FINA 4320: Investment Management
Efficient Diversification

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Two-Security Portfolio: Return

- Expected Return: $E[r_p] = w_1 E[r_1] + w_2 E[r_2]$
- Expected Return of portfolio with n securities

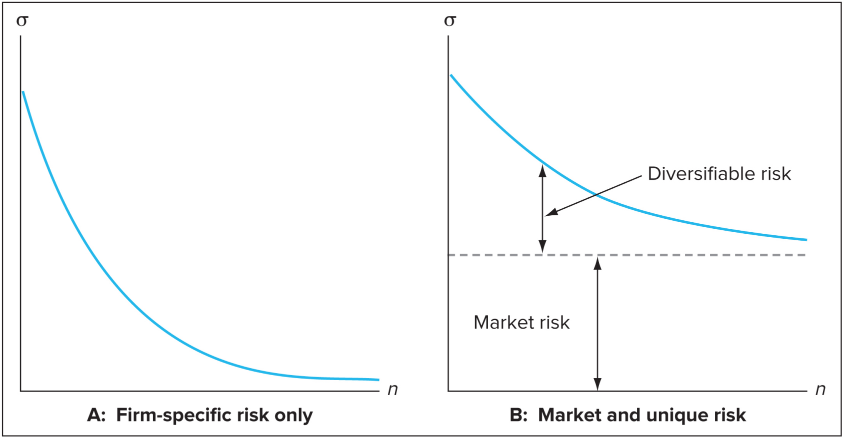
$$E[r_p] = \sum_{i=1}^{n} w_i E[r_i]$$

$$\sum_{i=1}^{n} w_i = 1$$

Portfolio Risk

- Risk factors common to the whole economy lead to market risk, also called systematic risk or nondiversifiable risk
- Risk that can be eliminated by diversification is called firm-specific risk, also called unique risk, idiosyncratic risk, residual risk, nonsystematic risk or diversifiable risk

Portfolio Risk as a Function of the Number of Stocks in the Portfolio



Portfolio Variance and Covariance

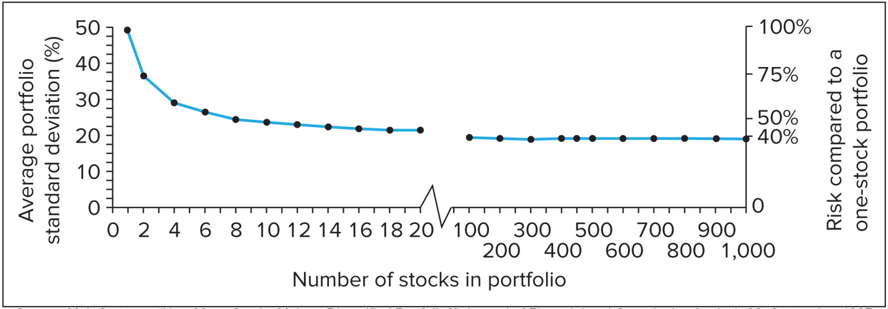
- Investing in more securities reduces risk:
  - Portfolio variance:

$$\sigma_p^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j Cov(r_i, r_j)$$

  - How to compute Covariance (Cov)

$$Cov(r_i, r_j) = \sum_{t=1}^{n} \frac{(r_i(t) - E[r_i])(r_j(t) - E[r_j])}{n-1}$$

Naive Diversification



Source: Meir Statman, "How Many Stocks Make a Diversified Portfolio?" *Journal of Financial and Quantitative Analysis* 22, September 1987

Correlation and Diversification

	Good	Bad	Ugly
Stock A	25%	10%	-25%
T-bills	5%	5%	5%
Stock B	1%	-5%	35%
Probability	0.5	0.3	0.2

- Examples of portfolios:

	E[r]	σ
All in A	10.5%	18.9%
Half A, half T-bills	7.75%	9.45%
Half A, half B	8.25%	4.83%

- Intuition:

$$\begin{aligned} Cov(A, B) &= -240.5 \\ \rho(A, B) &= -0.86 \end{aligned}$$

Covariance and Correlation

- Problem with Covariance
  - Covariance does not tell us the intensity of the comovement of the stock returns, only the direction.
  - We can standardize the covariance however and calculate the correlation coefficient which will tell us not only the direction but provides a scale to estimate the degree to which the stocks move together.
- Correlation

$$\rho(r_i, r_j) = \frac{Cov(r_i, r_j)}{\sigma_i \sigma_j}$$

- Rewriting portfolio variance in terms of correlation

$$\sigma_p^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_i \sigma_j \rho(r_i, r_j)$$

Covariance and Correlation

- Two stocks case:

$$\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_1\sigma_2\rho(r_1, r_2)$$

- ρ and diversification in a 2 stock portfolio:
  - Diversification improves risk-return trade-off. The lower the correlation b/w assets, the greater the gain from diversification (risk $\downarrow$)
  - Portfolios of $\rho < +1$ always offer better risk-return opportunities than individual securities on their own
  - With perfect positive correlation b/w assets, $\rho = +1$, no gain from diversification – just weighted avg. of individual security std. deviation $\sigma_p = w_1\sigma_1 + w_2\sigma_2$

Covariance and Correlation

- Perfect negative correlation b/w assets, $\rho =$ -1, is extreme case of diversification; $\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 - 2w_1w_2\sigma_1\sigma_2 = (w_1\sigma_1 - w_2\sigma_2)^2$
  - Create perfectly hedged position with zero portfolio risk using portfolio weights:

$$
\begin{aligned}
w_1 &= \frac{\sigma_2}{\sigma_1 + \sigma_2} \\
w_2 &= \frac{\sigma_1}{\sigma_1 + \sigma_2}
\end{aligned}
$$

- $\rho(r_1, r_2) = \rho(r_2, r_1)$ (same is true for covariance)

Summary

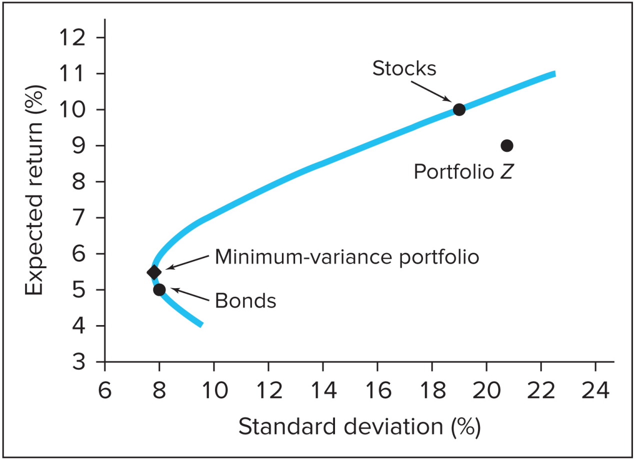
Portfolio Risk/Return Two Security Portfolio

- Amount of risk reduction depends critically on *correlation or covariances*
- Adding securities with correlations < 1 will result in risk reduction
- If risk is reduced by more than expected return, what happens to the return per unit of risk (the Sharpe ratio)?

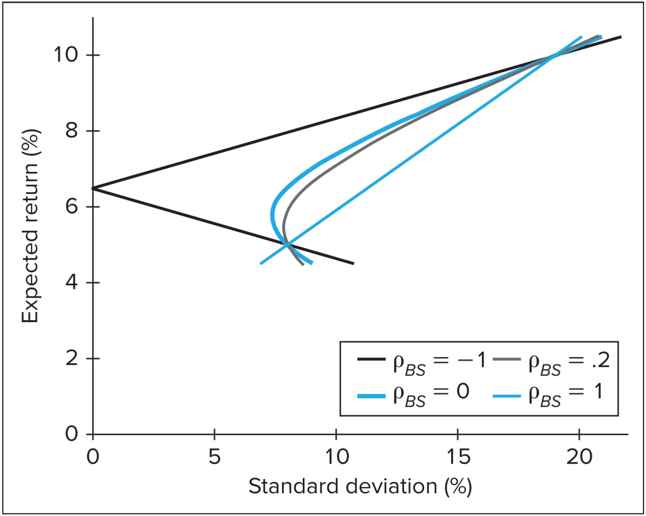
Asset Allocation (2 Risky Assets)

- Portfolio Opportunity Set: Graph of possible combinations of risk and expected return
- Consider the following example:
- Bonds: $E[r_B] = 5\%$; $\sigma_B = 8\%$
- Stocks: $E[r_S] = 10\%$; $\sigma_S = 19\%$
- Excel:

The Investment Opportunity Set with the Stock and Bond Funds (Assuming Correlation $\rho_{BS} = 0.2$)



The Investment Opportunity Sets for Bonds and Stocks with Various Correlation Coefficients



Minimum Variance Portfolio (MVP)

- Minimum Variance Portfolio (MVP)
  - Gives lowest risk portfolio
  - Formal optimization problem: Choose w_1 such that portfolio variance is lowest.
  - Solution: Set derivative (with respect to w_1) to 0

$$
\begin{aligned}
w_1 &= \frac{\sigma_2^2 - Cov(r_1, r_2)}{\sigma_1^2 + \sigma_2^2 - 2Cov(r_1, r_2)} \\
w_2 &= 1 - w_1
\end{aligned}
$$

- Diversification: For $\rho < +1$, minimum variance portfolio has less risk than either asset

Efficient and Dominated Portfolios

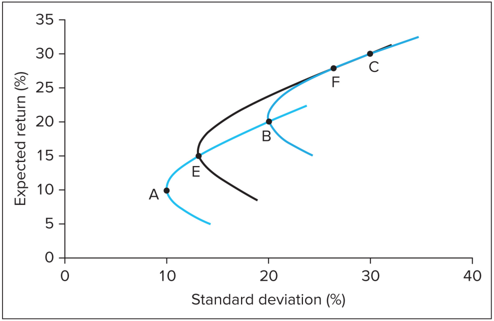
- Portfolios that lie below the minimum-variance portfolio in the figure (on the downward-sloping portion of the curve) are inefficient.
- Any such portfolio is dominated by the portfolio that lies directly above it on the upward-sloping portion of the curve since that portfolio has higher expected return and equal standard deviation
- Efficient Set: Portfolios that are not dominated
- The best choice among the portfolios on the upward-sloping portion of the curve is not as obvious, because in this region higher expected return is accompanied by greater risk

Extending Concepts to All Securities (Many Risky Assets)

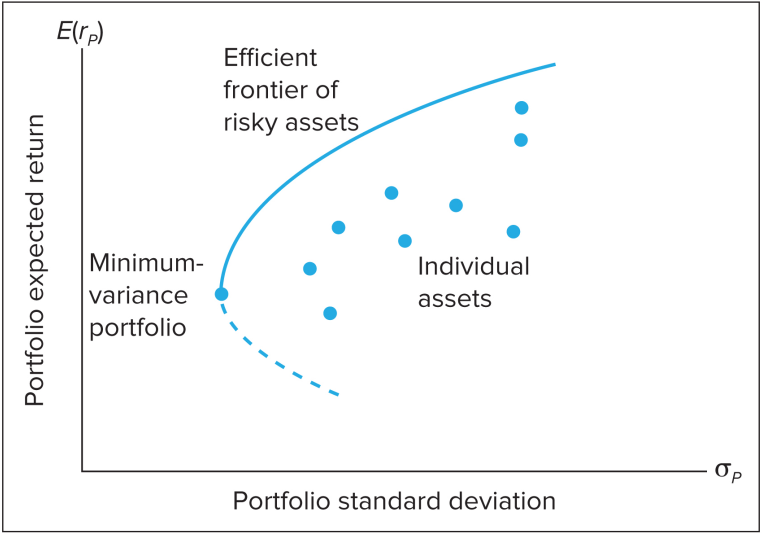
- Consider all possible combinations of securities, with all possible different weightings and keep track of combinations that provide more return for less risk or the least risk for a given level of return and graph the result
- The set of portfolios that provide the optimal trade-offs are described as the *efficient frontier*
- The efficient frontier portfolios are *dominant* or the best diversified possible combinations
- All investors should want a portfolio on the efficient frontier (... Until we add the riskless asset)

Asset Allocation with Many Risky Assets

- Find portfolios that minimize risk for given expected return (use computer)
  - Minimum-variance frontier: Looks like previous opportunity set with 2 assets
- Efficient frontier: Minimum variance portfolios that are not dominated (looks like previous efficient set)



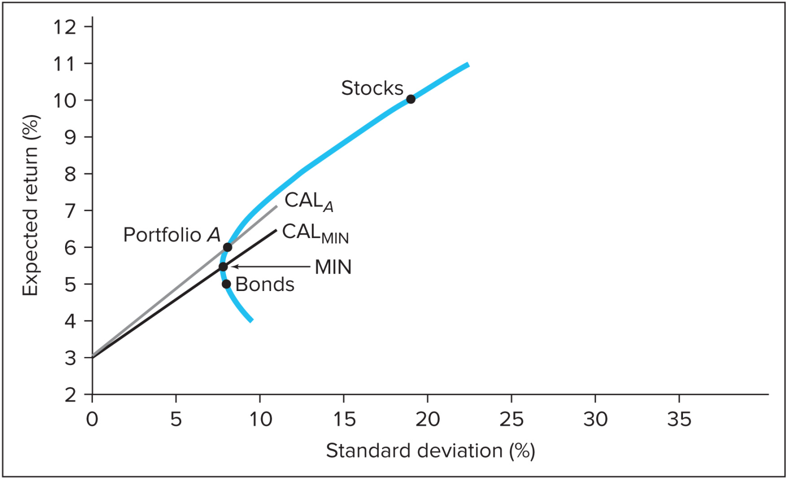
The Efficient Frontier of Risky Assets



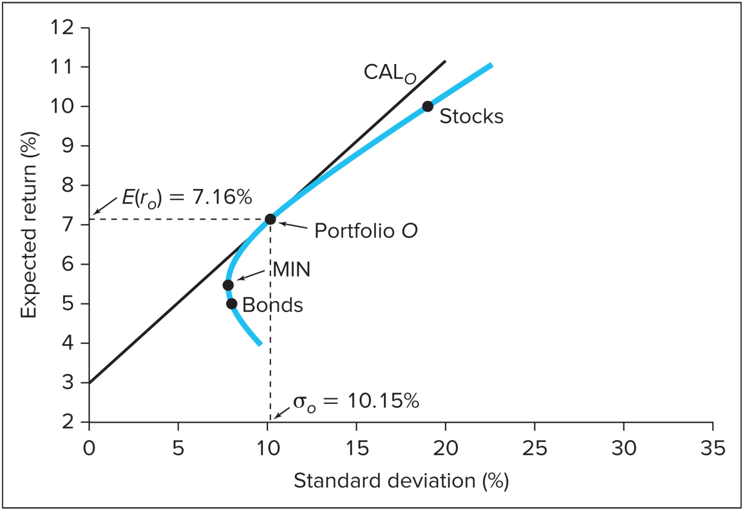
Asset Allocation with Many Risky Assets and a Risk-free Asset

- Combinations of riskless asset + any risky portfolio = Straight line (CAL)
- Each risky portfolio $\longrightarrow$ Different CAL
- Optimal risky portfolio $\longrightarrow$ Point of tangency between CAL & efficient frontier
    - Gives highest feasible reward-to-variability ratio (slope of CAL)
- Note: Optimal risky portfolio is independent of risk aversion!!! Portfolio manager offers same risky portfolio to all = separation property

The Opportunity set of Stocks, Bonds and a Risk-free Asset with two Capital Allocation Lines



The Optimal Capital Allocation Line with Stocks, Bonds and a Risk-free Asset



The Optimal Capital Allocation Line with Stocks, Bonds and a Risk-free Asset

- Formally: Find weights that result in highest slope of CAL

$$\max_{w_B} \frac{E[r_p - r_f]}{\sigma_p} \; s.t. \sum_i w_i = 1$$

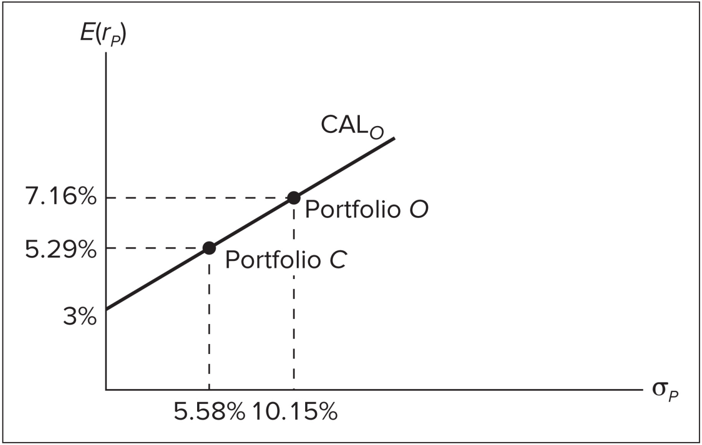
- Solution: w_B is given by the expression

$$\frac{(E[r_B] - r_f)\,\sigma_S^2 - (E[r_S] - r_f)\sigma_B\sigma_S\rho_{BS}}{(E[r_B] - r_f)\,\sigma_S^2 + (E[r_S] - r_f)\sigma_B^2 - (E[r_B] - r_f + E[r_S] - r_f)\sigma_B\sigma_S\rho_{BS}}$$

- $w_S = 1 - w_B$

Combining the Optimal Risky Asset with the Risk-free Asset

One possible choice for the preferred complete portfolio, C where the investor places 55% of wealth in portfolio O and 45% in Treasury bills



Summary: Asset Allocation with Many Risky Assets and a Risk-free Asset

- Specify risk-return characteristics of securities and calculate the portfolio opportunity set
  - Statistical task
- Find the optimal risky portfolio P (same for all investors)
  - Maximize reward-to-variability ratio
- Combine the optimal risky portfolio P and the riskless asset
  - This is the optimal complete portfolio C for the given level of risk-aversion

Discussion

- Mean-variance analysis is one of the crown jewels of finance theory (It got Harry Markowitz the Nobel Prize)
- But there are implementation problems
- Mean-variance analysis sometimes leads to large short positions in some assets. Is this reasonable/appropriate?
  - Solution: Constrain the weights to be positive

Underlying Reason for the Implementation Problems?

- Precision in estimating the inputs
  - What if you have an asset with interesting risk/return trade-off or correlation properties but these inputs are estimated imprecisely?
  - Also, large covariance matrices are hard to work with.
- Time variation in the inputs
  - For variances, historical averages are usually okay
  - Covariances are trickier and change over the business cycle. They also change in periods of market turmoil.
  - The hardest part is to estimate expected returns !!! Expected returns are not constant over time, so historical averages are not always helpful.

A Single-Index Stock Market

- *Index model* is a model that relates stock returns to returns on both a broad market index and firm-specific factors.

$$R_i = \beta_i R_M + e_i + \alpha_i$$

- R_i: Excess return of the firm's stock (stock return−risk-free rate)
- $\beta_i R_M$: Component of return due to movements in overall market
  - β_i : Sensitivity of security's returns to market factor
  - R_M: Excess return of the market index (market index return−risk-free rate)
- e_i: Component attributable to unexpected events relevant only to this security (firm-specific or residual risk)
- α_i: Stock's expected excess return if the market factor is neutral, i.e. if the market-index excess return is zero

Variance of the Excess Return of the Stock

$$
\begin{aligned}
\text{Variance}(R_i) &= \text{Variance}(\beta_i R_M + e_i + \alpha_i) \\
&= \text{Variance}(\beta_i R_M) + \text{Variance}(e_i) \\
&= \beta_i^2 \sigma_M^2 + \sigma^2(e_i) \\
&= \text{Systematic risk} + \text{ Firm-specific risk}
\end{aligned}
$$

The total variance of the rate of return of each security is a sum of two components:

1. The variance attributable to the uncertainty of the entire market. This variance depends on both the variance of R_M, denoted by σ_M^2, and the β of the stock on R_M.
2. The variance of the firm-specific return, e_i, which is independent of market performance.

Statistical and Graphical Representation of Single-Index Model

- $R_i = \beta_i R_M + e_i + \alpha_i$ may be interpreted as a single-variable regression equation of R_i on the market excess return R_M
- The regression line is called the Security Characteristic Line (SCL)
- *Security Characteristic Line* is a plot of a security's expected excess return over the risk-free rate as a function of the excess return on the market
- Slope of the SCL is the regression coefficient β and the intercept is the α for the security
- The algebraic representation of the regression line is

$$E(R_i|R_M) = \alpha_i + \beta_i R_M$$

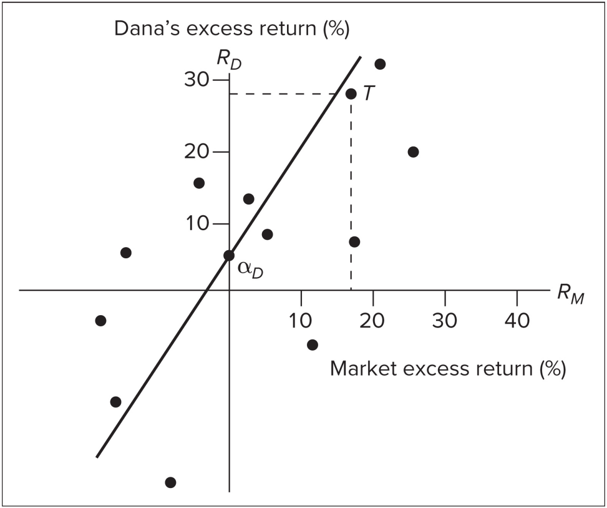
Relative Importance of Systematic Risk

- One way to measure the relative importance of systematic risk is to measure the ratio of systematic variance to total variance
- Ratio of systematic variance to total variance is the square of the correlation coefficient ρ between R_i and R_M

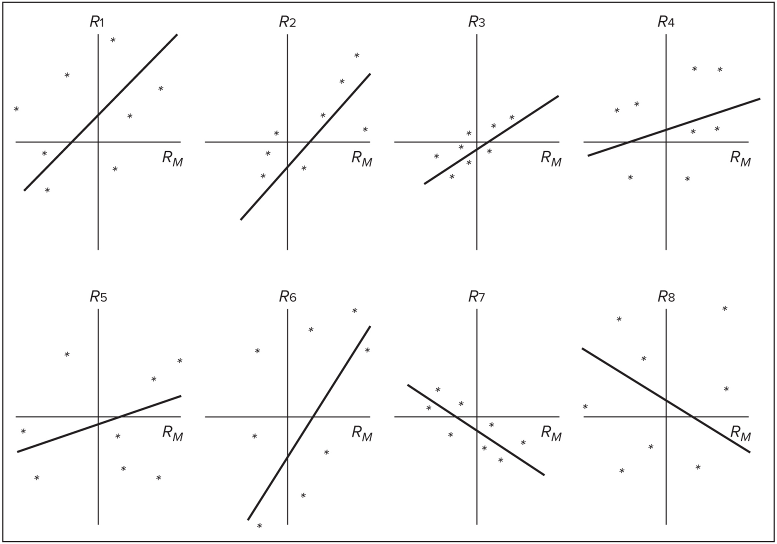
$$\rho^2 = \frac{\text{Systematic (or explained) variance}}{\text{Total variance}}$$
$$= \frac{\beta_i^2 \sigma_M^2}{\sigma_i^2} = \frac{\beta_i^2 \sigma_M^2}{\beta_i^2 \sigma_M^2 + \sigma_{e_i}^2}$$

- A large correlation coefficient (in absolute value terms) means systematic variance dominates the total variance; that is, firm-specific variance is relatively unimportant
- When the correlation coefficient is small (in absolute value terms), the market factor plays a relatively unimportant part in explaining the variance of the asset, and firm-specific factors dominate

Scatter Diagram for Dana Computer Corp.



Various Scatter Diagrams



Diversification in a Single-Index Security Market

- Imagine a portfolio P that is divided equally among securities whose returns follow the single-index model
- The β of the portfolio is a simple average of the individual security βs
- Hence, there are **no diversification effects on systematic risk** no matter how many securities are involved
- Intuition: The systematic component of each security return, $\beta_i R_M$, is driven by the market factor and therefore is perfectly correlated with the systematic part of any other security's return

Diversification in a Single-Index Security Market

- Consider a portfolio P of n securities with weights w_i $(\sum_{i=1}^{n} w_i = 1)$
- Nonsystematic risk of each security is $\sigma_{e_i}^2$
- Nonsystematic portion of portfolio P's return is

$$e_P = \sum_{i=1}^{n} w_i e_i$$

- Portfolio P's nonsystematic variance is

$$\sigma_{e_P}^2 = \sum_{i=1}^{n} w_i^2 \sigma_{e_i}^2$$

- The sum is far less than the average firm-specific variance of the stocks in the portfolio

Diversification in a Single-Index Security Market

- **The impact of nonsystematic risk becomes negligible as the number of securities grows** and the portfolio becomes more diversified
- The number of securities counts more than the size of their nonsystematic variance
- Sufficient diversification can virtually eliminate firm-specific risk. Only systematic risk (market risk) remains.
- **For diversified investors, the relevant risk measure for a security is the security's** β, since firms with higher β have greater sensitivity to market risk
- The systematic risk, $\beta^2\sigma_M^2$, will be determined by both the market volatility, σ_M^2, and the firm's β