Equivalent Martingale Methods

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We have shown how you can determine the value of a contingent claim by solving a PDE. There is another approach: the value of a contingent claim can be determined as the expected present value of the payoffs to the claim, where the expectation is taken with respect to something called an "equivalent martingale measure." An equivalent martingale measure ("EMM") is a probability measure that "shares sets of measure zero" with the "true" probability measure, but under this measure discounted asset prices are a martingale, i.e., they have no "drift."

How do you turn a discounted process into a martingale? Something called the Girsanov Theorem provides the roadmap.

Consider a stock price process, where W_t is a Brownian motion:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$$

This gives the dynamics of the stock price under the true/physical measure. Under this measure, the expected return is μ and the variance of the return is σ^2 (both annualized).

Under the Girsanov Theorem, one can define a new process, $\tilde{W}$, by adjusting the drift of the original B.M.:

$$d\tilde{W}_t = W_t + \theta(W_t, t)dt$$

Note: the $\theta(.,.)$ function is arbitrary: you can choose it.

Given a choice of $\theta(.,.)$, the density of $\tilde{W}_t$ is the probability density of W_t, multiplied by:

$$Z_t = exp\{-\int_0^t \theta(W_t, t)dW_t - .5\int \theta^2(W_t, t)dt\}$$

(Look familiar?)

Now consider the discounted price process $X_t = e^{-rt}S_t$. According to Ito's Lemma:

$$dX_t = -re^{-rt}S_t dt + e^{-rt}dS_t = e^{-rt}S_t[-rdt + \mu dt + \sigma dW_t]$$

Substitute $dW_t = d\tilde{W}_t - \theta dt$:

$$dX_t = e^{-rt}S_t[(-r + \mu - \sigma\theta)dt + \sigma d\tilde{W}_t]$$

To make this a martingale, the terms involving dt must go away, so:

$$\theta = \frac{\mu - r}{\sigma}$$

which just happens to be the Sharpe Ratio for this stock. That is, θ has the interpretation of a risk premium, A/K/A the "market price of risk."

Note that this θ is unique. By the Second Fundamental Theorem of Asset Pricing, this means that markets are "complete," and that all contingent claims on S_t can be hedged with a dynamic trading strategy. This in turn implies that value of any such contingent claim is the expectation of the discounted value of the payoffs, *where this expectation is taken with respect to the equivalent probability measure.*

What is that EM? (Sometimes called the $Q-$measure or the tilde-measure.)

There are two ways of figuring this out. The first is to multiply the density of the log stock price by Z_t. The density of the log stock price is:

$$P(\ln S_t) = \frac{exp\{\frac{-.5(\ln S_t - \mu t - .5\sigma^2 t)^2}{\sigma^2 t}\}}{\sqrt{2\Pi}}$$

by Z_t (see above).

The easier way is to substitute for W_t. Consider a stock price process, where W_t is a Brownian motion:

$$\frac{dS_t}{S_t} = \mu dt + \sigma(d\tilde{W}_t - \frac{\mu - r}{\sigma}dt) = rdt + \sigma d\tilde{W}_t$$

Thus, the distribution of the log stock price $\ln S_t$ is normal, with mean $(r - .5\sigma^2)t$ and variance $\sigma^2 t$.

$$\tilde{P}(\ln S_t) = Q(\ln S_t) = \frac{exp\{\frac{-.5(\ln S_t - rt - .5\sigma^2 t)^2}{\sigma^2 t}\}}{\sqrt{2\Pi}}$$

That is, under the equivalent measure, the expected return is r and the variance is σ^2. That is, the drift of the process is different under the EMM than under the physical/true measure, but the variance is the same.

So I can hear you saying: "OK, prof. Nice trick. Why should I care?" Well, it turns out there is a connection between the existence of an EMM and the absence of arbitrage. Further, there is a linkage between the uniqueness of an EMM and the ability to determine a unique price for derivatives–which just so happens (fortunately!) to be the same value we get from the Black-Scholes-Merton PDE.

An arbitrage exists if you can devise a trading strategy with payoff X_T such that $\mathcal{P}(X_T \geq 0) = 1$, and $\mathcal{P}(X_T > 0) > 0$. Here, $\mathcal{P}$ is the "true" or "physical" probability measure. In owrds, this definition says that an arbitrage exists if the strategy never loses money, and makes money with positive probability.

Call D_t the discount factor. In what we've done so far, $D_t = e^{-rt}$, but as we will see we can choose other discount factors.

Let $\tilde{\mathcal{P}}$ denote an EMM, and $\tilde{E}_t$ be the expectation taken at time t with respect to this measure. Since $\tilde{\mathcal{P}}$ is an EMM, if a trading strategy X involves zero investment at $t = 0$, $\tilde{E}_0(D_T X_T) = 0$. Assume an arbitrage exists. Thus, $\mathcal{P}(X_T > 0) > 0$. By the definition of equivalence, $\tilde{\mathcal{P}}(X_T > 0) > 0$. Moreover, arbitrage implies $\mathcal{P}(X_T \geq 0) = 1$, and thus equivalence implies $\tilde{\mathcal{P}}(X_T < 0) = 0$. Since $D_T > 0$, these facts imply that $\tilde{E}(D_T X_T) > 0$. This is a contradiction. Therefore, if there exists an EMM, there is no arbitrage. (The converse is much harder to prove–and not particularly important.)

Now let's examine the pricing of contingent claims. Consider a simple economy with two assets, a stock and the money market (that pays r per annum). A trading strategy $\phi(t)$ is a $\mathcal{F}_t$ measurable strategy, that involves investment in $\phi_S(t)$ units of the stock and $\phi_M(t)$ units of the money market. This strategy is "self-financing," meaning that if the value of the portfolio implied by the strategy is $V_\phi(t)$, then:

$$dV_\phi = \phi_M dM + \phi_S dS_t$$

Note we have to rule out some trading strategies. If you had access to infinite funds, you could always design a trading strategy that would double down on losses, and lead to a profit (of arbitrary magnitude!) with probability 1.

Now consider contingent claim valuation. Consider a contingent claim (e.g., a call option) with a payoff of Y_T Further, assume that this payoff is "attainable" via a trading strategy ϕ. This means that this strategy also has a payoff of Y_T almost surely. (And don't call me Shirley!) This is a replicating trading strategy.

If this is all true, the value of the contingent claim $\Pi_Y(t)$ is:

$$\Pi_Y(t) = \frac{1}{D_t}\tilde{E}(Y_T D_T)$$

To see why, consider a strategy of selling the contingent claim, and buying the replicating trading strategy. Also, assume:

$$\Pi_Y(t) > \frac{1}{D_t}\tilde{E}(Y_T D_T)$$

Since the equivalent measure is a martingale measure, $D_t V_t = \tilde{E}(D_T V_T) = \tilde{E}(D_T Y_T)$, so this strategy generates a positive inflow at t. But at T, the strategy has a zero payoff. The portfolio is worth Y_T (since it is a replicating strategy) but the contingent claim you've sold pays off $-Y_T$.

Free money! But that contradicts the assumption of the existence of an EMM (which implies no free money).

Meaning that we can value a contingent claim by taking the expected present value of its payoffs, where the expectation is taken under *the equivalent measure,*not the true/physical measure.

You might have noticed a big assumption here: I've assumed you can replicate the payoffs to the contingent claim. This is sort of like an old comedy routine: "How to make $1 million with out paying taxes. First, make $1 million." How can we be sure we can replicate?

As it turns out, this will be true if and only if the EMM is unique. Sometimes this is easy to verify, as in the Black-Scholes model. (See the derivation of Θ above.) In other cases, it is easy to verify that the EMM is not unique.

To see this, consider a market with a single traded asset, a stock S_t:

$$\frac{dS_t}{S_t} = \mu dt + V_t^{.5} dW_{st}$$

The difference between this and the standard model is that the stock's variance, is a stochastic process:

$$dV_t = \alpha(V_t, t)dt + \sigma_\nu(V_t, t)dW_{vt}$$

The logic we went through above allows us to determine a market price of risk for the stock that turns it into a martingale:

$$\lambda(V_t) = \frac{\mu - r}{V_t^{.5}}$$

Using this, we can write:

$$\frac{dS_t}{S_t} = rdt + V_t^{.5} d\tilde{W}_{st}$$

Now consider another traded claim: a call option on this stock. The probability distribution of the stock's return at expiration will depend on the evolution of V_t, so plausibly $C(S_t, V_t, t)$. Using the multidimensional version of Ito's Lemma:

$$de^{-rt}C = e^{-rt}[-rCdt+C_tdt+C_SdS+C_VdV+.5C_{SS}dS^2+.5C_{VV}dV^2+C_{SV}dSdV]$$

Substituting in for dS in the equivalent measure, we get:

$$de^{-rt}C = e^{-rt}[-rCdt+C_tdt+C_S[rS_tdt+V_td\tilde{W}_t)+C_V(\alpha(V_t,t)dt+\sigma_\nu(V_t,t)dW_{vt})+.5C_{SS}S_t^2V_t+.5C_{VV}\sigma$$

Now do a Girsanov substitution:

$$d\tilde{W}_{Vt} = dW_{Vt} + \Lambda dt$$

$$de^{-rt}C = e^{-rt}[-rCdt+C_tdt+C_S[rS_tdt+V_td\tilde{W}_t)+C_V((\alpha(V_t,t)-\Lambda)dt+\sigma_\nu(V_t,t)d\tilde{W}_{vt})+.5C_{SS}S_t^2V_tdt+$$

For this to be a martingale, the terms multiplying dt must sum to zero:

$$0 = -rC+C_t+rS_tC_S+C_V(\alpha(V_t,t)-\Lambda)+.5C_{SS}S_t^2V_t+.5C_{VV}\sigma_\nu^2+C_{SV}\rho S_tV_t\sigma_\nu$$

Note that this is a PDE. Great! If we know Λ, we can solve it. But note, Λ is completely arbitrary. We can choose it to be anything we want, and for each different choice we get a different solution for $C(S_t, V_t, t)$. Thus, the EMM is not unique, and there is not a unique solution to the value of the option.