage restrictions apply to futures/forward option prices.
- The most important is put-call parity. It holds as an equality only for European options.
- Consider European options on the same futures contract (e.g., July 2023 crude oil futures) with the same strike price *K*. The futures price at *t* is F_t. The continuously compounded interest rate is *r*. The options expire at *T*.
- The call price is $c(F_t, K, t, T)$. The put price is $p(F_t, K, t, T)$

Put-Call Parity

- Consider the following strategy: buy the call, sell the put, and sell the future. The long call-short put strategy creates a synthetic long forward position with forward price *K*:

$$\max[F_T - K, 0] - \max[K - F_T, 0] = F_T - K$$

- The short futures (with payoff $F_t - F_T$) hedges this synthetic long exposure, giving a certain payoff of:

$$F_t - K$$

- This payoff is received at T, so discounted back to the present, the value is

$$e^{-r(T-T)}(F_t - K)$$

Put-Call Parity (2)

- Since the mark-to-market value of the futures contract is zero, value of the portfolio of a long call, short put, short futures is *c-p*. Since the payoff to this portfolio is certain, to prevent arbitrage its current value must equal the present value of its (riskless) payoff, that is:

$$c - p = e^{-r(T-T)}(F_t - K)$$

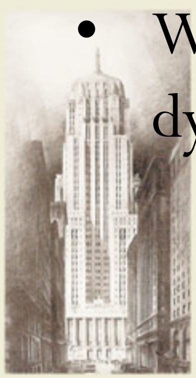
- This is the put-call parity relationship.
- For American options, we cannot construct a riskless payoff, so we can only bound the values:

$$C - P < e^{-r(T-T)}(F_t - K)$$

THE BINOMIAL MODEL

Options Pricing Models

- PCP only gets us so far: it tells us about the relationship between put and call prices. But we want to know what a put price should be, or what a call price should be.
- To do this, we need to add more structure. PCP made no assumptions about how the underlying futures price behaves. To derive an option pricing model, we will need to make such assumptions.
- Every different set of assumptions about price behavior gives us a different model.
- We will still use arbitrage principles, but our hedges will have to be dynamic, not static like in cash-and-carry or PCP.



The Binomial Model Framework

- The binomial model is a very stick-figure-like characterization of how futures prices move over time.
- We first divide time into equal increments Δt in length. We can choose Δt. The smaller it is, the more accurate our model.
- Over each time interval, the futures price can either move up by proportion $u>1$ or down by proportion $d<1$. So if the futures price at the beginning of an interval is F, at the end of the interval it is either uF or dF. We'll see how to choose u and d later.
- The probability of an up move is q. We will see this doesn't matter (perhaps surprisingly to you).

Constructing a Riskless Portfolio

- Just like with cash-and-carry or PCP, we will try to create a riskless portfolio. But we will have to create a different portfolio at the beginning of every time interval of time and adjust it dynamically.
- In our portfolio, we will hold a short position in 1 call. At the end of the next time interval, the value of the short call will be either $-c_u$or $-c_d$.
- We will also hold Δ futures contracts. At the end of the interval, if the futures price is now *F*, the futures position will be worth $\Delta(uF-F)$ or $\Delta(uF-F)$.

Choose the Futures Position

- To make our portfolio riskless, it must have the same value regardless of whether the futures price goes up or down. Therefore:

$$\Delta(uF - F) - c_u = \Delta(dF - F) - c_d$$

- Note we have one equation in one unknown, Δ.
- Solving for Δ we get:

Choose the Futures Position

- To make our portfolio riskless, it must have the same value regardless of whether the futures price goes up or down. Therefore:

$$\Delta(uF - F) - c_u = \Delta(dF - F) - c_d$$

- Note we have one equation in one unknown, Δ.
- Solving for Δ we get:

$$\Delta = \frac{c_u - c_d}{F(u - d)}$$

- This is a “hedge ratio. Note that it is like a slope: the difference in option values per unit difference in the futures price.

Solving For the Option Value

- The riskless cash flow is received in Δt years. The current value of the portfolio is $-c$ (because the futures position has no value). To avoid arbitrage the future value of the portfolio value must be the same as the value of the riskless cash flow:

$$-e^{r\Delta t}c = \frac{(c_u - c_d)(u-1)}{u-d} - c_u$$

- Simplifying:

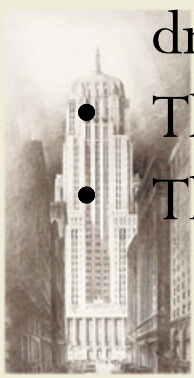
$$c = e^{-r\Delta t}[pc_u + (1-p)c_d]$$

Where

$$p = \frac{1-d}{u-d}$$

Some Observations

- This formula for the value of the call today looks like an expected present value because *p* is like a probability: it is a number between 0 and 1. Furthermore, it is the probability of an up move
- However, this probability is NOT the probability in the real world that the futures price moves up.
- Instead, it is the probability that would obtain if the expected futures price at the end of the next time interval is equal to the current futures price.
- That is, if *p* is the probability of the up move, the futures price has no drift.
- This is only true in a risk neutral world.
- Thus, we can price options as if we are in this world.



This Makes Option Pricing Practical

- Since we can price options assuming there is no risk premium in the futures price, we don't have to estimate the risk premium.
- This is a good thing! Because we can't!
- But that doesn't matter. For the purpose of pricing, we can pretend we live in an alternate universe in which everyone is risk neutral.
- This is a consequence of our ability to hedge perfectly.
- This perfect hedging is a consequence the assumptions about how the futures price moves over time.
- We leave Nirvana if we make different assumptions.

Proof of the Risk Neutrality Result

$$E(F_{t+\Delta t}) = puF_t + (1-p)dF_t$$

$$E(F_{t+\Delta t}) = F_t\left[\frac{(1-d)(u-d)}{u-d} + d\right]$$

$$E(F_{t+\Delta t}) = F_t$$

Using the Binomial Model

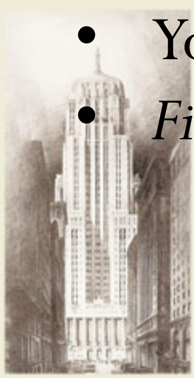
- You might think that the formula derived above isn't much help: it tells us about how today's option price relates to the possible prices next period, but if we don't know those what help is that:
- Well, there is a time we know that: one time interval before expiration.
- If the futures price is *F* then, for a call we know that $c_u = \max[uF - K, 0]$ and $c_d = \max[dF - K, 0]$.
- This is news we can use.
- Build a "binomial tree" starting at the current futures price and ending at expiration. Then work backwards through the tree applying our formula at each step.

Choosing *u* and *d*

- Almost done!
- To choose u and *d*, we want to match the dynamics of our theoretical futures price it what we believe the actual dynamics are.
- We know the expected change doesn't matter. So let's match the variance.
- The variance of the percentage futures change is:

$$p(u-1)^2 + (1-p)\,(d-1)^2$$

- We set this equal to our estimate of the actual return variance, $\sigma^t \Delta t$.
- Note we have 1 equation and 2 unknowns. So set d=1/u.
- You can show if Δt is small enough, $u = e^{\sigma \Delta t}$.
- *Finis*!



What the Model Tells Us

- The solution to the binomial model gives us the value of the option, *i.e.*, the price we should be willing to pay for it if we buy it, or the price we get if we sell it.
- This price is frequently referred to as the "premium."
- The model can also quantify how the option value changes over time in response to changes in the underlying futures price and the passage of time.

Where Does σ Come From, Daddy?

- All of the inputs to the model are known but one.
- We know the current futures price, the strike price, and the time to expiration, and can choose the time interval. Only σ is not known.
- σ is referred to as "volatility," and is the secret sauce of option pricing. Options trading is effectively taking a view on volatility.
- This is a parameter that describes the statistical behavior of the futures price, which is not objectively knowable.
- Later we'll discuss in some detail how to estimate volatility.
- The model assumes volatility is constant. We'll also see that this isn't a realistic description of the world, but that more realistic models mean we aren't in a risk neutral world anymore. That creates complications!

Using the Binomial Model: Hedging

- The Δ derived above was chosen to hedge the risk of the option
- That is, taking a position of Δ units of the futures contract offsets the risk of -1 unit of the option
- This means that given *F*, a *long position* in the option is equivalent to Δ futures contracts, and a short position is equivalent to −Δ futures
- Note Δ changes as F changes and time passes: therefore, hedging is *dynamic*
- The math of Δ explains this: it is effectively a slope
- Δ scales linearly: a position of *X* units of the option has a delta of XΔ
- You can add up the Δs of options with different strikes and maturities as long as they are on the same underlying (e.g., a July 2023 CL future)

Some Delta Basics

- Delta is a measure of the riskiness of an option because it measures the sensitivity of the option price to the (risky) futures price.
- All else equal, the bigger is delta in absolute value, the riskier is the option.
- Call delta is in [0,1] so a call is no riskier than a long futures position.
- Put delta is in [-1,0] so a put is no riskier than a short futures position.
- Delta is positive for calls and negative for puts. Therefore, long calls are equivalent to long positions in the underlying, and long calls are equivalent to short positions in the underlying.
- Delta is negative for a short call, and positive for a short put.

Using the Binomial Model: ‘Merican Options

- One advantage of the binomial model as opposed to the Black Model that we will examine next is that you can use the binomial model to price American options.
- You can do this by evaluating whether it is more profitable to exercise or hold the option at each “node” of the binomial tree
- The formula we derived above gives the value of holding the option one more period, and at each node $\max[F-K,0]$ or $\max[K-F,0]$ gives the value from exercise for a call and put respectively, when the futures price is *F*.

THE BLACK MODEL

Black Model Basics

- The binomial model is a "discrete time" model: we slice time into discrete increments Δt in length.
- The Black Model is a continuous time model: we slice time into infinitesimally small increments *dt* in length.
- The Black Model is the continuous time limit of the binomial model as $\Delta t \rightarrow dt$.
- This has pros and cons: the model is more accurate, but it cannot handle early exercise because we can't make the hold-or-exercise decision at an infinite number of points in time.

The Probabilistic Basis of the Black Model

- Recall that the binomial model implies that the value of the option is the expected present value of its payoffs when the futures contract has no trend/drift, that is, in the alternative risk neutral universe.
- The same thing is true in the Black Model. We can operate in the risk neutral universe.
- Calculating an expected value requires a probability distribution.
- In the Black Model the relevant probability distribution is the well-known "normal" ("Gaussian") distribution—the bell-shaped curve.
- The natural logarithm of the futures price at expiration has a normal distribution, with mean $\ln(F_{t,T})$ and standard deviation $\sigma\sqrt{T-t}$.

The Normal Distribution

- In the Black Model, the probability density of the futures price at expiration is given by:

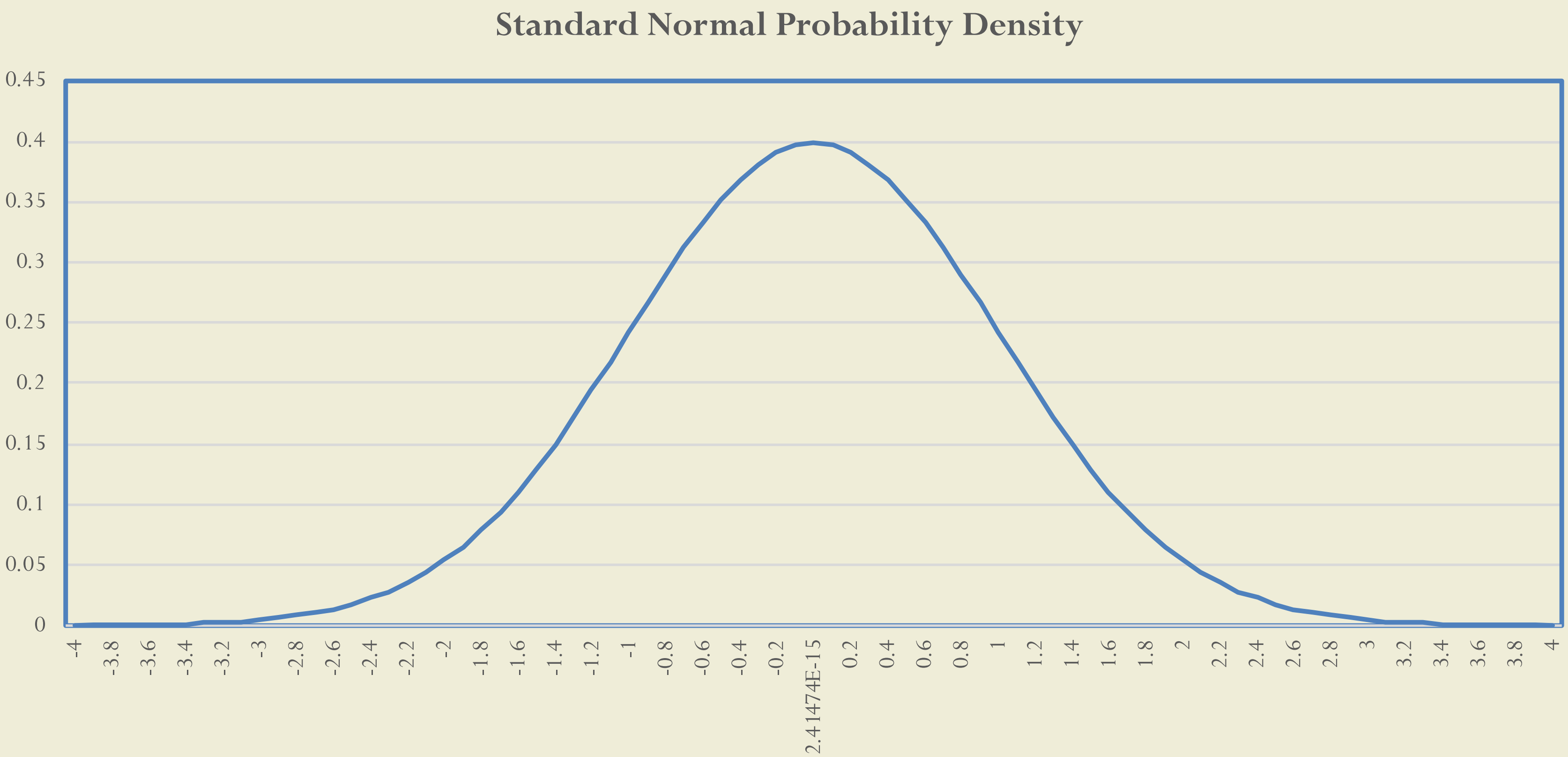
$$p\left(F_{T,T} \middle| F_{t,T}\right) = \frac{e^{-.5\left(\frac{\ln(F_{T,T}) - \ln\left(F_{t,T}\right)}{\sigma\sqrt{T-t}}\right)^2}}{\sigma\sqrt{2\pi(T-t)}}$$

Using the Black Model to Price a Call

- Since the price of an option is the expected present value of its payoff, the Black Formula value of an option is derived from this expression which gives that expected present value:

$$c(F_{t,T}, K, \sigma, r, T, t) = e^{-r(T-t)} \int_K^\infty \max[F_{T,T} - K, 0]\, p(F_{T,T} | F_{t,T})\, dF_{T,T}$$

The Normal Distribution Illustrated



The Black Call Formula

- In a separate note, as a service to the interested (or masochistic!) I go through the math to derive the Black Formula. (I used to be a sadist and go through the derivation in class.)
- We care about the bottom line, which is:

$$c\left(F_{t,T}, K, \sigma, r, T, t\right) = e^{-r(T-t)}[F_{t,T}N(d_1) - KN(d_2)]$$

- Where

$$d_1 = \frac{ln\frac{F_{t,T}}{K} + .5\sigma^2\,(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

The Black Put Formula

- You can use the same math to derive the Black Put Formula, or you can use put-call parity.
- This produces:

$$p\left(F_{t,T}, K, \sigma, r, T, t\right) = e^{-r(T-t)}[KN(-d_2) - F_{t,T}N(-d_1)]$$

- Note that the put and call formulae are essentially mirror images: you change the signs of everything inside the brackets.
- This isn't surprising, given the mirror image nature of the payoffs to puts and calls.

Interpretation

- The term

$$\frac{\ln(\frac{F_{t,T}}{K})}{\sigma\sqrt{T-t}}$$

Is the number of standard deviations the option is in the money.

- The term $.5\sigma\sqrt{T-t}$ is a "convexity" correction.

Options Risks: The “Greeks”

- Recall in the binomial model the option Δ gives the sensitivity of the option price to a change in the underlying futures price. This is a measure of the riskiness of the option.
- We can do the same thing in the Black Model.
- Delta is a first order measure of risk: “Gamma”-—the sensitivity of the Delta to the futures price—is a second order measure of risk.
- We can also determine the sensitivity of the option price to time to expiration. This “theta” isn’t really a risk measure because the passage of time is a certainty. It is a measure of the cost of holding the option.
- Somewhat inconsistently, we can calculate the sensitivity of the option value to σ. This is “lambda” (or “vega”).

Delta

- Recall that in the binomial model the option Δ gives the sensitivity of the option price to a change in the futures price. The same holds in the Black Model. For a call

$$\Delta = \frac{\partial c}{\partial F_{t,T}} = e^{-r(T-t)} N(d_1)$$

For a put:

$$\Delta = \frac{\partial p}{\partial F_{t,T}} = -e^{-r(T-t)} N(-d_1)$$

Delta Continued

- Just like in the binomial model, the Δ can be used to calculate the hedge ratio.
- Recall Δ is typically expressed as the sensitivity of a long position in a single option. For a position of X units of an option, the position delta is $X\Delta$.
- To hedge this position, take a futures position of $-X\Delta$.
- Just like with the binomial model, Δ changes as the futures price changes, and as time passes.
- Therefore, hedging is dynamic.

Gamma

- Gamma is the sensitivity (derivative) of the Δ to a change in the futures price:

$$\Gamma = \frac{\partial \Delta}{\partial F_{t,T}} = \frac{\partial^2 c}{\partial F_{t,T}^2} = \frac{e^{-r(T-t)} e^{-.5 d_1^2}}{F_{t,T} \sigma \sqrt{2\pi (T-t)}}$$

- Note that the gamma of puts and calls on the same underlying, strike, and time to expiration are the same.
- Like Δ, Γ scales with the size of the position. The Γ of a position of *X* units of an option is $X\Gamma$.
- Note that gamma is positive for both puts and calls. Gamma is a consequence of the convexity of options values/payoffs. Convexity is valuable: heads I win, tails I don't lose.

Hedging Gamma

- You can hedge gamma by taking a position in options on the same underlying: you can't hedge gamma with futures (which have a gamma of 0).
- See the next slide.
- If you are short options, buy options on the same underlying with the same expiration. The number of options you need to buy to hedge is the ratio of the gammas.

Hedging Gamma and Delta

- If you hedge Γ, the Δ of your position changes.
- But you can hedge gamma using options, and Δ using futures.
- Consider a position in X units of option 1, which has a gamma of Γ_1 and a delta of Δ_1.
- Use option 2 to hedge the gamma:

$$N_2\Gamma_2 + X\Gamma_1 = 0$$

Thus:

$$N_2 = \frac{X\Gamma_1}{\Gamma_2}$$

Then choose futures position N_F so that

$$N_F + X\Delta_1 + N_2\Gamma_2 = 0$$

More on Hedging Gamma and Delta

- The foregoing implies that you want to eliminate delta and gamma risk.
- However, you can construct a portfolio to achieve any targeted level of gamma and delta (which can differ from zero).
- Options traders view options as packets of delta and gamma that they can mix together to achieve target levels of these variables.
- That is, options traders bucket risks and target the size of the gamma and delta buckets.

Theta

- As noted before, theta is not properly a risk measure because the passage of time is inevitable. However, it measures "time decay" which is a cost of holding options: it measures how much an option value changes as time passes and prices don't change. This serves to emphasize how volatility benefits option value.

$$\Theta = \frac{\partial c}{\partial t}$$

- Unlike gamma, there are different formulae for European call and put thetas.

Theta Formulae

- European Call:

$$\Theta_c = -\frac{e^{-r(T-t)}F_{t,T}e^{-.5d_1^2}}{2\sqrt{2\pi(T-t)}} - rKe^{-r(T-t)}N(d_2)$$

- European Put:

$$\Theta_P = -\frac{e^{-r(T-t)}F_{t,T}e^{-.5d_1^2}}{2\sqrt{2\pi(T-t)}} + rKe^{-r(T-t)}N(-d_2)$$

Theta Interpretation

- Note that the first terms for both put and call theta are negative, and the same: this term measures the loss of "insurance value" due to the passage of time. This hurts both puts and calls.
- The second term is negative for a call, and positive for a put. This is a time value of money effect. For a put, the passage of time moves you closer to the receipt of a cash flow, which is beneficial. For a call, the passage of time moves you closer to the payment of a cash flow, which is detrimental.
- Thus, a call theta is always negative, but a European put theta can be positive (especially for deep in the money options).
- American thetas are always negative for both puts and calls.

Lambda

- Lambda is the sensitivity of an option price to a change in volatility:

$$\Lambda_C = \frac{\partial c}{\partial \sigma}$$

  And similarly for a put.
- Note there is an element of schizophrenia here. The Black Model assumes σ is constant, so what's the point of calculating lambda?
- As will be discussed, traders recognize that in reality volatility changes, and despite the logical inconsistency, they use the Black lambda to measure the volatility risk of options.
- Lambda is often called "vega," which isn't actually a Greek letter, but it sounds Greek and starts with "v." (Vega is actually a Chevy from the 70s.)

Lambda Formula

- The formula for lambda is:

$$\Lambda_c = \Lambda_P = e^{-r(T-t)} F_{t,T} \sqrt{T-t} \frac{e^{-.5 d_1^2}}{\sqrt{2\pi}}$$

- Note that the call and put lambdas are identical.
- The formulae for gamma and lambda share the normal distribution term, and hence these functions have somewhat similar shapes—distorted normal distribution curves.
- The similarity reflects the economics: volatility is valuable (i.e., lambda is positive) because of convexity (gamma). The more convexity, the more valuable is volatility.
- If you hedge gamma, you also hedge lambda.

VOLATILITY ESTIMATION

Estimating Sigma

- All of the inputs but one to the Black Model are known: the unknown is σ. It must be estimated or guessed.
- Estimation usually done using statistical methods.
- Volatility is the standard deviation of the change in the natural logarithm of the futures price.
- If you really believe the Black Model's constant volatility assumption, you should collect as much historical data as you can and use it to calculate that standard deviation.
- But we know volatility isn't constant. Volatility is volatile!
- Strictly speaking, this means we should junk the Black Model. But the alternative "stochastic volatility models" are mess—and we can't assume a risk neutral world because we can't hedge vol risk.

Dealing With Our Fall From Grace

- Even though it is logically inconsistent, it is still conventional to utilize the Black Model and adjust the volatility over time to reflect changes in volatility.
- There is an infinite variety of ways of doing this using historical data.
- The traditional historical vol estimate weights all past data equally. Time-varying volatility estimates weight more recent data more heavily than data from further in the past.
- "GARCH" is an example of this.
- Infinite number of possible weighting schemes. GARCH-type models use the historical data to estimate the weights.

Seasonal Commodities

- Volatility estimation is even more complicated for seasonal commodities, especially natural gas, because volatility varies seasonally. For NG, for instance, volatility is much higher in the winter than in the shoulder months.
- This means that you have to use the data for the futures contract of interest (*e.g.*, December futures to estimate December volatility), but this also raises complications.
- Volatility tends to increase as maturity nears. So if you are in June 2023, using the last 6 months of December 2023 futures prices probably understates NGZ3 volatility over the remaining life of the option.
- Can you use previous years' Z futures prices? Yes . . . But that poses dilemmas too. Are 2021 and 2020 winters really like 2023 winter?

Implied Volatility

- An alternative to using historical data to estimate volatility is to use "implied volatility."
- An implied vol is the value of σ that equates the Black Model value of an option to the market price for that option.
- Whereas you can use historical volatility to determine whether an option may be mispriced, you can't do that with implied vol: you *assume* the option is priced correctly.
- Perhaps you can use it to determine whether options are mispriced relative to one another.
- Can use implied vol when calculating Greeks.

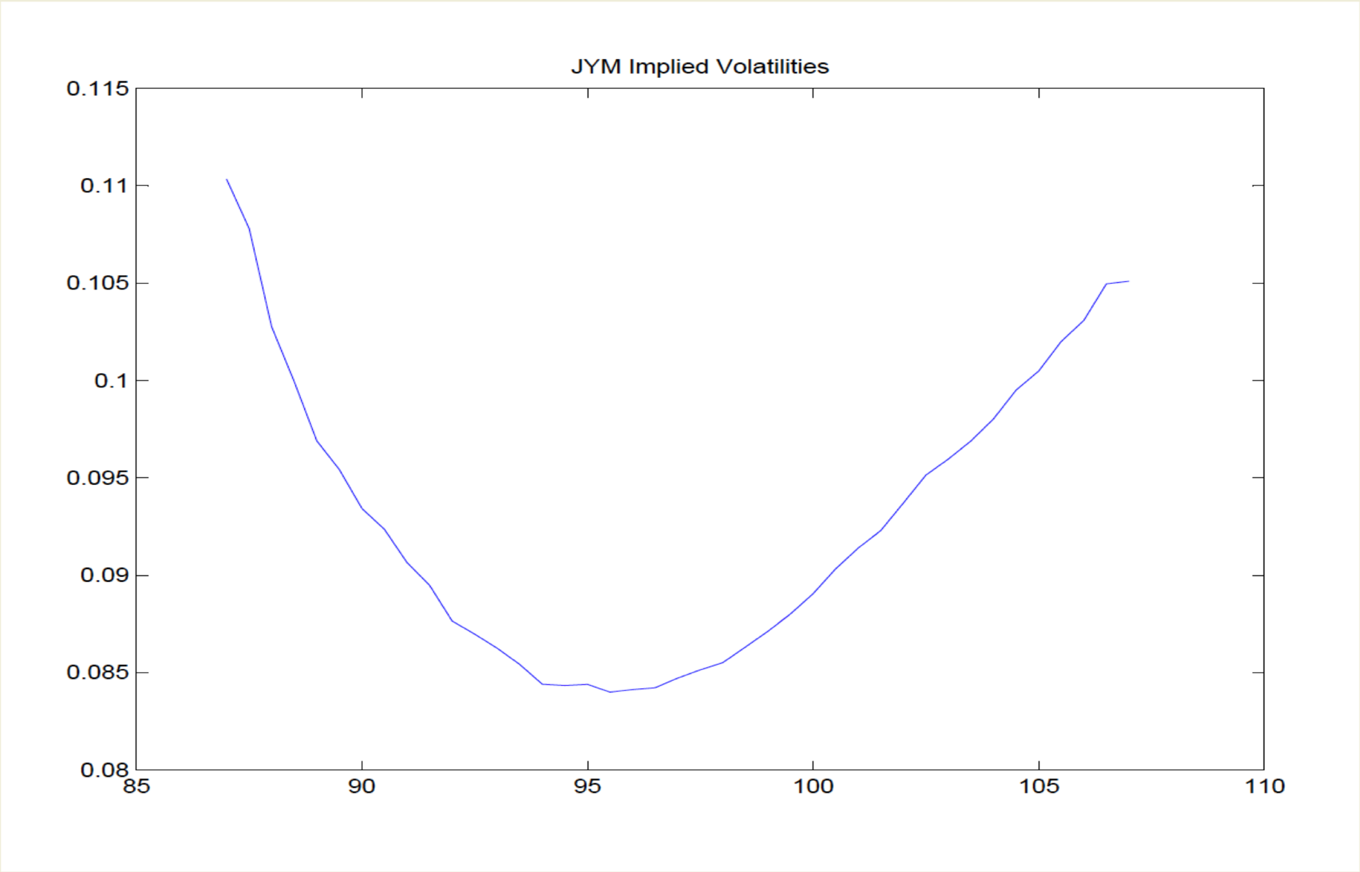
DOES THE BLACK MODEL WORK?

The Performance of the Black Model

- Models are representations of reality, not reality. They are hypothesis generating machines. The crucial question is how well they describe reality.
- One implication of the Black Model is that the implied volatility of options on a given underlying should not vary with strike price or time to expiration.
- This implication is clearly violated: there are volatility “smiles” and “skews.” That is, implied volatility is a nonlinear function of strike price.

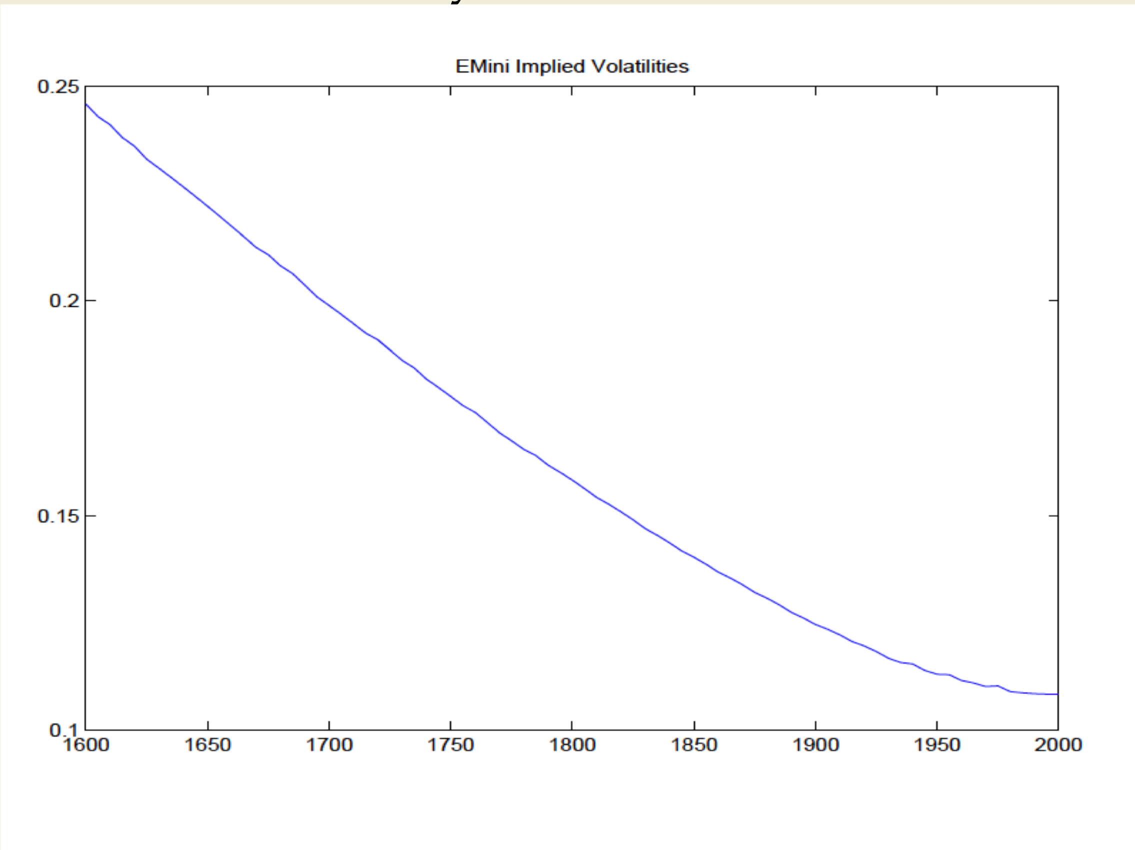
Currency "Smile"

- Currencies tend to have a u-shaped relation between vol and strike: this is the "volatility smile."



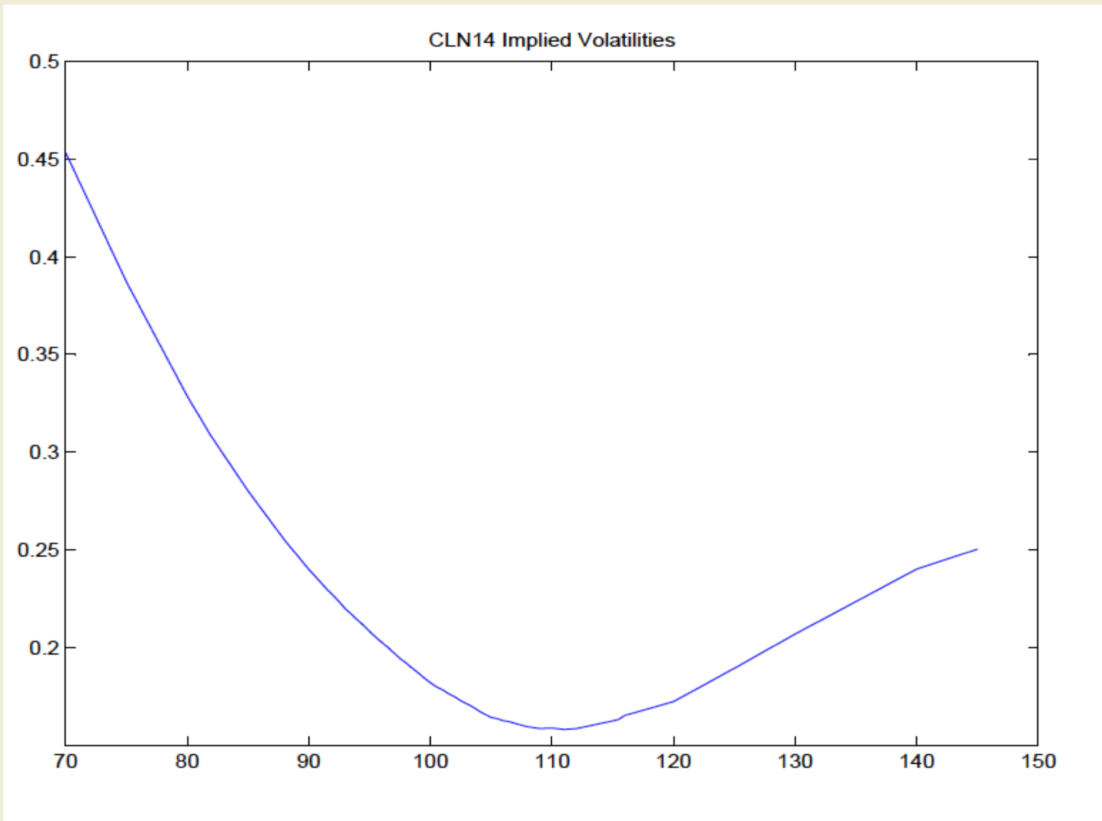
Stock Index "Skew"

- Stock indices tend to have a downward sloped relation between vol and strike: this is the "volatility skew" or "smirk."



Commodities

- Commodities can have a variety of shapes, and can flip over time. Here's July 2014 CL:

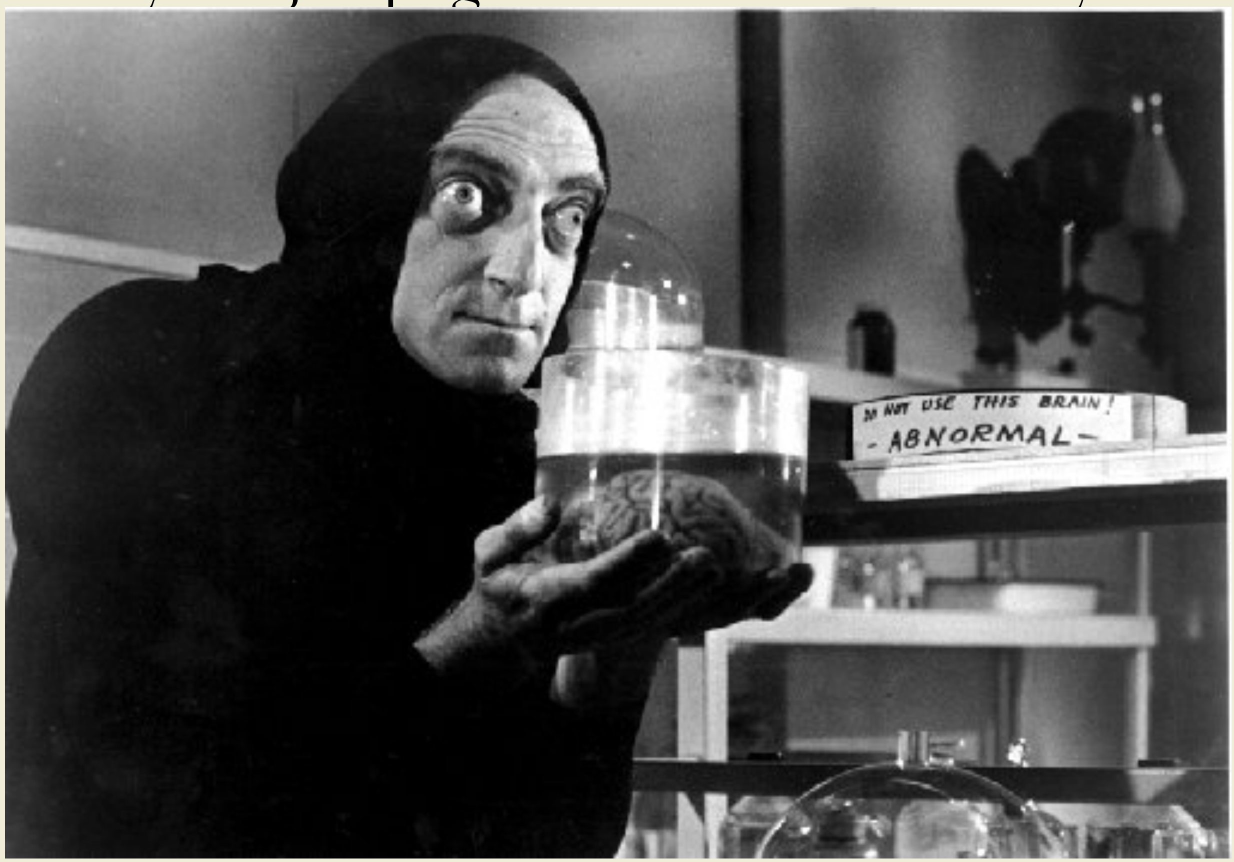


Why Are There Smiles and Skews?

- Smiles/skews exist because the real world does not adhere to the Black Model assumptions.
- Black Model assumes volatility is constant (or at least a deterministic function of the stock price and time): in the real world volatility is "stochastic" (*i.e.*, varies randomly).
- Black Model assumes that prices move continuously—there are no "jumps" or "gaps": in the real world, jumps—sometimes extreme—occur. Think of CL on 20 April 2020. Or the Black Monday stock market crash.

What Do Smiles and Skews Tell Us?

- Remember that in the Black Model, log futures prices are "normal." Stochastic volatility and jumps give rise to non-normality in returns.



Psychiatric Diagnosis Using Skews

- The smiles/skews tell us the nature of the "abnormality."
- Symmetric smile: price movements are symmetric, but extreme moves in either direction are more likely than the normal distribution allows.
- Skew to the "put wing": asymmetric price movements with down moves more likely than up moves. Also, typically extreme down moves more likely than the normal distribution implies.
- Skew to the "call wing": asymmetric price movements with up moves more likely than down moves. Also, typically extreme up moves more likely than the normal distribution implies.

Should We Institutionalize the Black Model?

- We know that the Black Model is an imperfect description of reality.
- However, alternatives raise grave complications. In particular, we aren't in a risk neutral world anymore because we can't hedge volatility changes or jumps. Therefore, any model must incorporate risk premia for these things, which are devilish hard to estimate.
- Also, underlying statistical models have a lot of parameters and are hard to estimate.
- So, in practice, we are schizophrenic, and adjust the Black Model to correct for things that the model says can't exist.
- Use "volatility surfaces" to adjust prices (and Greeks) for skews and smiles.