Auctions or Negotiations?
A Theory of How Firms are Sold

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Abstract

Consider a situation where a firm faces multiple acquirers interested in buying it. There are two common ways to proceed with selling the firm. The target can either do bilateral negotiations or run an auction. Which of these would the target prefer? I address this question by adding an important source of buyer uncertainty to prevailing models of the takeover process, namely that in many takeover contests, buyers may not know how many other buyers are participating in the sale, if any. Since the seller chooses an auction or negotiations after observing the number of buyers, the seller's choice of mechanism can signal information to the buyers about how many other buyers there are, possibly changing the buyers' prior beliefs about the competition they face, and lowering the expected revenue. Due to this signaling effect, in equilibrium, the seller chooses an auction if the buyers' belief about competition is above a certain threshold and conducts negotiations otherwise, irrespective of the actual number of buyers. Thus, the information about the number of buyers does not benefit the seller. Empirical implications of the theory are (i) The revenue is higher if buyer valuations are less volatile; (ii) More risk-averse sellers choose auctions more often; (iii) The average transaction price in auctions is greater than that in negotiations if and only if the seller's risk-aversion is above a threshold. These predictions are consistent with empirical findings on takeover contests. Committing to a mechanism before seeing the number of buyers increases the seller's revenue.

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> ... a sealed-bid process enables you to create the perception of competition when there isn't any. As a senior investment banker once told me:
>
>> We have run many auctions where we have one bidder. We never let the bidder into the central room, so the one bidder thinks they are bidding against somebody else.
>
> —Guhan Subramanian, *Dealmaking: The New Strategy of Negotiauctions*

1 Introduction

Consider a situation where a firm faces multiple buyers interested in purchasing it. There are two ways to sell the firm. The seller can either run an auction or conduct exclusive bilateral negotiations. Which of these would the seller prefer? This paper develops a theory that answers this question and provides empirical implications of the theory.

Recent studies show that neither auctions nor negotiations are universally chosen, with firm sales almost equally split between the two.[1] Another feature of firm sales is that the process is often completed before the merger is publicly announced.[2] Very often, buyers sign confidentiality agreements with the seller and are uncertain about the number of other buyers taking part. The uncertainty is significant if the number of participating buyers is low, which is precisely the case in corporate takeovers.[3]

Motivated by these empirical observations, I modify the standard framework in mechanism design to one where the buyers take part in the sale process without knowing how many other buyers are participating. Each buyer's belief about the competition is captured by two parameters - the maximum number of buyers who may enter the sale process and the probability with which each enters. These are common knowledge to the buyers and the seller. In addition, the seller observes the actual number of buyers that participate in the process. For most of the analysis, I assume a common values setting where the value of synergies in the transaction is the same for all the buyers.

[1] For instance, among recent studies, Boone and Mulherin (2007) conclude that 51% of the firms in their sample were sold through auctions, while Aktas et al. (2010) find that 52% were auctions. They classify a transaction as an auction if multiple potential buyers are mentioned in the merger background section of the SEC filings, and as a negotiation when there is only one buyer mentioned.

[2] To understand the distinction between the public part of the sale which happens after the merger is announced and that which happens before, contrast the results obtained by Boone and Mulherin (2007) and Aktas et al. (2010) with those from prior studies. Earlier studies classified a sale an auction if there was more than one publicly announced bidder *after the takeover is publicly announced*. For example, Moeller et al. (2004) find that only 1.21% of this sample were public auctions with multiple bidders, while Betton et al. (2008) report that only around 3.4% were public auctions. Applying this criterion, one would conclude that few targets are sold in an auction. Hence, it is imperative to look at the private part of the takeover process before the public announcement of the merger to provide a better classification.

[3] Boone and Mulherin (2007) report that in the auctions, the average number of bidders was 1.57. In many of the auctions, only one buyer submitted the final bid even though many buyers were contacted. The median number of buyers which submitted a final bid was in fact 1. It is worth mentioning that they only consider listed targets. The identity and number of the bidders are likely to be even more uncertain if the targets are unlisted.

Hence, the setting I consider has two-sided information asymmetry. The buyers know their valuations, but the seller doesn't. However, the seller knows the number of buyers taking part in the sale process, which the buyers do not. The question is how far the information advantage of the seller helps him mitigate the usual information disadvantage about buyer valuations in any selling mechanism. In this aspect, the informed principal is similar to that in Maskin and Tirole (1990, 1992).

In my model, the key difference between an auction and a negotiation is as follows. In an auction, the outcome depends on the bids of all the buyers participating in the process. In contrast, in a negotiation between the seller and a buyer, the outcome of the negotiation is independent of any other buyer's actions. I model these mechanisms in this manner for two reasons. First, this lets me capture the exclusivity inherent in negotiations where the competitive pressure is across the table between the seller and the buyer, in contrast to an auction where the competitive pressure is between buyers who are on the same side of the table among buyers. Second, it ties my model to empirical studies on the sale of firms where a takeover is classified as an auction if multiple potential buyers are mentioned in the filings and a negotiation when there is only a single buyer mentioned.

I assume that the auction is a first-price auction with a reserve price equal to the standalone value of the firm. Modelling negotiations also requires a few assumptions.[4] I assume that a negotiation involves a take-it-or-leave-it offer made by the seller. If a buyer rejects an offer during negotiations, the seller moves on to the next buyer and never goes back to it. This makes the take-it-or-leave-it offer credible. The result of the negotiation thus depends only on whether an agreement is reached between the seller and the buyer involved in the negotiation. Unlike in an auction, a competing buyer's strategy will not affect the result of a negotiation.

The main findings are as follows. The seller does not benefit from the ability to observe the number of buyers before choosing the mechanism. The buyers make inferences about the total number of buyers from the mechanism chosen by the seller. Because of the buyer inference, the seller cannot increase his revenue by making the choice between auctions and negotiations dependent on the number of buyers. There are multiple equilibria. In the best case for the seller, the seller does does not benefit from observing the number of buyers and in the worst case, his revenue decreases when he observes the number of buyers.

Thus, in equilibrium, the seller does not employ his knowledge to increase his payoff. But if this is the case, why should the seller not commit to choose the same mechanism irrespective of the actual number of buyers he observes? I show that if the seller can commit ex ante to choosing an auction or a negotiation irrespective of the number of buyers, the seller can increase the expected revenue. In the equilibria with commitment, the seller chooses negotiations up to a threshold probability of entry and chooses auctions above the threshold.

The threshold probability above which auctions are chosen decreases as the seller becomes more risk-averse or the variance of buyer valuations increases. The expected transaction price conditional on the sale being a negotiation is higher than that conditional on an auction for low

[4] **?** point out that negotiations *"can take a variety of observationally indistinguishable forms... An observed single bidder sale could, for example, reflect a successful one-shot negotiation or a successful first stage in a sequential negotiation, among other possibilities."*

levels of the relative risk aversion of the seller. If the seller's risk aversion is high, the transaction price in auctions is higher than in negotiations. The volatility of the price conditional on auctions is higher than that in negotiations.

I contribute to various strands of research. The central question in the paper is the choice between negotiations and auctions. I add two elements, which are very relevant in the context of the sale of firms, to previous studies examining the choice between auctions and negotiations. First, I assume that the buyers do not know the number of other buyers participating, a realistic assumption to model corporate takeovers. Second, I adapt the usual framework used to model negotiations to capture the exclusivity inherent in the negotiations during the sale of firms.[5]

These two elements add to and modify some of the results obtained by previous theoretical models in the literature. For example, Bulow and Klemperer (1996) concludes that sellers almost always choose auctions since additional competition (i.e. the presence of an extra bidder in an auction) is preferable to a negotiation. However, since the "negotiation" in their analysis is actually an optimal auction and the "auction" a second price auction, their conclusion is actually a comparison of an optimal auction to a second price auction with an extra bidder. It is not surprising that in reality, empirical studies have repeatedly found that takeover auctions are much less prevalent than they suggest and negotiations much more common. Bulow and Klemperer (2009) study the choice between a simultaneous auction, and a sequential process in which potential buyers decide in turn whether to enter the bidding, which they call "negotiations". My model differs from theirs since I assume an exogenous participation probability independent of whether the seller chooses an auction or negotiations. This enables me to show that even if the probability of entry is the same in an auction and negotiations, sellers may prefer one mechanism to the other.

The second contribution is to the literature that characterises equilibria in games where the principal possesses information that the agent does not. The "principal" corresponds to the seller in my model and "agent" to the buyers. The seller has information on the number of buyers, which the buyers do not know. The informed principal is similar to that in Maskin and Tirole (1990, 1992). I give the principal all the bargaining power in choosing the mechanism as in Maskin and Tirole (1992). Also, like in their game, the agent's expected payoff depends on his interim beliefs about the principal's type, where the interim beliefs are obtained by updating the prior beliefs using the information conveyed by the prinicipal's contract proposal. My setting is more restrictive than theirs since the principal is just choosing between two specific mechanisms. In addition, the private information the principal has is of a very specific kind, about the number of agents. I find that in this setting, the information asymmetry can lead to equilibria where the seller is *in fact at a disadvantage* due to possessing information.

I also contribute to the research on auctions by studying the bidding strategies in a setting with an unknown number of bidders when the values are common. Previous studies which look at auctions with an unknown number of bidders, for example McAfee and McMillan (1987) and Harstad et al. (1990), derive bidding strategies and seller revenues if the bidders have independent values. I derive closed-form expressions for the bidding strategies and seller

[5]In many studies, the only difference between auctions and negotiations seem to be that auctions are simultaneous and negotiations sequential.

revenues in a case when the bidder valuations are not independent but (perfectly) correlated. In addition, unlike these studies which assume that the seller always chooses an auction, I show that the presence of unknown number of buyers can lead the seller to choose negotiations.

The empirical implications of my paper can be compared with the results found in empirical studies of takeovers, for instance Boone and Mulherin (2007) and Aktas et al. (2010). They find that sellers employ both auctions and negotiations frequently. This can be explained by the variation in the competition, risk aversion and number of possible buyers, both across industries and temporally, leading sellers to choose either mechanism depending on the parameter values. Mulherin and Womack (2015) find that private auctions exist even in the takeover market for REITs. The median number of bidders in their sample is one. Therefore, it is reasonable to suppose that the REIT takeover market is also characterized by uncertainty about the number of bidders. They report that larger firms choose to negotiate more often than smaller firms, possibly because the sellers of larger firms are less risk-averse. This is consistent with what my model predicts about the effect of seller's risk aversion on negotiations. In addition to being consistent with existing empirical research, I also provide directions for future empirical research.

I analyse the effect of the volatility of buyer valuations on negotiations, which goes some way towards explaining another puzzling phenomenon frequently reported in studies of takeovers. Many studies (Chang (1998), Moeller et al. (2004) and Faccio et al. (2006) to name a few) find that the acquisitions of private targets are associated with higher abnormal returns for the acquirer than acquisitions of public targets. Officer et al. (2008) also documents that the returns to acquirers are higher when a target is difficult to value irrespective of the target's public status. While public and private targets may differ on many dimensions (size, for example), one of the most significant differences between them is how volatile the target's value is. My analysis shows that more volatile (or private) targets can extract less of the surplus in negotiations. Hence, acquiring private targets through negotiations will benefit the acquirer.

The rest of the paper is organised as follows. Section 2 introduces the baseline model. I derive the optimal offers made by the seller in the sequential negotiations subgame, the buyers' bidding strategies in the auction subgame, and then use these results to describe the equilibria of the overall game. Section 3 suggests a way for the seller to increase the expected revenue by credibly committing ex ante to choose one of the two mechanisms. Section 4 considers extensions to the basic model where the buyer valuations are independent rather than correlated, and also considers the effect of the volatility of buyer valuations on the choice of the mechanism. Section 5 outlines empirical implications of the theory. Section 6 concludes.

2 The Baseline Model Setup

A risk-averse seller is trying to sell a firm of unknown value to a set of unknown buyers. The seller and buyer are both risk-averse, with CRRA utility function. The relative risk aversion of the seller is γ_s and that of the buyer γ_b with both γ_s and γ_b less than 1. The number of potential buyers participating in the sale of the firm is n. Here, n can be interpreted as a measure of asset-specificity of the firm being sold. The more specific the assets of the firm, the lesser is the number of potential buyers. Each buyer participates in the sale with probability p. Here, p measures how competitive the sales process is. I assume that p is exogenous. Both n and p are common knowledge. The seller also observes the number of buyers who actually participate, m. None of the buyers observe m. The seller chooses a mechanism after he observes m.

The value of the object to the i^{th} buyer is V_i ($i = 1, 2, \ldots m$ for $m > 0$).[6] Each buyer knows their value. The seller does not know any of the values, but knows the distribution of the values. I consider two cases:

(a) Common values - The values are the same. $V_i = V$ with $V \sim U[0, 1]$. This is the case I consider in the baseline model.

(b) Independent private values - The values are *iid.* $V_i \sim U[0, 1]$.

In negotiations, the seller proceeds sequentially. In each round of the negotiations, the seller picks one of the buyers randomly and makes a take-it-or-leave-it offer. If the offer is accepted, the negotiations end. If the offer is refused, the seller goes on to negotiate with the next buyer, if there is one. If not, the negotiations end. The seller does not go back to a buyer who previously refused an offer.

In an auction, each buyer submits a sealed bid. It is a first- price auction with a reserve price equal to the standalone value of the firm.

I solve the game as follows. I first solve the negotiations subgame and the auctions subgame separately. Then, I solve the supergame where the seller makes the choice of mechanism by backward induction.

2.1 The Negotiations Subgame

With negotiations, the buyer's strategy is to accept the seller's offer if it is less than the value V_i. The buyer's strategy is independent of the beliefs about m, the number of other buyers participating.

Let $W(k, a)$ denote the expected utility for the seller from optimal sequential negotiations with k buyers when $V \sim U[0, a]$. Denote $W(k, 1)$ by W_k for the sake of simplicity.

Lemma 1. *The seller's expected utility from negotiations with k buyers when $V \sim U[0, a]$ is homogenous of degree $1 - \gamma_s$ in a, that is, $W(k, a) = W_k a^{1-\gamma_s}$*

Proof. First consider the case when $k = 1$. If a price $x \leq a$ is offered, the offer is accepted when $V \geq x$, which happens with probability $1 - \frac{x}{a}$. The seller's utility conditional on acceptance is $\frac{x^{1-\gamma_s}}{1-\gamma_s}$. So, the optimal offer price x_1^* maximises

$$W(1, a) = \underset{x \in [0,a]}{Max} \left(\frac{x^{1-\gamma_s}}{1 - \gamma_s} (1 - \frac{x}{a}) \right) \tag{1}$$

Differentiating and solving for x_1^* gives

$$x_1^* = \frac{1 - \gamma_s}{2 - \gamma_s} a \tag{2}$$

[6]If $m = 0$, the firm is not sold.

Substituting in equation 1 gives

$$W(1,a) = \frac{(1-\gamma_s)^{1-\gamma_s}}{(2-\gamma_s)^{2-\gamma_s}} \frac{a^{1-\gamma_s}}{1-\gamma_s} \tag{3}$$

This is indeed of the form $W_1 a^{1-\gamma_s}$ where $W_1 = \left(\frac{1-\gamma_s}{2-\gamma_s}\right)^{1-\gamma_s} \frac{1}{1-\gamma_s}$, so lemma 1 holds for $m = 1$. Now, I prove that if lemma 1 is true for $k = l-1$, it is also true for $k = l$. Consider the first stage of the negotiation when there are l buyers. An offer of x in the first stage is accepted when $V_i \geq x$, which happens with probability $1 - \frac{x}{a}$. The seller's utility conditional on acceptance is $\frac{x^{1-\gamma_s}}{1-\gamma_s}$. If the offer is rejected, which happens with probabililty $\frac{x}{a}$, there are $l-1$ buyers left to negotiate with. We also know that conditional on the offer being rejected, $V \sim U[0,x]$. So, the expected seller utility conditional on rejection is obtained by substituting $k = l-1$ and $a = x$ in $W(k,a)$, that is, $W(l-1,x)$. This gives us the recursive equation

$$W(l,a) = \underset{x\in[0,a]}{Max} \left(\frac{x^{1-\gamma_s}}{1-\gamma_s}(1-\frac{x}{a}) + W(l-1,x)\frac{x}{a} \right) \tag{4}$$

By assumption, lemma 1 is true for $k = l-1$. In other words,$W(l-1,x) = V_{l-1}x^{1-\gamma_s}$. Substituting in equation 4, we have

$$W(l,a) = \underset{x\in[0,a]}{Max} \left(\frac{x^{1-\gamma_s}}{1-\gamma_s}(1-\frac{x}{a}) + W_{l-1}x^{1-\gamma_s}\frac{x}{a} \right) \tag{5}$$

Differentiating equation 5 w.r.t. x and solving for the optimal offer, we have

$$x(l,a) = \frac{1-\gamma_s}{2-\gamma_s} \frac{a}{1-(1-\gamma_s)W_{l-1}} \tag{6}$$

Substituting back in equation 5, we get that

$$W(l,a) = \left(\frac{1-\gamma_s}{2-\gamma_s} \frac{1}{1-(1-\gamma_s)W_{l-1}} \right)^{1-\gamma_s} \left(\frac{1}{2-\gamma_s} \right) \frac{a^{1-\gamma_s}}{1-\gamma_s} = W_l a^{1-\gamma_s} \tag{7}$$

This is indeed of the form $W_l a^{1-\gamma_s}$ where $W_l = \frac{(1-\gamma_s)^{1-\gamma_s}}{(2-\gamma_s)^{2-\gamma_s}} \left(\frac{1}{1-(1-\gamma_s)V_{k-1}} \right)^{1-\gamma_s} \frac{1}{1-\gamma_s}$. In other words, lemma 1 is also true for $k = l$. Since lemma 1 is true for $k = 1$, this means that by induction, lemma 1 is true for all k. $\square$

Theorem 1. *The expected utility for the seller from optimal sequential negotiations with k buyers when $V \sim U[0,1]$ is given by the recursion*

$$W_k = \frac{1}{(1-\gamma_s)^{\gamma_s}(2-\gamma_s)^{2-\gamma_s}} \left(\frac{1}{1-(1-\gamma_s)W_{k-1}} \right)^{1-\gamma_s} \tag{8}$$

with the boundary condition

$$W_0 = 0 \tag{9}$$

The offer in the i^{th} round of the negotiations is given by

$$x(k,i) = ((1-\gamma_s)(2-\gamma_s))^{\frac{i}{1-\gamma_s}} \prod_{j=0}^{i-1} W_{k-j}^{\frac{1}{1-\gamma_s}} \tag{10}$$

Proof. The proof follows from lemma 1, specifically equations 6 and 7, by substituting $a = 1$. □

Panel A of figure 1 plots the optimal offers in each round when the number of buyers is 4 as a function of the seller's coefficient of relative risk aversion. The graph shows that as the seller becomes more risk-averse, the offers in each round decrease. The intuition behind this is that as the risk-aversion increases, the seller increasingly prefers selling the firm even for a low price rather than being left with the firm as a result of the negotiations breaking down. So, the seller offers low prices to increase the probability of the transaction going through.

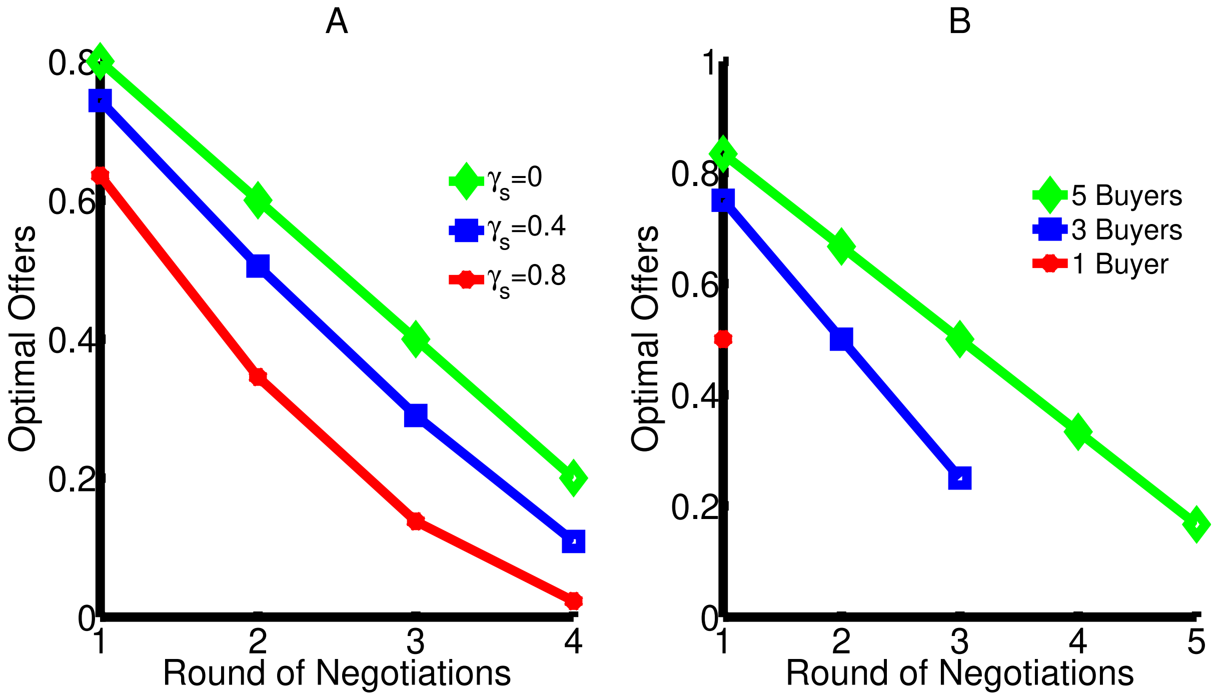

Figure 1: The optimal offers in each round of negotiations
Panel A shows the offers when the number of buyers is 4 for varying levels of the seller's coefficient of relative risk aversion. Panel B plots the offers made by a risk-neutral seller for varying number of buyers.

If the seller is risk-neutral, the expressions for the seller utility and the offers in each round simplify considerably.

Corollary 1. *If the seller is risk-neutral, the expected utility for the seller from optimal sequential negotiations with m buyers is given by*

$$W_m = \frac{1}{2}\frac{m}{m+1} \tag{11}$$

The offer in the i^{th} round is given by

$$x(m, i) = \frac{m+1-i}{m+1} \tag{12}$$

Proof. Risk-neutrality corresponds to the case where $\gamma_s = 0$. Substituting $\gamma_s = 0$ in equation 8, we obtain

$$V_m = \frac{1}{4(1 - V_{m-1})} \tag{13}$$

with the boundary condition $V_0 = 0$. The last step consists in showing that this simplifies to $\frac{1}{2}\frac{m}{m+1}$. The proof, by induction, is provided in Appendix A. $\square$

The i possible offers that the seller makes in the m rounds of the negotiations are equally spaced in the range of possible values $(0, 1)$, with a maximum offer of $\frac{m}{m+1}$ in the first round and a minimum offer of $\frac{1}{m+1}$ in the m^{th} round.

Panel B of figure 1 plots the offers made by a risk-neutral seller in each round of the negotiations. Separate graphs are plotted for different values of k, the realised number of buyers in the sale. The seller starts by offering a high price in the initial rounds of the negotiation. In the later rounds, he offers lower prices since he learns that the value is less than the amount offer rejected by the buyers in the earlier rounds. The information revelation in negotiations is thus gradual in contrast to an auction where it is a simultaneous process.[7]

The range of offers made by the seller increases as the number of buyers increases because the maximum offer increases and the minimum offer decreases. As the number of buyers becomes very large, the maximum approaches 1 and the minimum 0. In the limit, the seller is able to offer prices ranging from 0 to 1, the entire range of possible values. Figure 1 shows that even in negotiations, as the competition increases, the seller can extract more and more surplus.

Figure 2 shows how the expected revenue of the transaction varies with the number of buyers m for a risk-neutral seller. As $m \to \infty$, there is perfect learning, and the seller is able to realize all the value of synergies, which equal 0.5 in expectation. Thus, as the number of buyers increases, the informational rent extracted by them decreases. Conditional on m, the seller's expected revenue does not depend on n and p, the maximum number of bidders and their chance of participation.

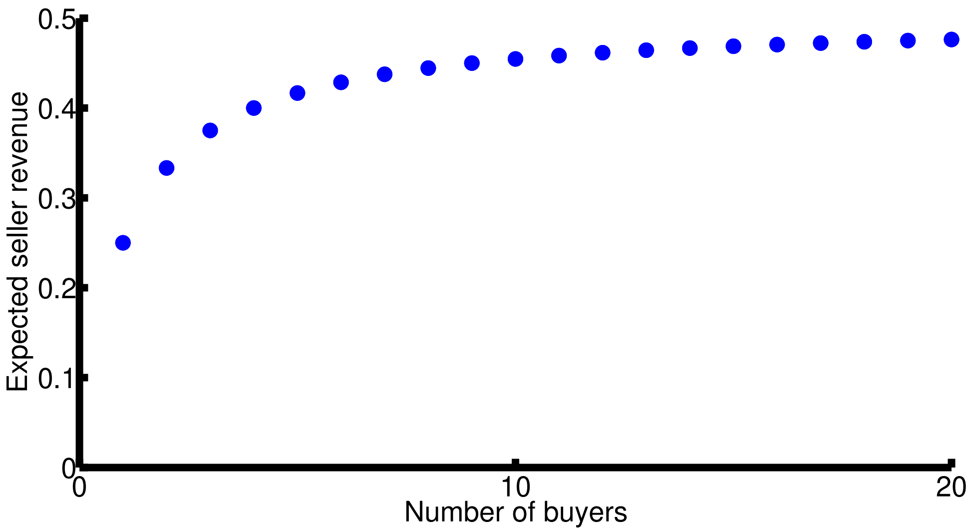

Figure 2: Expected seller revenue from negotiations

[7]In this model, gradual revelation of information does not make a difference since there is no discounting and nothing happens between two rounds of negotiation. But in a richer model where firms decide to enter based on the history of the previous round of negotiations, it does make a difference.

2.2 The Auctions Subgame

I construct a symmetric equilibrium where each bidder randomizes his bid x in the interval $[0, \bar{V}(n,p,\gamma_b)]$ for some maximum bid $\bar{V}(n,p,\gamma_b)$. The cumulative distribution function of x, $F(x)$, can have neither gaps nor atoms in this interval. A sketch of the reasoning is as follows.

If there is a gap, it is not optimal to bid just above the gap. Decreasing the bid by ϵ keeps the probability of winning constant, but makes the payment on winning lesser. So, the gap would decrease till it shrinks to zero.

There cannot be an atom because in the presence of an atom in the other bidders' CDFs, a bidder would not bid in a suitably chosen small interval below the atom. Shifting his bid from this interval to another just above the atom would decrease his payoff only by an order less than it increases his probability of winning. So, he would prefer to bid above the atom. But this leads to a gap which we have already proved is impossible. So, the bidding is continuous in the interval $[0, \bar{V}(n,p,\gamma_b)]$.

Lemma 2. *$\bar{V}(n,p,\gamma_b)$, the maximum bid, is $(1-(1-p)^{\frac{n-1}{1-\gamma_b}})V$*

Proof. Bidding 0 gives the buyer an expected payoff of $(1-p)^{n-1}\frac{V^{1-\gamma_b}}{1-\gamma_b}$. So, no buyer would bid above $(1-(1-p)^{\frac{n-1}{1-\gamma_b}})V$ since even if he wins the auction with certainty, his payoff will not be greater than $(1-p)^{n-1}\frac{V^{1-\gamma_b}}{1-\gamma_b}$ which he could have got by bidding 0. Hence, $\bar{V}(n,p,\gamma_b) \leq (1-(1-p)^{\frac{n-1}{1-\gamma_b}})V$. If $\bar{V}(n,p,\gamma_b)$ were less than $(1-(1-p)^{\frac{n-1}{1-\gamma_b}})V$, there is a profitable deviation for any buyer to bid at $\bar{V}(n,p,\gamma_b)+\epsilon$ with probability 1. □

Lemma 3. *The bidding is continuous in the interval $[0, \bar{V}(n,p,\gamma_b)]$. The cumulative distribution function of each buyer's bid x, $F(x)$, is given by*

$$F(x) = \frac{1-p}{p}\left(\left(\frac{V}{V-x}\right)^{\frac{1-\gamma_b}{n-1}} - 1\right) \tag{14}$$

Proof. To derive the functional form of $F(x)$, equate the expected utility from bidding any x in the interval to the expected utility from bidding 0. (These have to be equal to keep the buyer indifferent throughout the interval over which he mixes).

If a bidder bids x, he wins only if all other $n-1$ bidders bid less than x. For any bidder to bid less than x, he either does not enter, which happens with a probability $1-p$ or, if he enters, bid less than x, which happens with probability $pF(x)$. The probability that any one bidder bids less than x is $(1-p)+pF(x)$. The probability that all $n-1$ bidders bid less than x is $((1-p)+pF(x))^{n-1}$. So, if the bidder bids x, he wins with probability $((1-p)+pF(x))^{n-1}$ and gets utility $\frac{(V-x)^{1-\gamma_b}}{1-\gamma_b}$. The expected utility has to be equal to $(1-p)^{n-1}\frac{V^{1-\gamma_b}}{1-\gamma_b}$.

$$((1-p)+pF(x))^{n-1}\frac{(V-x)^{1-\gamma_b}}{1-\gamma_b} = (1-p)^{n-1}\frac{V^{1-\gamma_b}}{1-\gamma_b} \tag{15}$$

Simplifying this equation leads to the expression for $F(x)$ in equation 14. □

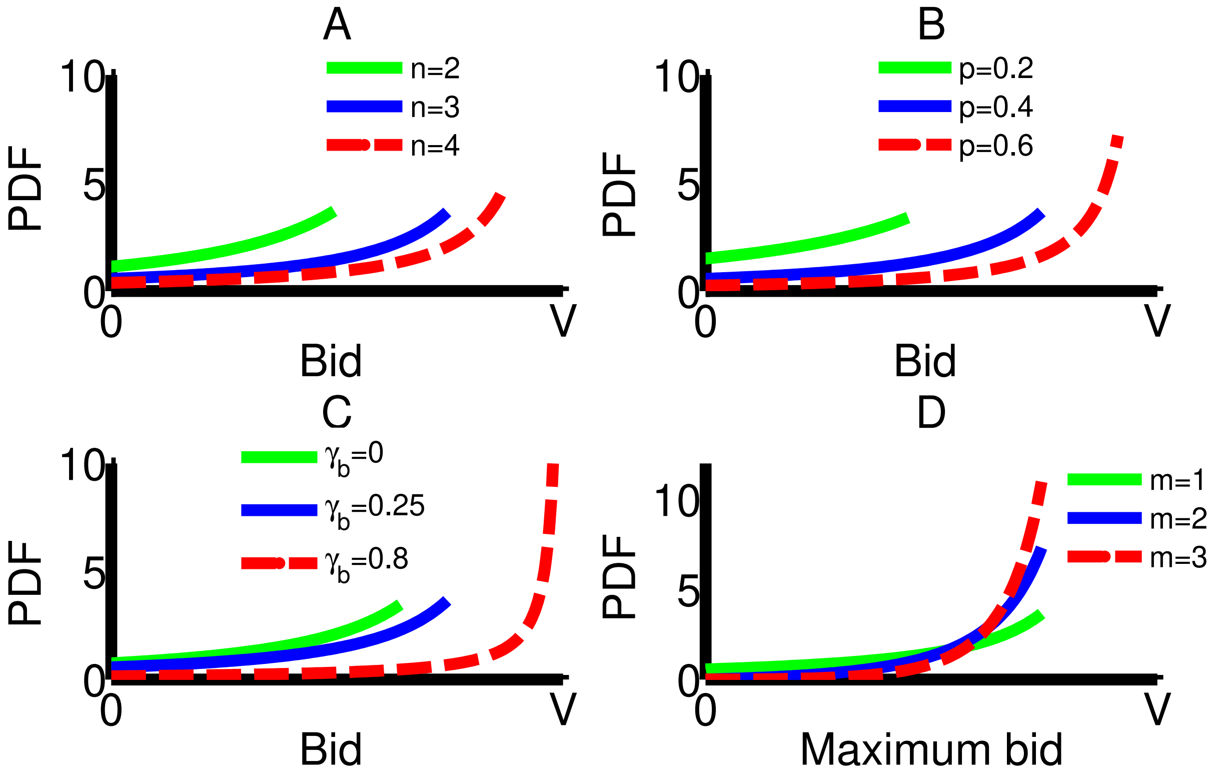

Figure 3: PDF of the bid for different values of n, p, γ_b and m
Panels A, B and C plot the the probability density function of each buyer's bid for different values of the maximum number of buyers n, probability of entry p, and buyer's relative risk-aversion coefficient γ_b. The default values are $n = 3$, $p = 0.4$ and $\gamma_b = 0.25$. In each of the panels A, B and C, a parameter is varied keeping the other 2 parameters at default levels. Panel D plots the PDF of the maximum bid when the actual number of buyers m varies from 1 to 3.

Theorem 2. *The distribution of the bids for a given value of the maximum number of buyers n, probability of entry p and the buyers' coefficient of risk aversion γ_b dominates the distribution of bids for a lower value of n, p or γ_b in the sense of first-order stochastic domiance*

Proof. When n, p or γ_b increase, the value of $F(x)$, given in equation 14, weakly decreases for all values of x. This is because the ratio $\frac{V}{V-x} \geq 1$ and the exponent $\frac{1-\gamma_b}{n-1}$ decreases when γ_b or n increases. So, the new distribution of bids dominates the old one in the sense of first- order stochastic dominance □

Panels A, B and C of figure 3 plot the probability density function of the bid for different values of n, p and γ_b. The default values are assumed to be $n = 3$, $p = 0.4$ and $\gamma_b = 0.25$. In each figure, two parameters are held fixed at the default values and the third varied. The graphs of the PDF clearly show the shift in the distribution as n, p and γ_b increase. Not only does the probability mass of the distribution shift to the right as first-order stochastic dominance implies, the interval of bidding extends to the right too. In other words, not only are higher bids more likely, but also amounts which were not bid earlier are now being bid.

The intuition for the shift in the distribution of bids is as follows. As n increases, each buyer considers it more possible that there are competing buyers present in the auction and shifts his bid higher to increase the probability of being the maximum bidder. The effect of an increase in p is very similar. In both cases, the competition increases, due to more number of potential entrants or higher entry probability of each.

The effect of an increase in the relative risk-aversion coefficient of the buyers is not due to an increase in perceived competition since the competition is held constant in each of the graphs in panel C of figure 3. The increase in γ_b shifts the distribution of bids because the buyers' incentives to take a chance by bidding low decreases as the buyers become more risk-averse, their utility from bidding high and winning the auction with a high probability becomes higher than that of bidding low and winning the auction with a low probability.

Lemma 4. *If the actual number of buyers is m, the cumulative distribution function of the highest bid is $(F(x))^m$ where $F(x)$ is the cumulative distribution function of each buyer's bid.*

Proof. The m bids are all independent random variables and are identically distributed. The probability that all m bids are less than x is the product of the individual cumulative probability distribution functions, $(F(x))^m$. □

Corollary 2. *The distribution of the highest bid for a given value of the maximum number of buyers n, probability of entry p, actual number of buyers participating m and the buyers' coefficient of risk aversion γ_b dominates the distribution of highest bid for a lower value of n, p or γ_b in the sense of first-order stochastic dominance.*

Proof. From lemma 4, the CDF of the maximum bid is $F(x)^m$. Since $F(x) \leq 1$, $F(x)^m$ decreases as m increases. In addition, $F(x)$ decreases as n, p or γ_b increase from theorem 2. So, $F(x)^m$ decreases as any of n, p , m or γ_b increase. □

The pdf of the maximum bid is plotted in panel D of figure 3. Notice the difference between an auction and a negotiations in this aspect. In negotiations, we have seen that the actual number of buyers m is a sufficient statistic for the seller's expected utility since, conditional on m, the seller's expected utility does not depend on n and p. In an auction, even conditional on knowing m, the expected utility of the seller depends on n and p. The m buyers participating bid based on their beliefs about the competition they face, which depends on n and p. I now derive the expected utility for the seller in an auction as a function of m, n and p.

Theorem 3. *$\Pi_V(m, n, p, \gamma_s, \gamma_b)$, the expected utility for the seller for a realisation of the value V when m buyers participate is given by*

$$\Pi_V(m,n,p,\gamma_s,\gamma_b) = (1-\gamma_b)m\left(\frac{1-p}{p}\right)^m \frac{V^{1-\gamma_s}}{1-\gamma_s}\left(\int_1^{\frac{1}{(1-p)^{\frac{1}{1-\gamma_b}}}} (y^{1-\gamma_b}-1)^{m-1}y^{-\gamma_b}(1-\frac{1}{y^{n-1}})^{1-\gamma_s}\mathrm{d}y\right) \tag{16}$$

Proof. The expected utility for the seller is the expected utility from the maximum bid. Since we know the probability density function of the maximum bid, the expected utility can be calculated. The proof is provided in Appendix E. □

Corollary 3. *The expected utility of the seller* $\Pi_V(m,n,p,\gamma_s,\gamma_b)$ *is increasing in* m,n,p *and* γ_b

Proof. This follows from corollary 2, which proves that the cumulative distribution function of the maximum bid undergoes a shift in the sense of first-order stochastic dominance as m,n,p or γ_b increase. Because of the shift, any decision maker with an increasing utility function would prefer the new distribution. So, the seller's expected utility is higher under the new distribution. □

It is useful to compare the expression in equation 16 to the maximum expected utility that the seller can extract from the sale. This corresponds to the case where the seller has perfect information and complete bargaining power. In that case, the seller would know V and would be able to extract the entire suplus due to the bargaining power. The seller's utility conditional on V would be $\frac{V^{1-\gamma_s}}{1-\gamma_s}$.

Corollary 4. *The expected utility of the seller* $\Pi_V(m,n,p,\gamma_s,\gamma_b)$ *is a fraction* ϕ *ofthe benchmark case where the seller has perfect information and complete bargaining power. The fraction* ϕ *is independent of the realised value* V.

Proof. The expected utility of the seller $\Pi_V(m,n,p,\gamma_s,\gamma_b)$ is is the benchmark $\frac{V^{1-\gamma_s}}{1-\gamma_s}$ multiplied by the term

$$\phi = (1-\gamma_b)m\left(\frac{1-p}{p}\right)^m \left(\int_1^{\frac{1}{(1-p)^{\frac{1}{1-\gamma_b}}}} (y^{1-\gamma_b}-1)^{m-1}y^{-\gamma_b}(1-\frac{1}{y^{n-1}})^{1-\gamma_s}\mathrm{d}y \right)$$

This term is independent of V □

Corollary 5. *The unconditional expected utility of the seller* $\Pi(m,n,p,\gamma_s,\gamma_b)$ *is increasing in* m,n,p *and* γ_b

Proof. Since $\Pi_V(m,n,p,\gamma_s,\gamma_b)$ is increasing in m,n,p and γ_b for any value of V, the unconditional expectation is also increasing in m,n,p and γ_b

□

Theorem 4. $\Pi(m,n,p,\gamma_s,\gamma_b)$, *the expected utility for the seller when* m *buyers participate is given by*

$$\Pi(m,n,p,\gamma_s,\gamma_b) = \frac{1}{(2-\gamma_s)(1-\gamma_s)}\phi \tag{17}$$

Proof. This follows from $V \sim U[0,1]$, so the

$$\mathbb{E}_V\left[\frac{V^{1-\gamma_s}}{1-\gamma_s}\right] = \int_0^1 \frac{\mathrm{V}^{1-\gamma_s}}{1-\gamma_s}\mathrm{d}V = \frac{1}{(2-\gamma_s)(1-\gamma_s)}$$

□

Theorem 5. *$\Pi(m, n, p, \gamma_s, \gamma_b)$ is dependent on the distribution of the value of synergies only through the certainty equivalent of the distribution. Consequently, any two distributions which have the same certainty equivalent give the seller the same expected utility.*

Proof. The only dependence of $\Pi(m, n, p, \gamma_s, \gamma_b)$ on the distribution of V is through the expression $\mathbb{E}_V\left[\frac{V^{1-\gamma_s}}{1-\gamma_s}\right]$ The certainty equivalent of the distribution is defined as the constant value c which satisfies

$$\frac{c^{1-\gamma_s}}{1-\gamma_s} = \mathbb{E}_V\left[\frac{V^{1-\gamma_s}}{1-\gamma_s}\right]$$

So, $\Pi(m, n, p, \gamma_s, \gamma_b)$ can be rewritten as

$$\Pi(m, n, p, \gamma_s, \gamma_b) = \phi\frac{c^{1-\gamma_s}}{1-\gamma_s}$$

which is the same for any distributions of the value of synergies which have the same certainty equivalent □

Corollary 6. *If the seller is risk-neutral, any two distributions of the value of synergies which have the same mean give the seller the same expected utility.*

Proof. If the seller is risk-neutral, the certainty equivalent is the same as the expected value. The proof then follows from theorem 5. □

Corollary 7. *The expected revenue from the auction $R(m, n, p, \gamma_b)$ is given by*

$$R(m, n, p, \gamma_b) = (1-\gamma_b)m\left(\frac{1-p}{p}\right)^m \frac{1}{2}\left(\int_1^{\frac{1}{(1-p)^{\frac{1}{1-\gamma_b}}}} (y^{1-\gamma_b}-1)^{m-1}y^{-\gamma_b}(1-\frac{1}{y^{n-1}})\mathrm{d}y\right) \tag{18}$$

Proof. The expected revenue is obtained by setting $\gamma_s = 0$ in equation 17 □

How does the expected seller utility change with n, the maximum number of buyers, m, the number of buyers in play and p, the belief each buyer has about the latent competition? Figure 4 plots the expected seller utility for different values of n, m and p when the buyers and the seller are risk-neutral. Panel A plots the expected revenue against p for $n = 2$ and $m = 1, 2$ and Panel B for $n = 3$ and $m = 1, 2, 3$. (I derive closed form expressions for these cases in Appendix C).

Figure 4 shows that the expected revenue is increasing in the three parameters n, m and p when the other two are held constant. As one would expect, the seller revenue increases when the perception of competition among the buyers is high (n and p) and also when the number of buyers that participate (m) is high. The expected revenue can be concave, linear or convex in p for different combinations of m and n.

However, for all values of m and n, as p approaches 1, the expected revenue approaches $\frac{1}{2}$, the expected value of synergies. This is because when $p \to 1$, each buyer is virtually certain that the other will enter. So, both bid V and the seller is able to extract all the surplus.

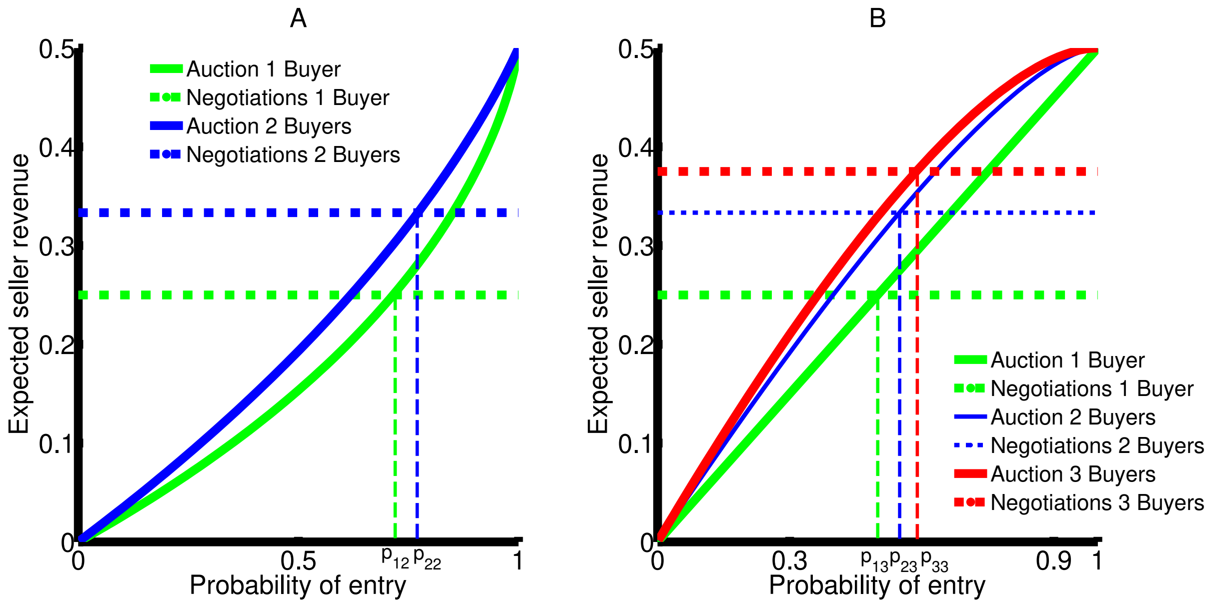

Figure 4: Expected seller revenue from an auction and negotiations for $n = 2$ and $n = 3$ The figure shows the expected revenue from an auction and negotiations as a function of the probability of entry p and the actual number of buyers participating m. Buyers are assumed to be risk-neutral. Panel A is for a maximum of 2 buyers, that is $n = 2$, and Panel B for a maximum of 3 buyers, that is $n = 3$. The dotted lines show the revenue from negotiations, the solid curves those from auctions. The probability at which the expected revenue from the auction equals that from negotiations is labelled p_{mn}.

On the other extreme, as p approaches 0, each buyer is almost sure that he is the only buyer and bids 0. The seller suffers from the perceived lack of competition and his expected payoff approaches 0.

2.3 Seller's Choice of Mechanism in the Supergame

So far, I have derived expressions for the expected seller utility in the negotiations and auction subgames. I now consider the question of whether the seller would choose auctions or negotiations in the supergame, which would determine which of these subgames occurs.

Since the seller observes m before he chooses the mechanism, the choice of the mechanism can depend on n, m and p. The seller can also employ a mixed strategy, which involves randomising between an auction or negotiations with different probabilities based on the number of buyers. Figure 5 depicts the relevant part of the game tree for $n = 2$ and m at least 1.

I begin the analysis by assuming that the seller and the buyers are risk-neutral. I relax this assumption later, and show that the conclusions remain broadly the same even when the buyers and seller are risk-averse.

The solution concept I use is perfect Bayesian equilibrium. In any perfect Bayesian equilibrium, the following twin requirements have to be satisfied-

- The strategies have to be sequentially rational given the beliefs of the players and
- The beliefs in any information set which is on the equilibrium path have to be consistent

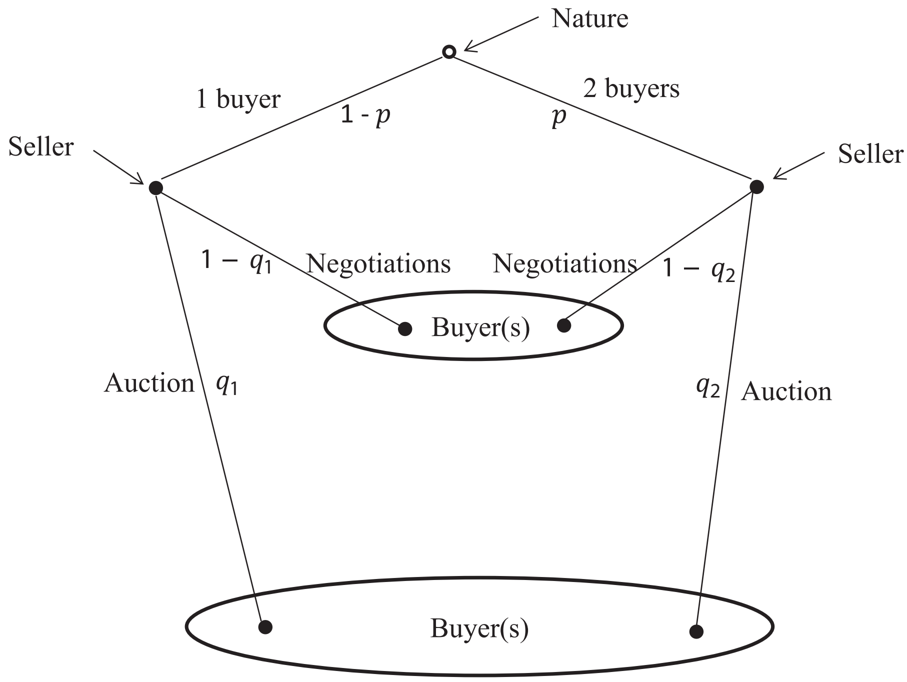

Figure 5: The evolution of the supergame
Nature moves first and reveals the number of buyers. The seller moves next after seeing the number of buyers. The seller randomizes between an auction and negotiations, choosing an auction with probability q_1 when there is 1 buyer and with probability q_2 when there are two buyers. The information sets of the buyers are not singleton sets since the buyers do not see whether there is another buyer or not.

with the strategies chosen by the players.

In other words, the buyers update their prior belief about there being another buyer based on the strategy of the seller. It is useful to illustrate this with an example.

Say there are 2 possible buyers, and each buyer's prior probability that there is another buyer participating is p. Say the seller's strategy is to choose negotiations when there is one buyer ($m = 1$) and auctions when there are two buyers ($m = 2$). If a buyer sees an auction, he will update the probability that there is another buyer from the prior of p to the posterior of 1. Similarly, conditional on seeing a negotiation, a buyer's posterior belief that there is another buyer is 0 and not p.

2.3.1 A Maximum of 2 Buyers ($n = 2$)

I start with the case of $n = 2$. Panel A of figure 4 shows the revenues from auction and negotiation for $n = 2$ and $m = 1, 2$ as a function of p in the same graph for convenience.

The expected revenue from negotiations is independent of p while that with auctions increases with p. Define p_{12} as the prior probability at which the expected revenue from the auction equals that from negotiations for $m = 1$ and p_{22} as the prior probability at which the expected revenue from the auction equals that from negotiations for $m = 2$. As can be seen from panel A of figure 4, $p_{12} < p_{22}$.

If the seller chooses to negotiate, the posterior beliefs of the buyer are not relevant to the seller revenue since it is a take-it-or-leave-it offer. So, the beliefs play no part in the sequential rationality of the seller.

However, if the seller chooses an auction, the expected revenue will depend on the prevailing belief that there are two buyers given that he has chosen an auction. Denote this belief by p^*. Let the probability of the seller choosing an auction when he sees one buyer be q_1 and when he sees two buyers be q_2. The belief p^* is given by a Bayesian updating of the prior belief p as per the formula

$$p^* = \frac{pq_2}{(1-p)q_1 + pq_2} \text{ if at least one of } q_1 \text{ or } q_2 \text{ is not zero} \tag{19}$$

$$p^* \in [0,1] \text{ if both } q_1 \text{ and } q_2 \text{ are zero} \tag{20}$$

where equation 20 follows from the fact that if both q_1 and q_2 are zero, the information set corresponding to auctions is reached only with zero probability. So, we are free to assign any belief to it since it is off the equilibrium path, *provided that given* p^*, $q_1 = q_2 = 0$ *is sequentially rational for the seller.*

Theorem 6. ***The equilibria for*** $n = 2$

If $p < p_{22}$, *that is, if the probability of each buyer participating is low, the unique equilibrium is that the seller chooses negotiations independent of the actual number of buyers in play. If the buyers see an auction, which happens off the equilibrium path, they believe that there is only one buyer.*

If $p \geq p_{22}$, *that is, if the probability of each buyer participating is high, there are multiple equilibria which fall into one of three categories*

1. *The seller chooses negotiations independent of the actual number of buyers in play. If the buyers see an auction, which happens off the equilibrium path, they believe that there is only one buyer.*

2. *The seller chooses an auction independent of the actual number of buyers in play*

3. *The seller chooses an auction if there is one buyer in play and mixes between an auction and negotiations if there are two buyers in play. The probability of choosing an auction when there are two buyers,* q_2, *is given by the solution to* $p_{22} = \frac{pq_2}{1-p+pq_2}$

In equilibria 1 and 2, the buyer's posterior belief that there is another buyer conditional on seeing an auction, p^*, *is the same as* p, *the prior belief, since the buyers learn nothing about* m *from the seller's choice. In equilibrium 3,* $p^* = p_{22}$, *since the buyers update their prior probability* p *to reflect the seller's choice. In equilibrium 3, the posterior belief that there is another buyer on seeing a negotiation is 1.*

Proof. See Appendix D □

Theorem 7. *When $p \geq p_{22}$, the expected revenue of the seller is highest in the "always-auction" pure strategy equilibria.*

Proof. The ex-ante belief of there being another buyer is p. If auctions are always chosen irrespective of the actual number of buyers m, the posterior belief of there being another buyer, p^* is also p, since there is no Bayesian updating. If $p^* > p_{22}$, from figure 4, the seller revenue on choosing auctions is higher than from choosing negotiations even when $m = 2$.

In the mixing equilibria, the seller is mixing between auctions and negotiations when $m = 2$. So, the seller is indifferent between the payoffs from choosing either. So, the two payoffs must be equal, which implies that the seller revenue from choosing auctions is the same as that from negotiations. Hence, the payoff from the mixing equilibria must be less than that from the "always-auction" pure strategy equilibria when $m = 2$.

When $m = 1$, the seller chooses auctions in both equilibria and gets the same payoff.

Hence, the overall payoff must be greater in the "always-auction" pure strategy equilibria □

Theorem 8. *In any pure strategy equilibria, the seller's choice of an auction or negotiations depends only on the ex-ante probability of entry and is independent of actual number of buyers.*

Proof. This follows from the fact that in one pure strategy equilibrium, the seller always chooses auctions and in the other, the seller always chooses negotiations. □

2.3.2 A Maximum of 3 Buyers ($n = 3$)

The expected revenue from negotiations is independent of p while that with auctions increases with p. Define p_{13} as the prior probability at which the expected revenue from the auction equals that from negotiations for $m = 1$, p_{23} as the prior probability at which the expected revenue from the auction equals that from negotiations for $m = 2$ and p_{33} as the prior probability at which the expected revenue from the auction equals that from negotiations for $m = 3$. As can be seen from Panel B of figure 4, $p_{13} < p_{23} < p_{33}$.

The expected revenue in any equilibrium can depend on the prevailing equilibrium beliefs that there are 1, 2 or 3 buyers. While the ex-ante probabilities each buyer assigns to there being 0, 1 or 2 other buyers can be characterised in terms of the entry probability p,[8] the posterior probabilities depend on the probability that the seller uses to randomise between an auction and a negotiation. They cannot be represented by a single parameter p.

Theorem 9 gives the expected seller utility when the maximum number of buyers is n for a general probability vector $(p_1, p_2, \ldots, p_n)$, $\sum_{i=1}^{n} p_i = 1$ and $p_i \geq 0$, where p_i denotes the equilibrium probability each of the m participating buyers assigns to there being a total of i buyers in the auction.

[8] These are the binomial probabilities $(1-p)^2$, $2p(1-p)$ and p^2 respectively.

Theorem 9. *Let n be the maximum number of buyers. The expected utility of the seller for a given V when there are m actual buyers is given by*

$$\Pi_V(m,n,p,\gamma_s,\gamma_b) = m\frac{V^{1-\gamma_s}}{1-\gamma_s}\int_0^1 \left[1-\left(\frac{p_1}{p_1+p_2y+p_3y^2+\cdots+p_ny^n}\right)^{\frac{1}{1-\gamma_b}}\right]^{1-\gamma_s} y^{m-1}\mathrm{d}y$$

Proof. See Appendix E □

Corollary 8. *When the seller and buyers are risk-neutral, the expected utility for the seller (which is also the expected revenue) is given by*

$$R(m,n,p) = \frac{1}{2}\left[1-mp_1\int_0^1 y^{m-1}\left(\frac{1}{p_1+p_2y+p_3y^2+\cdots+p_ny^n}\right)\mathrm{d}y\right]$$

Proof. The result follows from setting $\gamma_s = \gamma_b = 0$ in theorem 9 and then computing an expectation across the probability distribution of V □

Now that we have the expression for seller revenues for negotiations and auctions including in equilibria that feature mixing, we can characterise the equilibria for $n = 3$.

Theorem 10. ***The equilibria for $n = 3$***

If $p < p_{33}$, that is, if the probability of each buyer participating is low, the unique equilibrium is that the seller chooses to negotiate independent of the actual number of buyers in play. If the buyers see an auction, which happens off the equilibrium path, they believe that there is only one buyer.

If $p \geq p_{33}$, that is, if the probability of each buyer participating is high, there are multiple equilibria which fall into one of three categories

1. *The seller chooses negotiations independent of the actual number of buyers in play. If the buyers see an auction, which happens off the equilibrium path, they believe that there is only one buyer.*
2. *The seller chooses an auction independent of the actual number of buyers in play.*
3. *The seller chooses an auction if there is one or two buyers in play and mixes between an auction and negotiations if there are three buyers in play. The probability of choosing an auction when there are three buyers, q_3, is unique for any given p .*

Proof. The proof consists of numerically checking for equilbria which obey both sequential rationality and the consistency conditions for belief in a PBE for various values of p.

A sketch of the proof is as follows. I divide p into the intervals $[0, p_{13}]$, $[p_{13}, p_{23}]$, $[p_{23}, p_{33}]$ and $[p_{33}, 1]$. For each interval, I rule out candidate equilibria which violate either sequential rationality or consistency of beliefs. For example, if p is in the range $[0, p_{13}]$ and the seller chooses negotiations when $m = 1$, he must choose negotiations for $m = 2$ and $m = 3$ as well due to sequential rationality. In other words, all equilibria where he chooses negotiations for $m = 1$ and an auction when $m = 2$ or $m = 3$ can be ruled out. I repeat this process for all the sub-intervals of p till I am left with the equilibria listed. □

When $p \geq p_{33}$, the expected revenue of the seller is highest in the "always-auction" pure strategy equilibria. Also, in any pure strategy equilibria, the seller's choice of an auction or negotiations depends only on the ex-ante probability of entry and is independent of actual number of buyers.

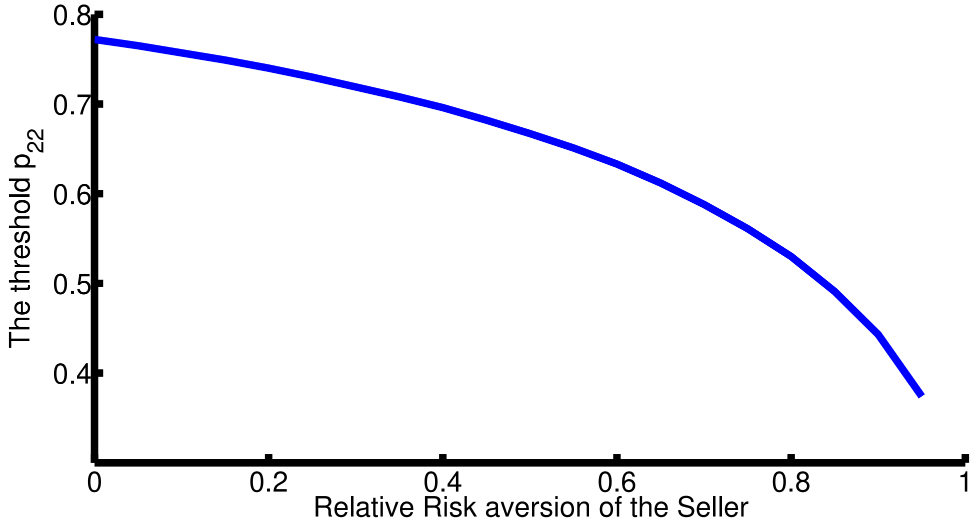

Figure 6: The threshold probability for auctions, p_{22} as a function of the seller's relative risk aversion γ_s

Figure 6 shows how the threshold probability p_{22} changes as the seller's risk-aversion increases.[9] As the seller becomes more risk-averse, the threshold probability above which he chooses auctions decreases. The intuition behind this is that the offers made by the seller during negotiations decreases as the risk-aversion increases. In auctions, the buyers' bid depend on their risk aversion, not the seller's, since they are competing with each other and not the seller. Hence, an increase in the seller's risk aversion has no effect on the bidding behaviour of the buyers in the auction. So, the effect of the risk-aversion on the seller's utility is greater in negotiations than in auctions.

3 Committing Ex Ante to a Particular Mechanism Irrespective of the Actual Number of Buyers

The analysis till now shows that the seller does not benefit from being able to see the actual number of buyers m before he chooses the mechanism. In the revenue-maximising equilibria, the seller's choice depends only on the buyers' belief about the competition (n and p), not the actual competition (m) that he is able to observe. In fact, the ability to observe the number of buyers *decreases* the seller's revenue in the mixed-strategy equilibria. Thus, the buyer inference about the actual competition from the seller's choice makes the information advantage of observing m useless and in fact worsens the seller's situation.

I next examine whether the seller can benefit from committing ex ante to choosing the same mechanism irrespective of how many buyers the seller sees. If the seller does so, his choice

[9] The buyer is assumed to be risk-neutral.

of the mechanism will reveal no extra information to the buyers about their competition since the choice is made before the seller observes the number of buyers. In this case, the expected revenue from auctions and negotiations depend only on n and p but is independent of m.[10]

3.1 Expected Revenue from Ex Ante Commitment

Auction We know the expression for $R(m, n)$ the expected revenue for each value of m when the maximum number of buyers is n. To calculate the expected revenue ex-ante, we need to multiply this by the probability that m buyers participate, which is just the binomial probability ${}^nC_m p^m(1-p)^{n-m}$

$$\text{Expected Revenue from an auction} = \sum_{m=1}^{n} ({}^nC_m)p^m(1-p)^{n-m}R(m,n)$$

Since this is independent of m, we drop the suffix m and denote this by R_n.

Theorem 11. *R_n, the expected revenue from committing to an auction ex ante when there is a maximum of n buyers, is given by*

$$R_n = \frac{1}{2}\left(1-(1+(n-1)p)(1-p)^{n-1}\right) \tag{21}$$

Proof. See Appendix F for the proof □

Negotiations

$$\text{Expected Revenue from negotiations} = \frac{1}{2}\sum_{m=0}^{n} ({}^nC_m)p^m(1-p)^{n-m}\frac{m}{m+1} \tag{22}$$

3.2 The benefits of commitment for $n = 2$ and $n = 3$

I illustrate the benefits of commitment for $n = 2$ and $n = 3$.

First, consider $n = 2$. Substituting in equations 21 and 22 above gives

$$\begin{aligned}
\text{Expected revenue from an auction} &= \frac{1}{2}\left(1-(1+p)(1-p)\right) \\
&= \frac{1}{2}p^2 \\
\text{Expected revenue from negotiations} &= \frac{1}{2}\left(2p(1-p)\frac{1}{2}+p^2\frac{2}{3}\right) \\
&= \frac{1}{2}p(1-\frac{p}{3})
\end{aligned}$$

The revenue from the two mechanisms is plotted against p in Panel A of Figure 7. Though the revenue from both auctions and negotiations increases as p increases, the graph exhibits single crossing. Hence, the seller chooses an auction if p is above a threshold p_2, and a negotiation otherwise. The point of indifference p_2 is the solution to the equation

$$\frac{1}{2}p_2^2 = \frac{1}{2}p_2(1-\frac{p_2}{3}) \implies p_2 = \frac{3}{4}$$

[10] In this section, I assume that the sellers and buyers are risk-neutral, but the results for risk-averse sellers and buyers are similar.

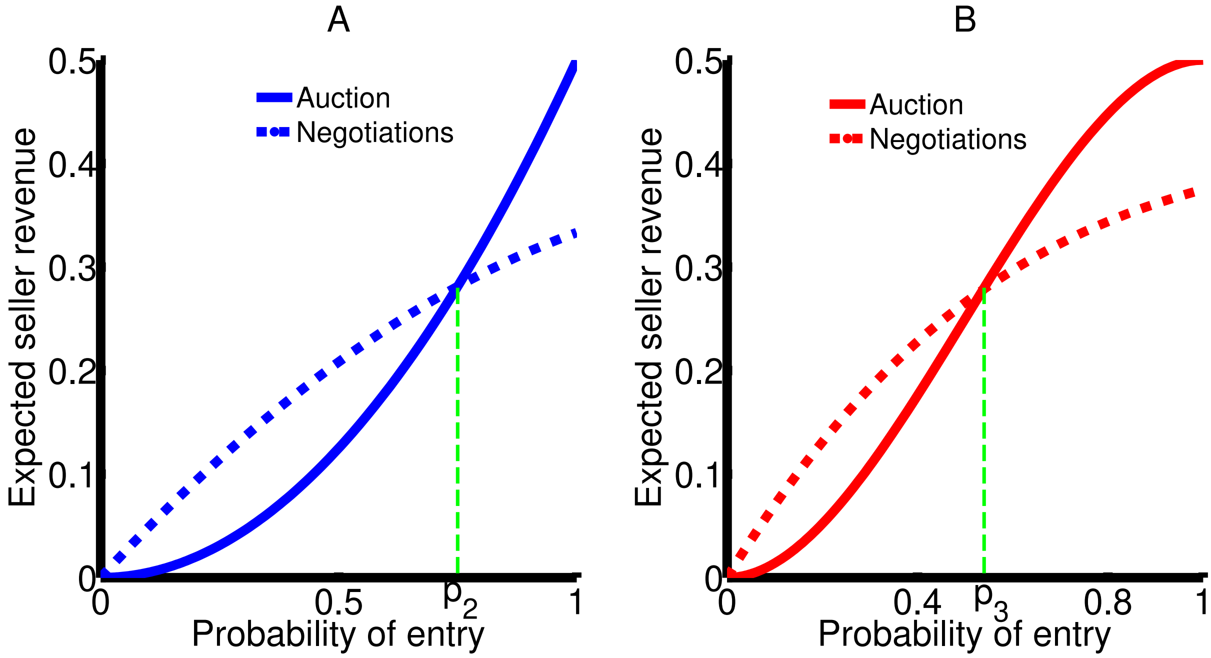

Figure 7: Expected revenue with commitment
The figure plots the expected seller revenue from an auction or from negotiations if the seller can credibly commit ex ante that he will choose the same mechanism irrespective of m. Panel A is for $n = 2$ and Panel B for $n = 3$. The probability at which the revenues from the auction and negotiations are the same is labelled p_n.

The seller benefits from the ability to commit since the threshold probability of entry above which he chooses an auction with commitment, p_2, is less than the threshold above which he chooses auctions without commitment, p_{22}. The seller is now able to employ auctions even at a lower level of competition, which the buyer inference prevented it from doing without commitment. Thus, committing to a mechanism ex-ante increases the seller's revenue since he can choose auctions in a greater range.

Next, consider $n = 3$. Substituting in equations 21 and 22 above gives

$$\begin{aligned}
\text{Expected Revenue from an auction} &= \frac{1}{2}\left(1-(1+2p)(1-p)^2\right) \\
&= \frac{1}{2}p^2\,(3-2p) \\
\text{Expected Revenue from negotiations} &= \frac{1}{2}\left(3p(1-p)^2\frac{1}{2}+3p^2(1-p)\frac{2}{3}+p^3\frac{3}{4}\right) \\
&= \frac{1}{2}p\left(\frac{p^2}{4}-p+\frac{3}{2}\right)
\end{aligned}$$

Panel B of figure 7 plots the revenues as a function of p. The target chooses an auction if p is above a threshold say p_3 and a negotiation otherwise. The point of indifference p_3 solves

$$\frac{1}{2}p_3^2(3-2p_3) = \frac{1}{2}p_3\left(\frac{p_3^2}{4}-p_3+\frac{3}{2}\right) \implies p_3 = 0.54 \tag{23}$$

Once again, we find that the seller benefits from the ability to commit, since the threshold above which he chooses an auction with commitment, p_3, is less than the threshold above which he chooses auctions with no commitment, p_{33}. The seller is now able to employ auctions over a greater range when p lies between p_{13} and p_{33}.

Theorem 12. *The seller can earn higher revenues if he can commit ex-ante to choosing a particular mechanism irrespective of the number of buyers that he observes. In equilibrium, the seller's choice of mechanism depends on the probability of entry p. The seller chooses auctions if the probability is greater than p_n and negotiations otherwise.*

Proof. The proof follows from the fact that without committment, the only equilibrium below p_{nn} was the seller choosing negotiations. With committment, choosing auctions strictly his revenue over the part of the range (p_n, p_{nn}). In the range $[p_{nn}, 1]$, the seller benefits from avoiding equilibria that feature mixing, which increases his revenue. □

What might commitment look like in practice? Instead of eliciting the participation decision from each potential buyer and *then* informing them of his choice of mechanism, the seller would first inform the potential buyers of whether he has chosen an auction or negotiation and then ask them whether they want to participate in the sale. In other words, with commitment, the timing of the process is flipped from finding out the value of m and then choosing a mechanism, to choosing the mechanism first and then finding out the value of m.

4 Extensions of the Baseline Model

In this section, I present extensions to the baseline model. I assume that the seller and the buyers are risk-neutral and that $n = 2$ for simplicity.

4.1 Independent Private Values

The analysis so far has been restricted to the case where the buyers have common values. I now examine how the conclusions change if the value to the buyers is private and independent, rather than common.

4.1.1 Expected Seller Revenue from Negotiations

Let the maximum expected revenue from sequentially negotiating with m buyers be equal to V_m and the corresponding offer be x^*_m

Consider the first stage. If a price x is offered, the offer is accepted when $V > x$, which happens with probability $1 - x$. The revenue conditional on acceptance is x. If the offer is rejected, which happens with probability x, there are $m - 1$ buyers left to negotiate with. So, the expected revenue conditional on rejection is V_{m-1}

This gives us the recursive equation

$$V_m = \underset{x\in[0,1]}{Max}\,(x(1-x) + (V_{m-1})x) \tag{24}$$

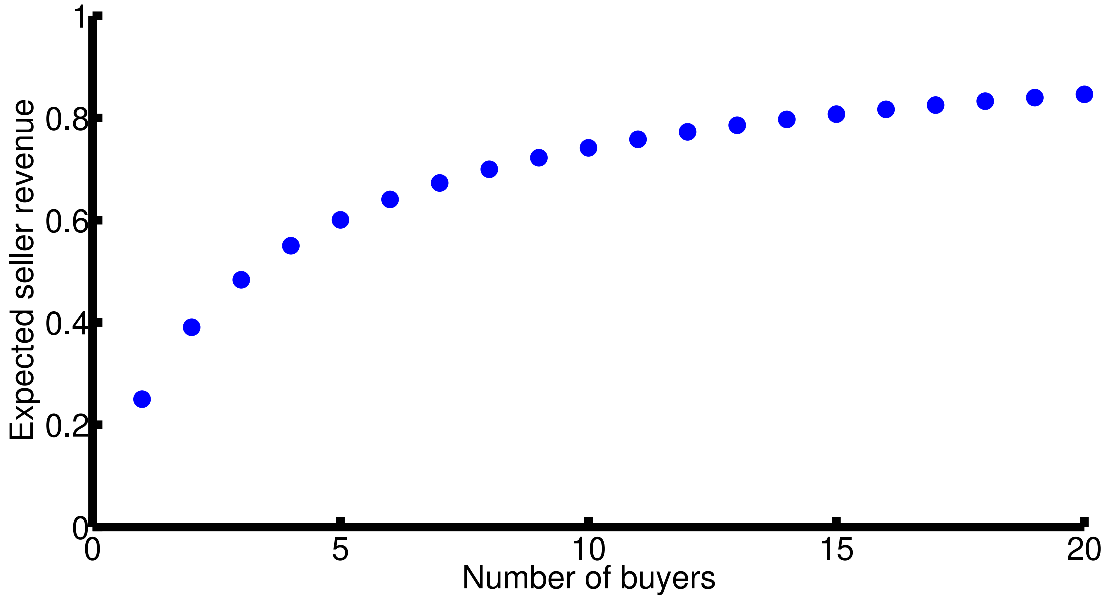

Figure 8: Expected seller revenue with independent buyer values $\sim U[0, 1]$

Differentiating w.r.t. x gives the optimal offer as

$$x^*_m = \frac{1 + V_{m-1}}{2} \tag{25}$$

Substituting in equation 24 , we obtain

$$V_m = \frac{(1 + V_{m-1})^2}{4} \tag{26}$$

$V_0 = 0$, so substituting recursively, we get the expected revenue as plotted in figure 8.[11] As the number of buyers increases, the seller can try offering higher amounts in the earlier stages. As $m \to \infty$, the seller is able to extract the maximum value from the transaction, which approaches 1, and the informational rent extracted by the buyers decreases.

Unlike in the common values case, there is no learning in the independent private values case. The rejection of an offer of x does not affect the probability of any subsequent offer being accepted. The expected revenue increases because of two factors. First, the seller has more rounds to negotiate as the number of buyers increases. Second, the expected maximum value for the m buyers is the expected maximum of m draws from $U[0, 1]$. This is given by the expression $\frac{m}{m+1}$, which approaches 1 as $m \to \infty$. Hence, there is a greater probability that a buyer will accept a higher offer.

4.1.2 Expected Seller Revenue from an Auction

Previous studies such as McAfee and McMillan (1987) and Harstad et al. (1990) derive bidding strategies and seller revenues for auctions with an unknown number of buyers and independent and identically distributed buyer valuations for any arbitrary distribution of the value. I

[11] Unlike the common values case, there is no closed form solution for V_m. Instead, the solution has to be obtained numerically by recursion.

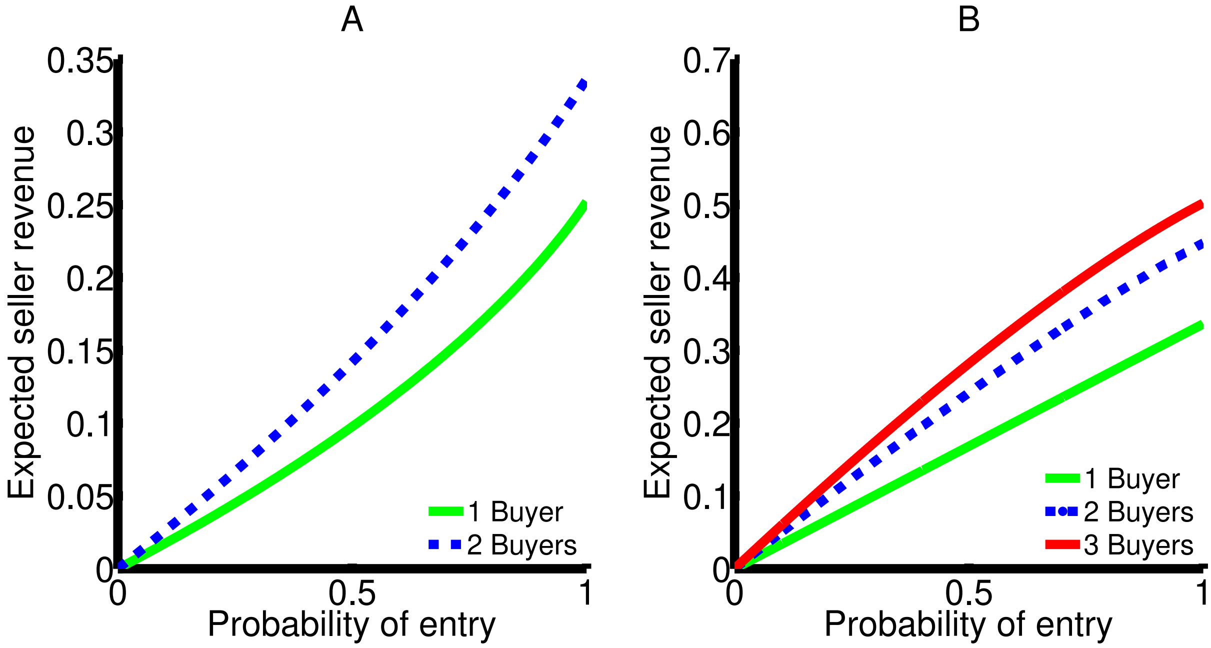

Figure 9: Expected seller revenue when the buyer values are independent
The figure plots the expected revenue when there is a maximum of n possible buyers whose values are independent, as a function of the probability of entry p and the number of buyers in play m. Panel A plots the graph for $n = 2$ and Panel B for $n = 3$

specialize their model to my setting of uniformly distributed buyer valuations to derive the expression for each buyer's bidding strategy.

With the values being independently distributed, the buyer bids are given by

$$B(V_i, n) = \frac{\sum\limits_{r=1}^{n-1} \binom{n-1}{r} p^r (1-p)^{n-1-r} V_i^r \frac{r}{r+1} V_i}{\sum\limits_{r=1}^{n-1} \binom{n-1}{r} p^r (1-p)^{n-1-r} V_i^r} \tag{27}$$

Each of the m buyers bid depending on their V_i. The expected revenue for the seller $\Pi(m, n)$ is the expected maximum bid, that is, the expected maximum of $B(V_i, n)$ for m draws of V_i. I compute this numerically as a function of m and n.

Panel A of figure 9 plots the expected revenue from an auction against p for $n = 2$ and $m = 1, 2$. As p approaches 1, each buyer is virtually certain that the other will enter. So, both bid $V_i/2$ and the expected revenue is $\frac{1}{3}$ as in a normal first price auction. On the other extreme, as p approaches 0, each buyer is almost sure that he is the only bidder and bids 0. Panel B shows the plot of the expected revenue from an auction against p for $n = 3$ and $m = 1, 2, 3$. Both graphs resemble the graphs for the expected revenue from an auction in the common values case.

The equilibrium characterisation and the benefits of commitment are similar to that in the common values case. The only change is that the threshold probability p_{22} is different.

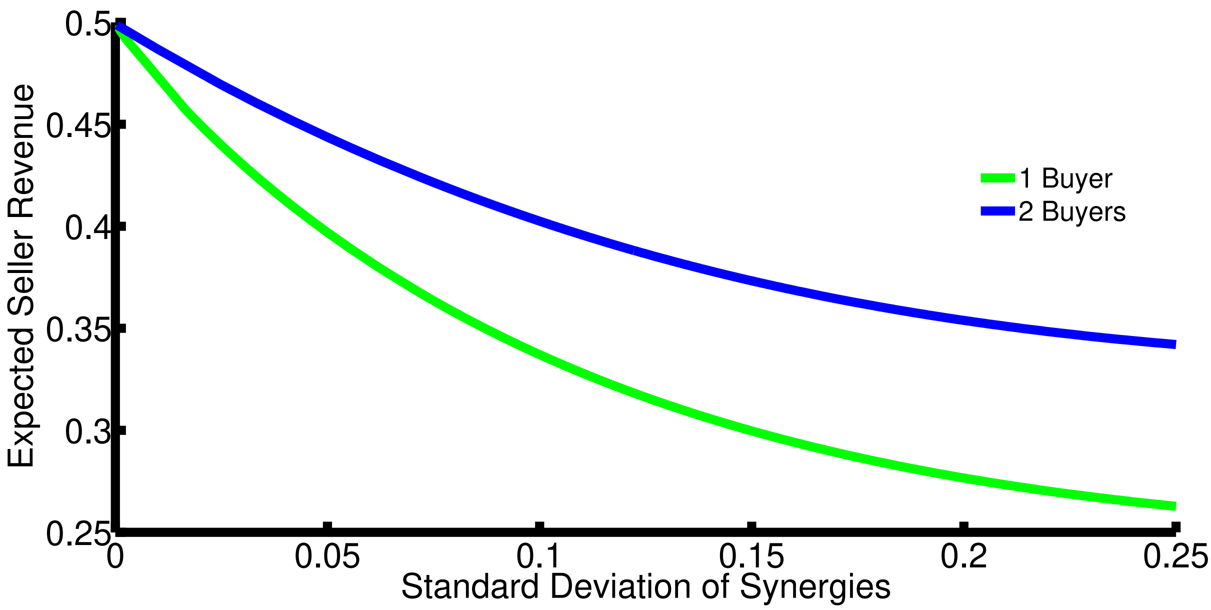

Figure 10: Expected revenue from negotiations as a function of the volatility of synergies. The distribution of synergies is truncated normal, with mean $\frac{1}{2}$ but various standard deviations. The revenue is plotted for negotiations when 2 buyers participate as well as when only 1 does.

4.2 Effect of the Volatility of the Synergies

So far, we have considered the value of the synergies to be uniformly distributed. The variance of $U[0,1]$ is $\frac{1}{2}$. To consider the effect of volatility of the synergies on negotiations, I change the distributional assumption.

I now assume that the synergies are normally distributed. I consider a series of truncated normal distributions with the same mean of $\frac{1}{2}$ but differing standard deviations.[12] I then look at how the optimal take-it-or-leave it offers in negotiations change with the volatility of the synergies. It is worth noting that the expected revenue from an auction depends only on the mean of the value of synergies, so the earlier analysis continues to hold for auctions even in this new setting.

4.2.1 Negotiations with 1 Buyer ($m = 1$)

If a price x is offered, it is accepted with probability $1 - F(x)$ and rejected with probability $1 - F(x)$. So, the optimal offer maximizes the following expression.

$$\underset{x\in[0,1]}{Max}\,[x(1-F(x))] \tag{28}$$

I solve this numerically. The results are plotted in figure 10. The expected seller revenue increases as the volatility of the synergies decreases. In the limiting case, as the volatility

[12] I truncate the distributions at 0 and 1 and assign the probabilities of the value being less than 0 and greater than 1 to atoms at 0 and 1. The standard deviations refer to those of the original normal distribution before truncation.

becomes zero, the seller will be able to extract all of the expected synergies in the transaction i.e. $\frac{1}{2}$. This is because the rent extracted by the buyer due to asymmetric information decreases with decrease in the volatility.

4.2.2 Negotiations with 2 Buyers ($m = 2$)

If the offer is accepted in the first stage, the expression of expected revenue is similar to that in equation 15. If the offer is rejected in the first stage, the seller still has one stage of negotiation left with an buyer. However, the value of the synergies conditional on the offer being rejected is a truncated normal with a one-sided truncation at x.

This problem is solved by backward recursion The payoff in the second stage is calculated as a function of the first stage offer. Then, the first stage offer is optimized using this payoff. The results are shown in figure 10. Two points are of interest. The first is that the revenue increases as the volatility on the synergies decreases. As the volatility becomes very low, the seller extracts all the synergies in the transaction. Second, the benefit of having an additional stage to learn about the value of synergies increases as the volatility increases. If the volatility is low, one stage is enough to extract most of the expected value because the asymmetry is low in magnitude. The higher the volatility is, the more the second stage of negotiation helps.

4.2.3 Threshold Probability of Choosing an Auction

The expected revenue from auctions remains the same independent of the volatility, but the expected revenue from negotiations changes with the volatility. So, the threshold probability above which the seller chooses an auction, p_{22}, would also change as the volatility increases. Since negotiations generate less revenue as the volatility increases, the threshold probability decreases with the volatility. Figure 11 shows the revenue from negotiations, auctions and threshold probability p_{22} when the volatility $\sigma = 0.1$. The threshold is higher than in the uniform distribution since the volatility is very low.

5 Empirical Implications

The vast amount of data available on takeovers enables the testing of empirical hypotheses generated from the model. I sketch a few of these below.

The theory specifies seller's choice of the mechanism as a function of the probability of entry p. However, sales of firms in some industries attract buyers with high probability and in some other industries with low probability. To derive empirical implications of the model in terms of the observed data on auctions and negotiations, I need to assume some distribution for p. This is because the aggregate data on auctions and negotiations pertains to a cross section of firms operating in different industries, each of which may have a different probability of entry. In the discussion that follows, I assume that $p \sim U[0, 1]$. I also assume that the equilibrium that will be realised is the one where the seller gets the maximum expected utility. This corresponds to equilibrium 2 in theorem 6 , that is, if $p \geq p_{22}$, the seller chooses an auction independent of the actual number of buyers in play.

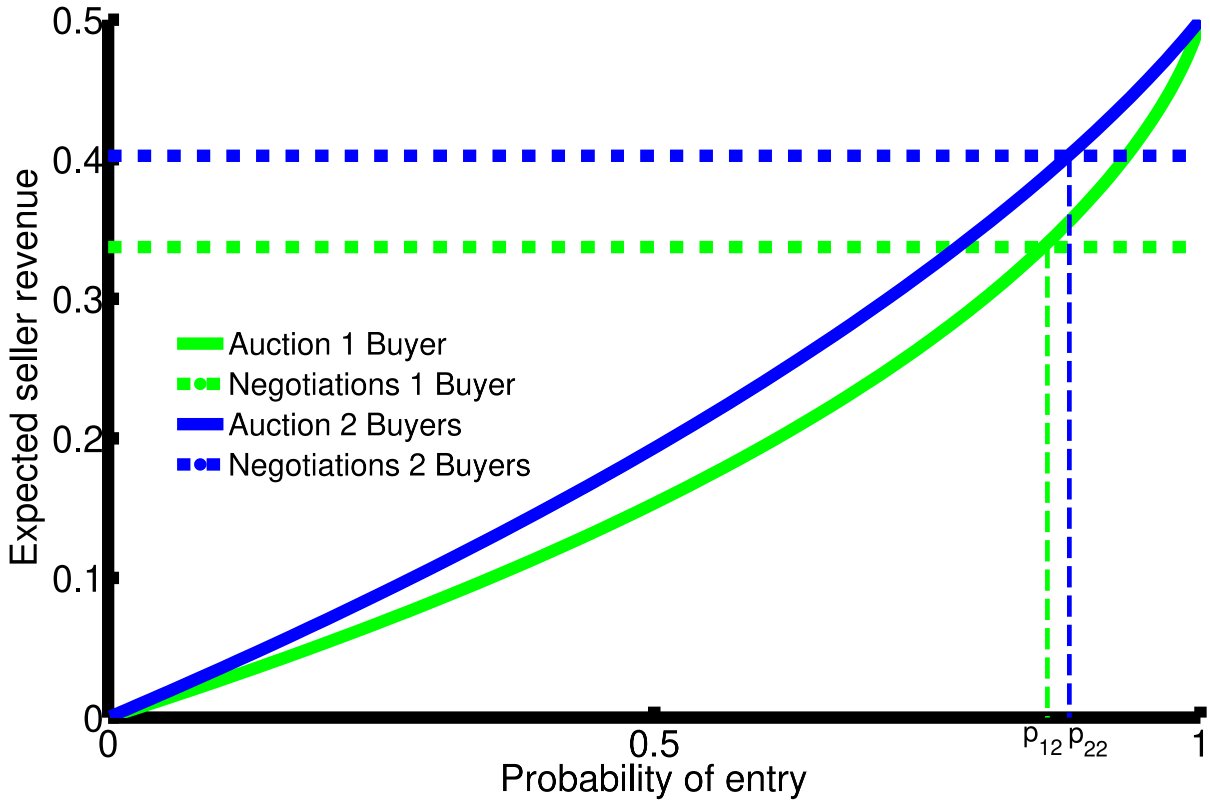

Figure 11: The threshold probability p_{22} when the volatility of synergies $\sigma = 0.1$

The empirical implications derived are for the observed transaction price. If the negotiation fails, the transaction will not be observed in the data. To take this into account, the expected transaction price is calculated as the expected offer made by the seller *conditional on the offer being accepted*. In the case of an auction, the firm is always sold, since the buyers always bid higher than the reserve price, which is the standalone value of the firm.

5.1 Mean and Volatility of the Transaction Price as a Function of the Probability of Entry

In this section, I examine how the mean and volatility of the transaction price depend on the probability of entry for a given level of the seller's and buyers' risk aversion. The relative risk aversion coefficient of the seller is assumed to be 0.5. The buyers are assumed to be risk-neutral.

The computation of expected transaction price for negotiations for a given value of p is as follows. When there is 1 buyer, the optimal offer for the seller is 0.33. So, the observed transaction price if there is 1 buyer is 0.33.

When there are 2 buyers, the optimal offer in the first round is 0.54 and in the second round 0.18. The firm is sold in the first round if the value is greater than 0.54, which happens with probability 1-0.54=0.46,[13] and in the second round if the value is between 0.18 and 0.54, which happens with probability 0.54-0.18=0.36. Conditional on the firm being sold, the probability that it was sold in the first round is $\frac{0.46}{0.46+0.36}$ and in the second round $\frac{0.36}{0.46+0.36}$. So, the observed

[13]The probability that the value V is greater than 0.56 is 1-0.56 because $V \sim U[0, 1]$

transaction price is

$$\frac{0.46}{0.46+0.36}(0.54)+\frac{0.46}{0.46+0.36}(0.18)=0.39.$$

The last step is to multiply the observed prices conditional on there being 1 or 2 buyers by the respective probabilities that there are 1 or 2 buyers, which are $\frac{2-2p}{2-p}$ and $\frac{p}{2-p}$. So, the observed transaction price for negotiations is given by the function

$$0.33(\frac{2-2p}{2-p})+0.39(\frac{p}{2-p}) \tag{29}$$

For auctions, the observed transaction price is the expected revenue from commitment, which was already calculated in equation 23 as $\frac{1}{2}p^2$, divided by the probability of there being at least one buyer, $1-(1-p)^2$.

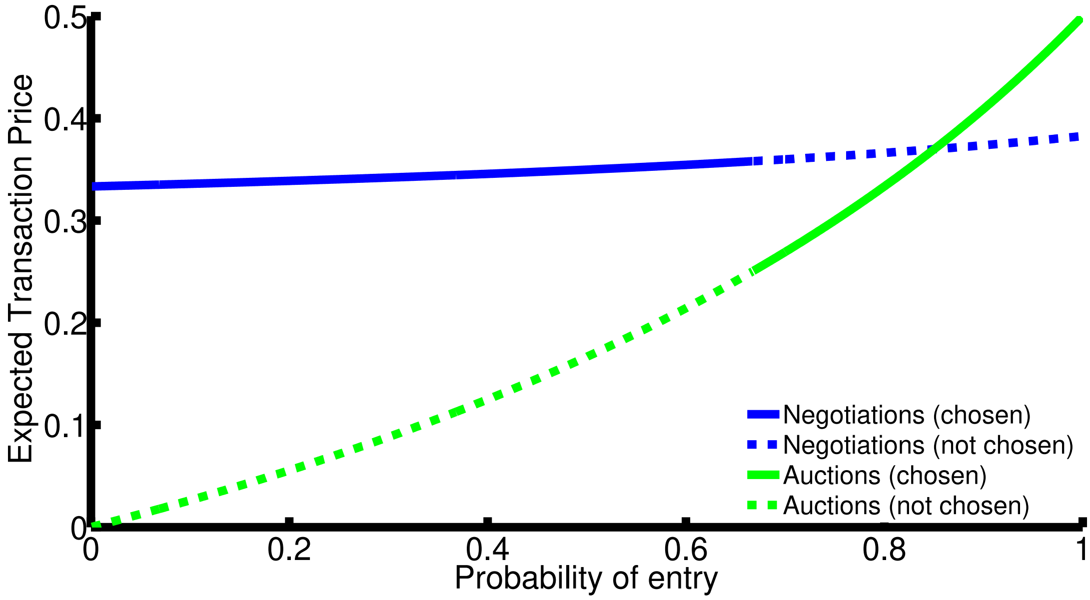

Figure 12: Mean of the transaction price as a function of the probability of entry
The solid part of the line and the curve correspond to regions where the mechanism is chosen; the dotted part to regions where it is not chosen. The risk-aversion of the seller is assumed to be 0.5, and that of the buyer 0.

Figure 12 plots the expressions in equations 29 and 23 as a function of the probability of entry for both auctions and negotiations. The straight line corresponds to negotiations and the curve to auctions. The solid part of the line and the curve correspond to regions where the mechanism is chosen; the dotted part to regions where it is not chosen.[14]

[14] It may seem paradoxical that the seller chooses auctions even in regions where the expected transaction price is lesser than that from negotiations. To reconcile this, recall that the seller's choice depends on his expected utility from the mechanism and not the expected transaction price. The expected transaction price differs from the expected seller utility due to two reasons. First, the seller is risk-averse, so the expected utility of the seller is not the same as the expected transaction price. Second, for negotiations, even if the seller is risk-neutral, the two are different because the negotiations can fail if the seller offers a price greater than the buyers' value. The expected utility is the the product of the probability of the offer being accepted and the expected transaction price conditional on the offer being accepted. So, the expected utility is less than the expected transaction price.

Empirical Implication 1. *For a given level of the seller's risk aversion, the mean transaction price for both auctions and negotiations increases as the as the probability of entry p increases.*

The intuition behind this result is as follows. For negotiations, the price increases because as p increases, it becomes increasingly likely that there are 2 buyers participating ($m = 2$). So, the seller is increasingly likely to have 2 rounds of negotiations than 1, which increases the expected offer price. I call this the effect of actual competition. For auctions, there are two effects in play. First, same as in negotiations, as p increases, it becomes increasingly likely that there are 2 buyers participating. This is the effect of actual competition. Second, unlike negotiations, the expected price increases with p also because the buyers believe it is more likely that $m = 2$, which leads them to bid higher whether $m = 1$ or $m = 2$. I call this the effect of latent competition. These effects combine to raise the expected transaction price.

I have so far ignored the fact that the selection of the mechanism depends on p. In the revenue-maximising equilibrium, auctions are chosen if $p \geq p_{22}$ and negotiations are chosen if $p \geq p_{22}$. Predictions about the observed transaction price *across mechanisms* must also account for selection. The observed transaction price will include only the solid line and curve in figure 12. It is immediately apparent that the observed transaction price has a discontinuity at the threshold probability due to the effect of selection.

Empirical Implication 2. ***The effect of selection on the mean transaction price***
For a given level of the seller's risk aversion, the mean transaction price increases as the industry becomes more competitive, till a cutoff level of competition. At the cutoff, the transaction price drops and then monotonically increases.

The plot for the volatility (standard deviation) of the transaction price as a function of the probability of entry is displayed in figure 13. The next empirical implication follows from the shape of the graph.

Empirical Implication 3. *For a given level of the seller's risk aversion, the volatility of the transaction price for both auctions and negotiations increases as the probability of entry p increases.*

The intuition for this result is as follows. For negotiations, recall that the offer prices are 0.33 if there is one buyer and 0.54 and 0.18 in the two rounds when there are two buyers. For very low values of p, that is as $p \to 0$, there is almost certainly only one buyer and the seller always offers 0.33. So, the observed transaction price is always 0.33. The standard deviation of the transaction price is 0. For very high values of p, that is as $p \to 1$, there are almost certainly two buyers.[15] The seller offers 0.54 and 0.18 in the two rounds and the standard deviation is 0.18. For values of p between 0 and 1, the offer can be any of 0.18, 0.33 or 0.54 and the volatility lies between 0 and 0.18.[16]

[15] Of course, the seller may never choose negotiations at such high values of probability p.

[16] Conditional on the offer being accepted, the probability that the transaction price 0.54 is $\frac{0.46}{0.46+0.36}$ and that the transaction price is 0.18 is $\frac{0.36}{0.46+0.36}$. The mean is 0.39 as calculated earlier in the section. The variance is

$$\frac{0.46}{0.46+0.36}(0.54-0.39)^2 + \frac{0.36}{0.46+0.36}(0.18-0.39)^2 = 0.032 \tag{30}$$

and the standard deviation is 0.18.

For auctions, the variance of the transaction price for a given value of p arises from 2 components, the within V variance and across V variance. For a given realisation of V, there is a range of bids since each buyer randomizes over an interval leading to within V variance. Then there is variance of the bids across Vs. For very low values of p, that is as $p \to 0$, there is almost certainly only one buyer and each buyer's bid approaches 0 for any value of V. Thus, the bid is constant and the variance is 0. For very high values of p, that is as $p \to 1$, there are almost certainly two buyers. So, each buyer's bid approaches V. The within V variance is 0 in the limit. The across V variance is just the variance of the maximum bid, which is V, when $V \sim U[0,1]$. The variance is $\frac{1}{12}$, and the corresponding standard deviation is 0.29. For values of p between 0 and 1, the variance lies between 0 and $\frac{1}{12}$.

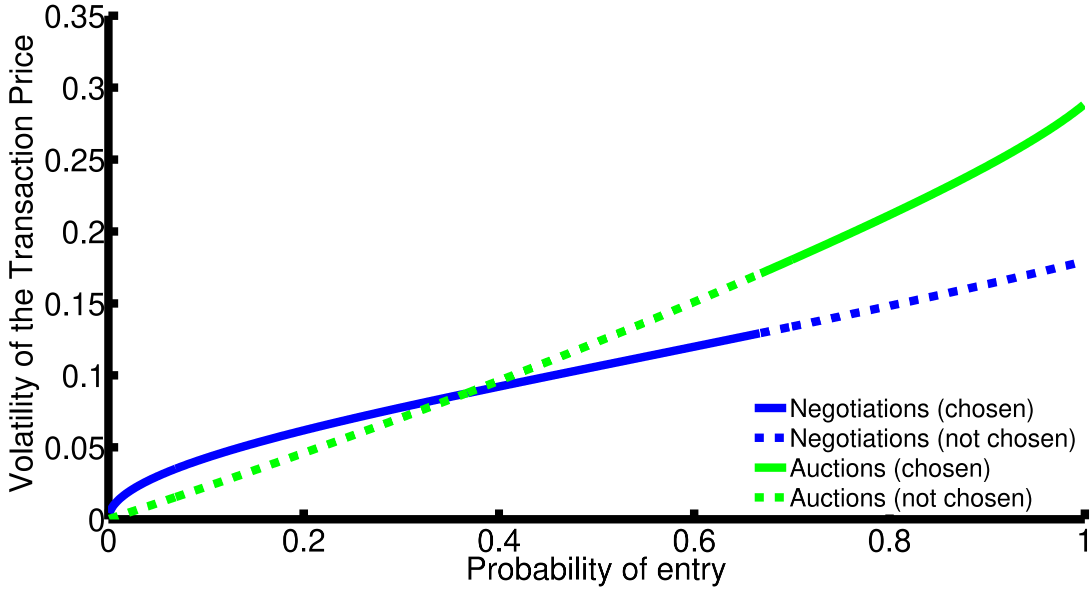

Figure 13: Volatility of the transaction price as a function of the probability of entry
The solid part of the line and the curve correspond to regions where the mechanism is chosen; the dotted part to regions where it is not chosen. The risk-aversion of the seller is assumed to be 0.5, and that of the buyer 0.

Once we account for selection of the mechanism, that auctions are chosen if $p \geq p_{22}$ and negotiations are chosen if $p \leq p_{22}$, the only relevant portions of the graph are the solid lines, not the dotted ones. The graph tells us that although the volatility monotonically increases, there is a discontinuity at the threshold probability $p \geq p_{22}$.

Empirical Implication 4. ***The effect of selection on the volatility of the transaction price***
For a given level of the seller's risk aversion, the volatility of the transaction price increases as the industry becomes more competitive. The volatility is continuous in p at all points except at the threshold probability where the seller starts choosing auctions.

5.2 Mean and Volatility of the Transaction Price as a Function of the Seller's Risk Aversion

In this section, I examine how the mean and volatility of the transaction price depend on the seller's risk aversion. The buyers are assumed to be risk-neutral. The probability of entry p is assumed to be $\sim U[0, 1]$

For each value of the seller's risk aversion, the average transaction price is computed by taking the expectation of the average transaction price for a given value of p across the entire ranges of p for which that mechanism is chosen.[17] The result is plotted in figure 14.

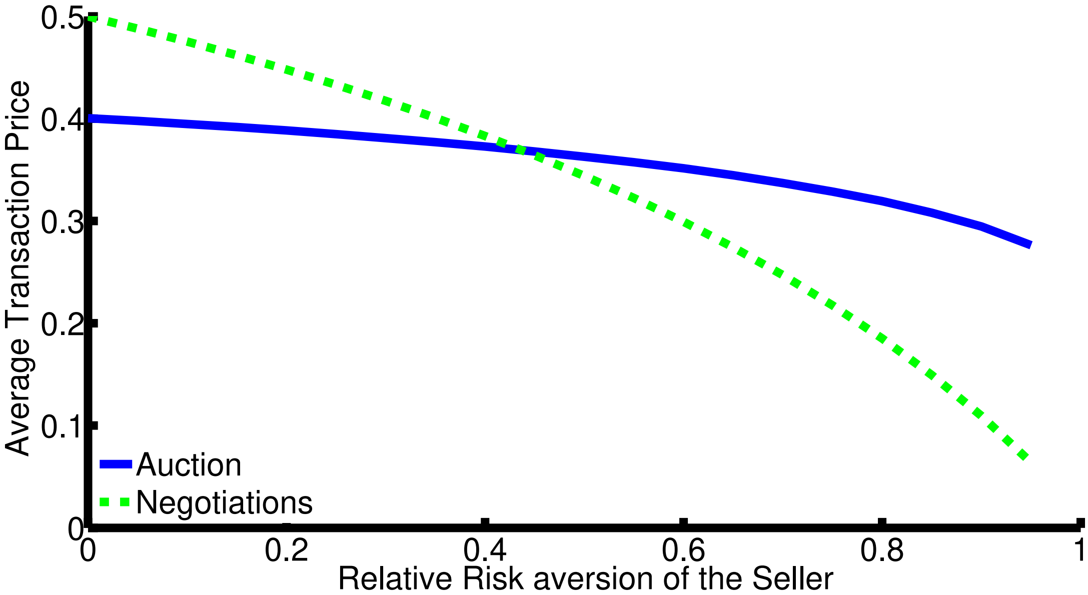

Figure 14: Mean of the transaction price as a function of the risk aversion of the seller
The buyers are assumed to be risk-neutral. The probability of entry p is assumed to be $\sim U[0, 1]$

Empirical Implication 5. *The average transaction price for both auctions and negotiations decreases as the risk aversion of the seller increases.*

The reason why the average transaction price decreases as γ_s increases is different for auctions and negotiations. For negotiations, decrease is because the seller makes lower offers as his risk aversion increases. However, in the case of auctions, the buyers bid the same irrespective of the seller's risk-aversion. This will change the expected utility of the seller as he becomes more risk-averse, but why does it change the expected transaction price? The reason is purely that the range of choosing auctions increases as the seller becomes more risk-averse. Since the average transaction price conditional on p is increasing in p, the average decreases across the range $[p_{22}, 1]$ as p_{22} decreases.

Also, from figure 14, the average transaction price for auctions and negotiations exhibits the property of single crossing in the seller's risk aversion. This leads to the next empirical implication.

[17]From figure 12, for auctions, this corresponds to the area under the graph from $[p_{22}, 1]$ scaled up by dividing by $1 - p_{22}$. For negotiations, it is the area under the graph from $[0, p_{22}]$ scaled up by dividing by p_{22}.

Empirical Implication 6. *Above a threshold level of the seller's risk-aversion, the expected transaction price from auctions is higher than that from negotiations. Below the threshold, the the expected transaction price from negotiations is higher than that from auctions.*

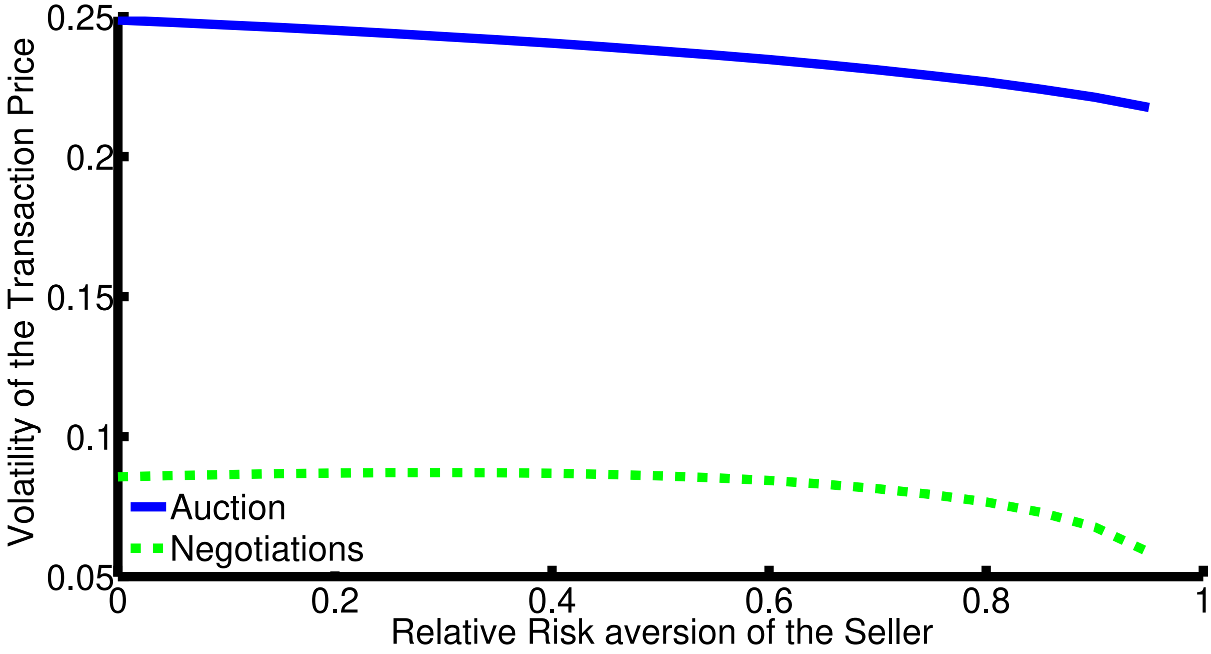

Figure 15: Volatility of the transaction price as a function of the risk aversion of the seller
The buyers are assumed to be risk-neutral. The probability of entry p is assumed to be $\sim U[0,1]$

Next, I extend this analysis to compute the volatility of the transaction price. As the seller becomes more risk-averse, p_{22} decreases which affects the interval over which the computation of the volatility is done.[18] The result is plotted in figure 15.

Empirical Implication 7. *The volatility of the transaction price for auctions decreases monotonically as the risk aversion of the seller increases. The volatility for negotiations is non-monotonic in the risk aversion of the seller.*

As the risk aversion of the seller approaches 1, the volatility of the transaction price for negotiations drops rapidly since the seller makes very low offers. This has two effects. First, the transaction almost always goes through and second, conditional on the transaction going through, the offer made is not very volatile.

However, for auctions, as the risk aversion of the seller approaches 1, the volatility of the transaction price does not drop to 0, since the bids of the buyers are not affected by the seller's

[18]Unlike the unconditional mean of the transaction price, the unconditional variance cannot be computed by taking the expectation of the conditional variance given p. To understand the computation of the volatility, refer back to figure 12 and 13 which give the volatility and mean conditional on p. The unconditional volatility has two components since

$$Var(Price) = \mathbb{E}[Var(Price \mid p)] + Var(\mathbb{E}[Price \mid p])$$

For auctions, the expectation is computed in the interval $[p_{22}, 1]$ where auctions are chosen. For negotiations, the expectation is computed in the interval $[0, p_{22}]$ where negotiations are chosen.

risk-aversion. The bids vary depending on the realised value of synergies unlike the seller's offers in negotiations which are independent of the realised value of synergies. So, the following empirical implication holds for all values of the coefficient of relative risk aversion of the seller in (0,1).

Empirical Implication 8. *The volatility of the transaction price for auctions is always greater than that for negotiations*

6 Concluding remarks

Recent empirical research has shown the importance of recognizing the intricacies of the sales process in analyzing corporate takeovers. Yet, prevailing models have not captured an important source of buyer uncertainty: buyers may not know how many other buyers are participating in the process. This assumes added significance in sales of firms because the average number of buyers is not just uncertain but very low. This is the first study to explicitly model how this uncertainty affects the seller's choice of mechanism between auctions and negotiations.

The main finding is that if the seller cannot commit to choosing the mechanism before he sees the number of buyers, the choice of the mechanism in the revenue-maximising equilibrium or any pure strategy equilibrium depends only on the buyers' prior beliefs about competition. It is independent of the actual competition in terms of the number of buyers who participate in the sale.

The seller can increase the revenue by committing to a choice of mechanism that depends only on the potential number of buyers and the likelihood of each entering, not on the actual number of buyers that participate. I show that the seller's choice can also be influenced by the volatility of synergies and whether the synergies are independent or correlated.

The model can be used to generate testable empirical hypotheses about the mean and volatility of the observed transaction prices. The model can also be extended to study situations that arise while selling assets rather than the whole firm. For example, consider a firm trying to spin off its assets. The firm has a choice of selling related subsidiary businesses together or separately in a sequential process. It is possible that the set of potential buyers is the same for all the subsidiaries. Whether the subsidiaries are sold separately can depend on how the buyers' beliefs about the competition in the later rounds of the sale change based on the transaction prices observed in the earlier rounds. If the competition is likely to be low and the buyers can use the prices to update their prior beliefs, a firm might prefer to sell the businesses together rather than separately. Empirical data on spinoffs can be analyzed to verify the predictions of the extension.

The theory developed in this paper sheds light on how firms are sold and goes some way towards explaining the various factors that underlie the choice of the sale mechanism. A lot of future work awaits, both in extending the model to make it richer and testing it empirically.

Appendices

A Expected Revenue from Negotiations- a Proof using Induction

The proof uses induction. We need to prove that

$$\text{If } W_m = \frac{1}{4(1-V_{m-1})} \text{ and } W_0 = 0, \text{ then } W_m = \frac{m}{2(m+1)}$$

To prove this by induction, assume the statement

$$W_m = \frac{m}{2(m+1)}$$

is true for $m = l-1$. If this is the case,

$$\begin{aligned} W_l &= \frac{1}{4(1-W_{l-1})} \\ &= \frac{1}{4(1-\frac{l-1}{2l})} \\ &= \frac{l}{2(l+1)} \end{aligned}$$

So, if the statement holds for $m = l-1$, it also holds for $m = l$.
$V_0 = 0$, which means that the statement holds for $m = 0$.
Hence, it must hold for $m = 1$. Extending the logic, it holds for all m

B Expected Revenue from an Auction

The cdf of each bid is

$$F(x) = \frac{1-p}{p}\left(\left(\frac{V}{V-x}\right)^{\frac{1-\gamma_b}{n-1}} - 1\right)$$

The pdf of the maximum bid is

$$G(x) = mF(x)^{m-1}F'(x)$$

Substituting for $F(x)$, this simplifies to

$$\begin{aligned} G(x) &= m\left(\frac{1-p}{p}\right)^m \left(\left(\frac{V}{V-x}\right)^{\frac{1-\gamma_b}{n-1}} - 1\right)^{m-1} \frac{1-\gamma_b}{n-1}\left(\frac{V}{V-x}\right)^{\frac{1-\gamma_b}{n-1}-1} \frac{V}{(V-x)^2} \\ &= m\frac{1-\gamma_b}{n-1}\left(\frac{1-p}{p}\right)^m V^{\frac{1-\gamma_b}{n-1}} \left(\left(\frac{V}{V-x}\right)^{\frac{1-\gamma_b}{n-1}} - 1\right)^{m-1} \left(\frac{1}{V-x}\right)^{\frac{n-\gamma_b}{n-1}} \end{aligned}$$

The expected utility of the seller for a given V,

$$\Pi_V(m,n) = \int_0^{\left(1-(1-p)^{\frac{n-1}{1-\gamma_b}}\right)V} G(x)\frac{x^{1-\gamma_s}}{1-\gamma_s}\mathrm{d}x$$

Substituting for $G(x)$, this simplifies to

$$\Pi_V(m,n) = m\frac{1-\gamma_b}{n-1}\left(\frac{1-p}{p}\right)^m V^{\frac{1-\gamma_b}{n-1}} \int_0^{\left(1-(1-p)^{\frac{n-1}{1-\gamma_b}}\right)V} \left(\left(\left(\frac{V}{V-x}\right)^{\frac{1-\gamma_b}{n-1}} - 1\right)^{m-1} \left(\frac{1}{V-x}\right)^{\frac{n-\gamma_b}{n-1}}\right) \frac{x^{1-\gamma_s}}{1-\gamma_s}\mathrm{d}x$$

Now, change the variable of integration to y where y is given by

$$
\begin{aligned}
y &= (\frac{V}{V-x})^{\frac{1}{n-1}} \\
\frac{1}{V-x} &= \frac{1}{V}(y)^{n-1} \\
x &= V(1-\frac{1}{y^{n-1}}) \\
\mathrm{d}x &= V(n-1)\frac{1}{y^n}\mathrm{d}y
\end{aligned}
$$

Substituting all these into the integral and changing the limits of integration yields

$$
\begin{aligned}
\Pi_V(m,n) &= m\frac{1-\gamma_b}{n-1}\left(\frac{1-p}{p}\right)^m V^{\frac{1-\gamma_b}{n-1}} \int_1^{\frac{1}{(1-p)^{\frac{1}{1-\gamma_b}}}} (y^{1-\gamma_b}-1)^{m-1}(\frac{1}{V})^{\frac{n-\gamma_b}{n-1}} y^{n-\gamma_b}\frac{V^{1-\gamma_s}}{1-\gamma_s}(1-\frac{1}{y^{n-1}})^{1-\gamma_s}V(n-1)(y)^{-n}\mathrm{d}y \\
&= (1-\gamma_b)m\left(\frac{1-p}{p}\right)^m \frac{V^{1-\gamma_s}}{1-\gamma_s}\left(\int_1^{\frac{1}{(1-p)^{\frac{1}{1-\gamma_b}}}} (y^{1-\gamma_b}-1)^{m-1}y^{-\gamma_b}(1-\frac{1}{y^{n-1}})^{1-\gamma_s}\mathrm{d}y\right)
\end{aligned}
$$

C Target's Expected Revenue from an Auction as a Function of p for $n = 2, 3$ and $m = 1, 2, 3$

C.1 $\Pi(1, 2)$

$$
\begin{aligned}
\Pi(1,2) &= \frac{1}{2}\frac{1-p}{p}\left(\int_1^{\frac{1}{1-p}}(1-\frac{1}{y})\mathrm{d}y\right) \\
&= \frac{1}{2}\frac{1-p}{p}\left(y - ln(y)\Big|_1^{\frac{1}{1-p}}\right) \\
&= \frac{1}{2}\frac{1-p}{p}\left(\frac{1}{1-p} - 1 - ln\left(\frac{1}{1-p}\right)\right) \\
&= \frac{1}{2}\left(1 + \frac{1-p}{p}ln\left(1-p\right)\right)
\end{aligned}
$$

C.2 $\Pi(2, 2)$

$$
\begin{aligned}
\Pi(2,2) &= \frac{1}{2}\left(\frac{1-p}{p}\right)^2 2\left(\int_1^{\frac{1}{1-p}}(y-1)(1-\frac{1}{y})\mathrm{d}y\right) \\
&= \frac{1}{2}\left(\frac{1-p}{p}\right)^2 2\left(\frac{1}{2}y^2 - 2y + ln(y)\Big|_1^{\frac{1}{1-p}}\right) \\
&= \frac{1}{2}\left(\frac{1-p}{p}\right)^2\left(\left(\frac{1}{1-p}\right)^2 - 1 - \frac{4}{1-p} + 4 + 2ln\left(\frac{1}{1-p}\right)\right) \\
&= \frac{1}{2}\left(3 - \frac{2}{p} + 2\left(\frac{1-p}{p}\right)^2 ln\left(\frac{1}{1-p}\right)\right)
\end{aligned}
$$

C.3 $\Pi(1,3)$

$$\begin{aligned}
\Pi(1,3) &= \frac{1}{2}\frac{1-p}{p}\left(\int_1^{\frac{1}{1-p}}(1-\frac{1}{y^2})\mathrm{d}y\right)\\
&= \frac{1}{2}\frac{1-p}{p}\left(y+\frac{1}{y}\Big|_1^{\frac{1}{1-p}}\right)\\
&= \frac{1}{2}\frac{1-p}{p}\left(\frac{p^2}{1-p}\right)\\
&= \frac{1}{2}p
\end{aligned}$$

C.4 Appendix $\Pi(2,3)$

$$\begin{aligned}
\Pi(2,3) &= \frac{1}{2}\left(\frac{1-p}{p}\right)^2 2\left(\int_1^{\frac{1}{1-p}}(y-1)(1-\frac{1}{y^2})\mathrm{d}y\right)\\
&= \frac{1}{2}\left(\frac{1-p}{p}\right)^2 2\left(\int_1^{\frac{1}{1-p}}(y-1-\frac{1}{y}+\frac{1}{y^2})\mathrm{d}y\right)\\
&= \frac{1}{2}\left(\frac{1-p}{p}\right)^2 2\left(\frac{1}{2}y^2-y-ln(y)-\frac{1}{y}\Big|_1^{\frac{1}{1-p}}\right)\\
&= \frac{1}{2}\left(\frac{1-p}{p}\right)^2\left(\left(\frac{1}{1-p}\right)^2+1-\frac{2}{1-p}+2ln\left(1-p\right)+2p\right)\\
&= \frac{1}{2}\left(2p+\frac{2}{p}-3+2\left(\frac{1-p}{p}\right)^2 ln\left(1-p\right)\right)
\end{aligned}$$

C.5 Appendix $\Pi(3,3)$

$$\begin{aligned}
\Pi(3,3) &= \frac{1}{2}\left(\frac{1-p}{p}\right)^3 3\left(\int_1^{\frac{1}{1-p}}(y-1)^2(1-\frac{1}{y^2})\mathrm{d}y\right)\\
&= \frac{1}{2}\left(\frac{1-p}{p}\right)^3 3\left(\int_1^{\frac{1}{1-p}}(y^2-2y+1-1+\frac{2}{y}-\frac{1}{y^2})\mathrm{d}y\right)\\
&= \frac{1}{2}\left(\frac{1-p}{p}\right)^3\left(y^3-3y^2+6\left(\frac{1-p}{p}\right)^3 ln(y)+\frac{3}{y}\Big|_1^{\frac{1}{1-p}}\right)\\
&= \frac{1}{2}\left(\frac{1-p}{p}\right)^3\left(\frac{1}{(1-p)^2}(\frac{1-3+3p}{1-p})+2+6ln(\frac{1}{1-p})-3p\right)\\
&= \frac{1}{2}\left(3p-11+\frac{15}{p}-\frac{6}{p^2}+6\left(\frac{1-p}{p}\right)^3 ln(\frac{1}{1-p})\right)
\end{aligned}$$

D The unique equilibrium when $n = 2$

$$p^* = \frac{pq_2}{(1-p)q_1+pq_2} \text{ if at least one of } q_1 \text{ or } q_2 \text{ is not zero} \tag{31}$$

$$p^* \in [0,1] \text{ if both } q_1 \text{ and } q_2 \text{ are zero} \tag{32}$$

Sequential rationality of the seller implies that if there is only one buyer in play, the posterior belief has to be greater that p_{12} for the seller to choose an auction over a negotiation. If there are two buyers in play, the posterior belief has to be greater that p_{22} for the seller to choose an auction over a negotiation.

$$q_1 = 1 \text{ if } p^* > p_{12} \tag{33}$$
$$q_1 = 0 \text{ if } p^* < p_{12} \tag{34}$$
$$q_1 \in [0,1] \text{ if } p^* = p_{12} \tag{35}$$
$$q_2 = 1 \text{ if } p^* > p_{22} \tag{36}$$
$$q_2 = 0 \text{ if } p^* < p_{22} \tag{37}$$
$$q_2 \in [0,1] \text{ if } p^* = p_{22} \tag{38}$$

We also know that

$$p_{22} > p_{12} \tag{39}$$

Now, we just consider the various strategies of the seller i.e. possible values of q_1 and q_2.

First, consider the subset of equilibria in which the seller doesn't always choose negotiations i.e. at least one of q_1 or q_2 is not zero.

- First, we can rule out the case where $q_1 = 0$. From equation 31, $p^* = 1$. But if $p^* = 1$, from equation 33, $q_1 = 1$, a contradiction. Hence, $q_1 = 0$ is not an equilibrium.
- Next, we can rule out the possibility that $q_2 = 0$. If $q_2 = 0$, from equation 31, $p^* = 0$. If $p^* = 0$, from equation 33, $q_1 = 0$. This contradicts the assumption that at least one of q_1 or q_2 is not zero. So, $q_2 = 0$ is not an equilibrium.
- Next, consider $q_1 \in (0,1)$. From equation 35, $p^* = p_{12}$. If $p^* = p_{12}$, from equation 37, $q_2 = 0$. But if $q_2 = 0$, from equation 31, $p^* = 0$ which contradicts $p^* = p_{12}$. So, $q_1 \in (0,1)$ is not an equilibrium.
- Next, consider $q_2 \in (0,1)$. From equation 38, $p^* = p_{22}$. If $p^* = p_{22}$, from equation 33, $q_1 = 1$. This is indeed a possible equilibrium.

  The probability q_2 for any value of p is pinned down by substituting $q_1 = 1$ in equation 31

  $$p_{22} = \frac{pq_2}{(1-p) + pq_2} \tag{40}$$

  This equation can only be satisfied for $p > p_{22}$. So, this class of equilibria exist only for $p > p_{22}$.
- Last, consider $q_2 = 1$. From equations 36 and equations 38, $p^* \geq p_{22}$. If $p^* \geq p_{22}$, from equation 33, $q_1 = 1$.This is indeed a possible equilibrium. Since the seller always chooses auctions, the posterior probability p^* is equal to the prior probability p. So, this class of equilibria exist only for $p \geq p_{22}$.

Note that for all these equilibria, $p \geq p_{22}$.

Now consider the case we have ignored till now, i.e. both q_1 and q_2 are zero. In this case, no Bayesian updating happens since the posterior belief is equal to the prior belief. This is an equilibrium for any value of p, with the appropriate beliefs off the equilibrium path which fulfil the sequential rationality criterion for the seller. An example of these beliefs would be that if the seller chooses auctions, the buyers believe that the number of buyers is 1.

E Expected Revenue from an Auction for a General Probability Distribution

Let p_i denote the equilibrium probability that there are i buyers in the auction.

Let $F(x)$ denote the cdf of the bid in the interval $[0, V - p_1 V]$.

Then, the cdf of the maximum bid with m buyers is $(F(x))^m$.

The pdf is $\mathrm{d}(F(x))^m$

The expected utility of the seller $\Pi_V(m, n, \gamma_s, \gamma_b)$ is given by

$$\int\limits_0^{(V-p_1V)} \frac{x^{1-\gamma_s}}{1-\gamma_s}\mathrm{d}(F(x))^m$$

Denote $F(x)$ by y where y is defined implicitly by the equation

$$\left(p_1 + p_2 y + p_3 y^2 + \cdots + p_n y^n\right) \frac{(V-x)^{1-\gamma_b}}{1-\gamma_b} = p_1 \frac{V^{1-\gamma_b}}{1-\gamma_b}$$

Rearranging this equation gives

$$x = V\left[1 - \left(\frac{p_1}{p_1 + p_2 y + p_3 y^2 + \cdots + p_n y^n}\right)^{\frac{1}{1-\gamma_b}}\right]$$

Substituting in the original integral, chaging the variable of integration to y (including changing the limits) gives

$$\Pi_V(m, n, \gamma_s, \gamma_b) = \frac{V^{1-\gamma_s}}{1-\gamma_s}\int\limits_0^1 \left[1 - \left(\frac{p_1}{p_1 + p_2 y + p_3 y^2 + \cdots + p_n y^n}\right)^{\frac{1}{1-\gamma_b}}\right]^{1-\gamma_s} m y^{m-1}\mathrm{d}y$$

F Expected revenue from committing to an auction ex ante

$$
\begin{aligned}
\Pi_n &= \sum_{m=1}^{n} ({}^nC_m) p^m (1-p)^{n-m} \Pi(m,n) \\
&= \sum_{m=1}^{n} ({}^nC_m) p^m (1-p)^{n-m} \frac{1}{2} m \left(\frac{1-p}{p}\right)^m \left(\int_1^{\frac{1}{1-p}} (y-1)^{m-1} (1 - \frac{1}{y^{n-1}}) \mathrm{d}y \right) \\
&= \frac{1}{2}(1-p)^n \sum_{m=0}^{n} \left(\int_1^{\frac{1}{1-p}} ({}^nC_m) m (y-1)^{m-1} (1 - \frac{1}{y^{n-1}}) \mathrm{d}y \right) \\
&= \frac{1}{2}(1-p)^n \left(\int_1^{\frac{1}{1-p}} \left(\sum_{m=0}^{n} ({}^nC_m) m (y-1)^{m-1} \right) (1 - \frac{1}{y^{n-1}}) \mathrm{d}y \right) \\
&= \frac{1}{2}(1-p)^n \left(\int_1^{\frac{1}{1-p}} \left(\sum_{m=0}^{n} \frac{\mathrm{d}}{\mathrm{d}y} (({}^nC_m)(y-1)^m) \right) (1 - \frac{1}{y^{n-1}}) \mathrm{d}y \right) \\
&= \frac{1}{2}(1-p)^n \left(\int_1^{\frac{1}{1-p}} \left(\frac{\mathrm{d}}{\mathrm{d}y} \left(\sum_{m=0}^{n} ({}^nC_m)(y-1)^m \right) \right) (1 - \frac{1}{y^{n-1}}) \mathrm{d}y \right) \\
&= \frac{1}{2}(1-p)^n \left(\int_1^{\frac{1}{1-p}} \left(\frac{\mathrm{d}}{\mathrm{d}y} \left(\sum_{m=0}^{n} ({}^nC_m)(y-1)^m 1^{n-m} \right) \right) (1 - \frac{1}{y^{n-1}}) \mathrm{d}y \right) \\
&= \frac{1}{2}(1-p)^n \left(\int_1^{\frac{1}{1-p}} \left(\frac{\mathrm{d}}{\mathrm{d}y} ((y-1+1)^n) \right) (1 - \frac{1}{y^{n-1}}) \mathrm{d}y \right) \\
&= \frac{1}{2}(1-p)^n \left(\int_1^{\frac{1}{1-p}} \left(\frac{\mathrm{d}}{\mathrm{d}y} (y^n) \right) (1 - \frac{1}{y^{n-1}}) \mathrm{d}y \right) \\
&= \frac{1}{2}(1-p)^n \left(\int_1^{\frac{1}{1-p}} \left(n y^{n-1} \right) (1 - \frac{1}{y^{n-1}}) \mathrm{d}y \right) \\
&= \frac{1}{2}(1-p)^n \left(\int_1^{\frac{1}{1-p}} \left(n y^{n-1} - n \right) \mathrm{d}y \right) \\
&= \frac{1}{2}(1-p)^n \left(\left(\frac{1}{1-p} \right)^n - \frac{n}{1-p} - (1-n) \right) \\
&= \frac{1}{2} \left(1 - (1 + (n-1)p)(1-p)^{n-1} \right)
\end{aligned}
$$

An alternative way to see this is as follows.

A buyer's payoff if he participates in the auction is $V(1-p)^{n-1}$. So, the ex-ante payoff of each buyer is given by

$$(1-p)0 + pV(1-p)^{n-1} = pV(1-p)^{n-1}$$

The sum of the ex-ante payoffs of the n buyers is $pnV(1-p)^{n-1}$. It must be that the sum of the seller's payoff and all the buyers' payoffs equals the gains from trade. Trade happens if there it at least one bidder, which happens with probability $(1-(1-p)^n)$, so the gains from trade are $V(1-(1-p)^n)$.

$$\text{Seller's payoff} + pnV(1-p)^{n-1} = V(1-(1-p)^n)$$

Taking expectations, we get that the seller's expected payoff is

$$\frac{1}{2}\left(1-(1+(n-1)p)(1-p)^{n-1}\right)$$

exacly the same as derived earlier.

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