Chapter 2
Continuous Distributions

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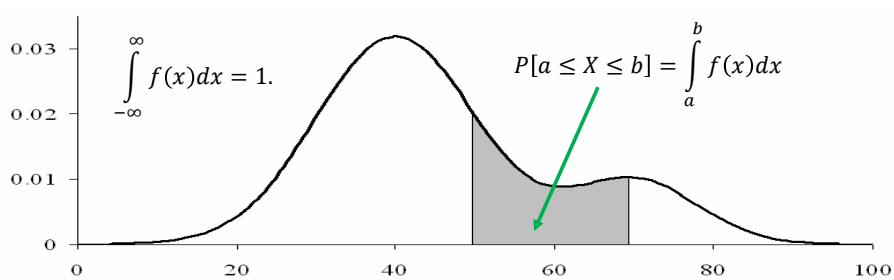
Continuous random variables: Review

• For a continuous random variable X the probability distribution is described by the probability density function $f(x)$, which has the following properties :

1. $f(x) \geq 0$.
2. $\int_{-\infty}^{\infty} f(x)dx = 1$.
3. $P[a \leq X \leq b] = \int_a^b f(x)dx$



Continuous random variables: Review

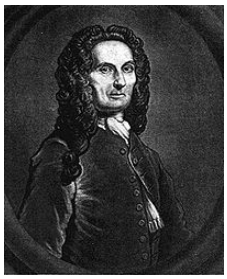
• Continuous distributions allow for a more elegant mathematical treatment.

• They are especially useful to *approximate* discrete distributions. Continuous distributions are used in this way in most economics and finance applications, both in the construction of models and in applying statistical techniques.

• Most used continuous distributions:
- Uniform
- Normal
- Exponential
- Weibull
- Gamma

The Uniform distribution from *a* to *b*

Abraham de Moivre (1667-1754)

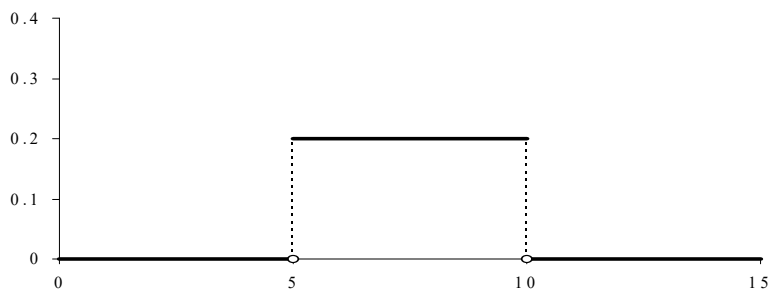


Uniform Distribution

- A random variable, X, is said to have a *Uniform distribution* from a to b if X is a continuous RV with probability density function $f(x)$:

$$f(x) = \begin{cases} \dfrac{1}{b-a} & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

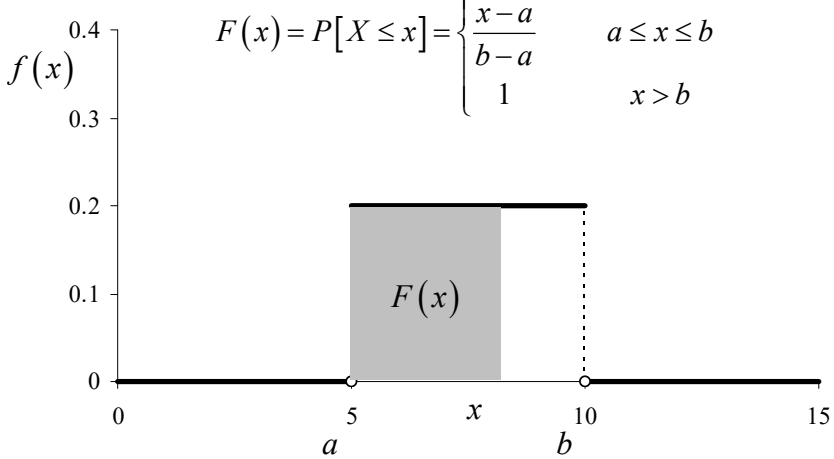
- PDF: Uniform distribution (from a to b)



Uniform Distribution: CDF

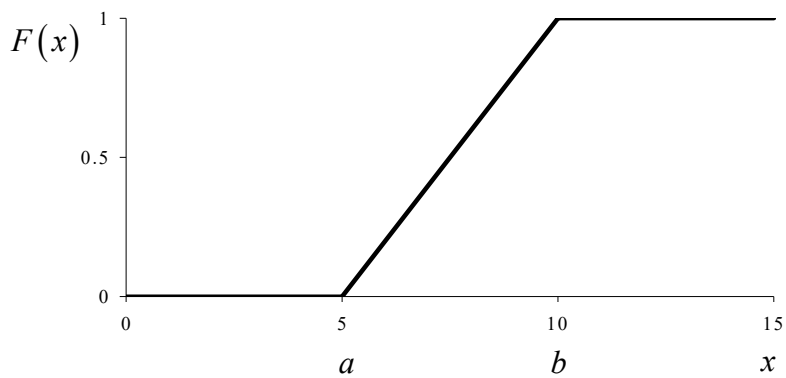
- The *CDF*, $F(x)$, of the uniform distribution from a to b:

$$F(x) = P[X \le x] = \begin{cases} 0 & x < a \\ \dfrac{x-a}{b-a} & a \le x \le b \\ 1 & x > b \end{cases}$$



Uniform Distribution function: Graph of $F(x)$

$$F(x) = P[X \le x] = \begin{cases} 0 & x < a \\ \dfrac{x-a}{b-a} & a \le x \le b \\ 1 & x > b \end{cases}$$



Uniform Distribution: Comments

• When a=0 and b=1, the distribution is called *standard uniform distribution,* which is a special case of the Beta distribution, with parameters (1,1)

• The uniform distribution is not commonly found in nature (but very common in casinos!). Because of its simplicity, it is used in theoretical work (for example, signals/private information are draw from a UD). It is used as prior distribution for binomial proportions in Bayesian statistics.

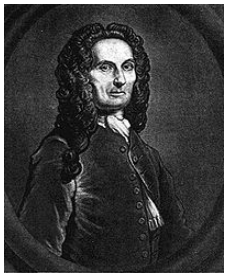
• It is also particularly useful for *sampling* from arbitrary distributions. A general method is the *inverse transform sampling method*, which uses the CDF of the target RV. This method is very useful in theoretical work.

Since simulations using this method require inverting the CDF of the target variable, alternative methods have been devised for the cases where the CDF is not known in closed form.

The Normal distribution

(Also called *Gaussian* or *Laplacian*)

Abraham de Moivre (1667-1754)



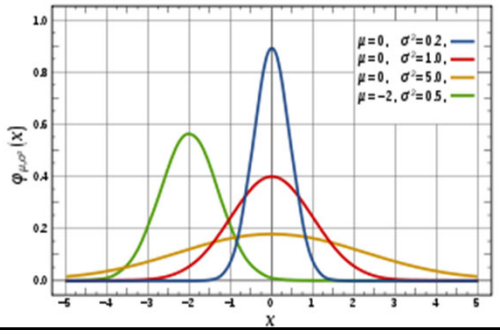
Carl Friedrich Gauss (1777–1855)

Pierre-Simon, m. de Laplace (1749–1827)

The Normal distribution: PDF

A random variable, X, is said to have a *normal distribution* with mean μ and standard deviation σ if X is a continuous random variable with probability density function $f(x)$:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$



Some characteristics of the Normal distribution

1. Location of maximum point of $f(x)$: parameter μ.

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$f'(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} * \left(-\frac{(x-\mu)}{\sigma^2}\right) = 0$$

This equality holds when $\mu = x$.

Therefore, the point μ is an extremum point of $f(x)$. (In this case a maximum = the mode.)

Some characteristics of the Normal distribution

2. Inflection point of $f(x)$.

$$f''(x) = -\frac{1}{\sqrt{2\pi}\,\sigma^3} \frac{d}{dx} e^{-\frac{(x-\mu)^2}{2\sigma^2}} (x-\mu)$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma^5} e^{-\frac{(x-\mu)^2}{2\sigma^2}} (x-\mu)^2 - \frac{1}{\sqrt{2\pi}\,\sigma^3} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma^3} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \left[\frac{(x-\mu)^2}{\sigma^2} - 1\right] = 0$$

if $\frac{(x-\mu)^2}{\sigma^2} = 1$ or $\frac{x-\mu}{\sigma} = \pm 1$ i.e. $x = \mu \pm \sigma$

Thus the points $\mu - \sigma$, $\mu + \sigma$ are points of inflection of $f(x)$.

Some characteristics of the Normal distribution

3. The pdf $f(x)$ integrates to 1.

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = 1$$

Proof:

To evaluate $$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

Make the substitution $z = \dfrac{x-\mu}{\sigma}$ $\qquad dz = \dfrac{1}{\sigma} dx$

When $x = -\infty, z = -\infty$ and when $x = \infty, z = \infty$.

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz$$

Some characteristics of the Normal distribution

Consider evaluating $$\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz = c$$

Note:

$$c^2 = \left(\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz\right)\left(\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz\right) = \left(\int_{-\infty}^{\infty} e^{-z^2} dz\right)\left(\int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} du\right)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} e^{-\frac{u^2}{2}} du dz = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\left(\frac{z^2+u^2}{2}\right)} du dz$$

Make the change to polar coordinates (R, θ):

$z = R\sin(\theta)$ and $u = R\cos(\theta)$

Some characteristics of the Normal distribution

Hence $R^2 = z^2 + u^2$ and $\tan(\theta) = \frac{z}{u}$

or $R = \sqrt{z^2 + u^2}$ and $\theta = \tan^{-1}\left(\frac{z}{u}\right)$

Using the following result

$$\boxed{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(z,u)\,dz\,du = \int_{0}^{2\pi}\int_{0}^{\infty} f\left(R\sin(\theta), R\cos(\theta)\right) R\,dR\,d\theta}$$

$$c^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\left(\frac{z^2+u^2}{2}\right)}\,du\,dz =$$

$$= \int_{0}^{2\pi}\int_{0}^{\infty} e^{-\left(\frac{R^2}{2}\right)} R\,dR\,d\theta = \int_{0}^{2\pi}\left[-e^{-\frac{R^2}{2}}\right]_0^{\infty} d\theta = \int_{0}^{2\pi} 1\,d\theta = 2\pi$$

Some characteristics of the Normal distribution

and $$c = \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}}\,dz = \sqrt{2\pi}$$

or

$$1 = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\,dz = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dz$$

Normal distribution: Comments

• The normal distribution is often used to describe or approximate any variable that tends to cluster around the mean. It is the most assumed distribution in economics and finance: rates of return, growth rates, IQ scores, observational errors, etc.

• The central limit theorem (CLT) provides a justification for the normality assumption when n is large.

• The normal distribution is often named after Carl Gauss, who introduced it formally in 1809 to justify the method of least squares. One year later, Pierre Simon (marquis de Laplace) proved the first version of the CLT. (In some European countries it is called *Laplacian*.)

<u>Notation</u>: PDF: $x \sim N(\mu,\sigma^2)$ & CDF: $\Phi(x)$

The Exponential distribution

Georges Teissier (1900-1972, France)

P. V. Sukhatme (1911-1997, India)

Exponential distribution: Derivation

• Consider a continuous RV, X with the following properties:

1. $P[X \geq 0] = 1$, and
2. $P[X \geq a + b] = P[X \geq a]\ P[X \geq b]$ for all $a > 0$, $b > 0$.

Let X= lifetime of an object. Then, (1) and (2) are reasonable to assume if X = lifetime of an object that does not age. Or (1) and (2) are reasonable for an object that lives a long time and a and b are small.

• The second property implies:

$$\frac{P\left[X \geq a+b\right]}{P\left[X \geq a\right]} = P\left[X \geq a+b \,\middle|\, X \geq a\right] = P\left[X \geq b\right]$$

$\Rightarrow$ Given the object has lived to age a, the probability that it lives b additional units is the same as if it was born at age a. Past history has no effect on x.

Exponential distribution: Derivation

Let $F(x) = P[X \leq x]$ and $G(x) = P[X \geq x]$.
Since X is a continuous RV $\Rightarrow G(x) = 1 - F(x)$

The two properties can be written in terms of $G(x)$:

1. $G(0) = 1$, and
2. $G(a + b) = G(a)\ G(b)$ for all $a > 0$, $b > 0$.

We can show that the only continuous function, $G(x)$, that satisfies (1) and (2) is the exponential function.

Exponential distribution: Derivation

From property 2 we can conclude

$$G(2a) = G(a + a) = G(a)G(a) = [G(a)]^2$$

Using induction

$$G(na) = G(a + \cdots + a) = G(a) \ldots G(a) = [G(a)]^n$$

Hence putting $a = 1$. $G(n) = \left[G(1)\right]^n$

Also putting $a = 1/n$. $G(1) = \left[G\left(\frac{1}{n}\right)\right]^n$ or $G\left(\frac{1}{n}\right) = \left[G(1)\right]^{1/n}$

Finally putting $a = 1/m$.

$$G\left(\frac{n}{m}\right) = \left[G\left(\frac{1}{m}\right)\right]^n = \left[\left[G(1)\right]^{\frac{1}{m}}\right]^n = \left[G(1)\right]^{\frac{n}{m}}$$

Exponential distribution: Derivation

Since $G(x)$ is continuous $\Rightarrow$ $G(x) = [G(1)]^x$ for all $x \geq 0$.

Recall: $0 \leq G(1) \leq 1$ and $G(0) = 1$, $G(\infty) = 0$.

If $G(1) = 0$ then $G(x) = 0$ for all $x > 0$ and $G(0) = 0$ if G is continuous. (a contradiction)

If $G(1) = 1$ then $G(x) = 1$ for all $x > 0$ and $G(\infty) = 1$ if G is continuous. (a contradiction)

$\Rightarrow G(1) \neq 0, 1$ and $0 < G(1) < 1$

Let $\lambda = -\ln(G(1))$, then $G(1) = e^{-\lambda}$

Thus $G(x) = \left[G(1)\right]^x = \left[e^{-\lambda}\right]^x = e^{-\lambda x}$

Exponential distribution: Derivation

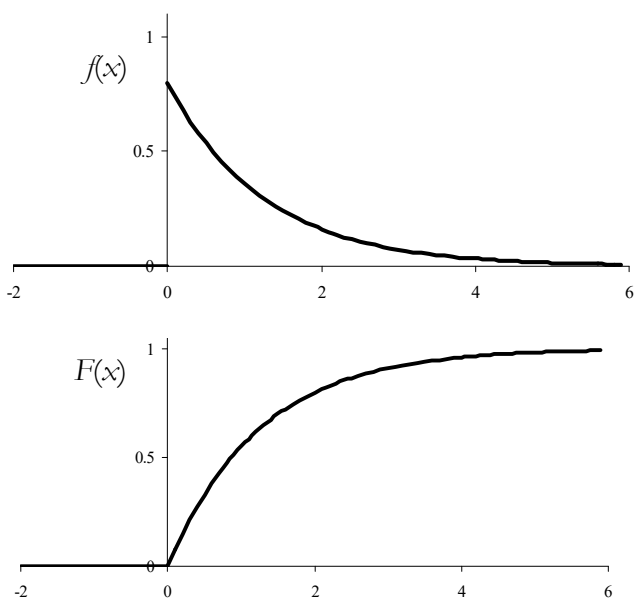
Finally $F(x) = 1 - G(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & x \geq 0 \end{cases}$

To find the density of X we use:

$$f(x) = F'(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

A continuous random variable with this density function is said to have the *exponential distribution* with parameter λ.

Exponential distribution: pdf and CDF



Exponential distribution: Alternative Derivation

Let $F(x) = P[X \le x]$ be the CDF of the exponential RV, X.
Then, $P[X \ge 0] = 1$ implies that $F(0) = 0$.

Also $P[x \le X \le x + dx \mid X \ge x] = \lambda\, dx$ implies

$$P\left[x \le X \le x + dx \middle| X \ge x\right] = \frac{P\left[x \le X \le x + dx\right]}{P\left[X \ge x\right]}$$

$$= \frac{F(x + dx) - F(x)}{1 - F(x)} = \lambda dx$$

$$\text{or} \quad \frac{F(x+dx) - F(x)}{dx} = \frac{dF(x)}{dx} = \lambda\left[1 - F(x)\right]$$

Exponential distribution: Alternative Derivation

Let's solve the ODE for the unknown F: $\dfrac{dF}{dx} = \lambda\left[1 - F\right]$

$$\frac{1}{1-F}\, dF = \lambda\, dx \qquad \text{or} \qquad \int \frac{1}{1-F}\, dF = \lambda \int dx$$

$$\text{and} \quad -\ln(1-F) = \lambda x + c \qquad \ln(1-F) = -\lambda x - c$$

$$\text{and} \quad 1 - F = e^{-\lambda x - c} \quad \text{or} \quad F = F(x) = 1 - e^{-\lambda x - c}$$

Now using the fact that $F(0) = 0$.

$$F(0) = 1 - e^{-c} = 0 \ \text{ implies } \ e^{-c} = 1 \ \text{ and } c = 0$$

$$\text{Thus} \quad F(x) = 1 - e^{-\lambda x} \ \text{ and } \ f(x) = F'(x) = \lambda e^{-\lambda x}$$

This shows that X has an *exponential distribution* with parameter λ.

Exponential distribution: Comments

• The exponential distribution is often used to model the failure time of manufactured items in production lines. Typical example is light bulbs.

• If X denotes the time to failure of a light bulb, then the exponential distribution says that the probability of survival of the bulb decays exponentially fast. More precisely:

$$P(X > x) = \lambda \, exp(-\lambda x)$$

<u>Note</u>: The bigger the value of λ, the faster the decay. This indicates that for large λ, the average time of failure of the bulb is smaller. This is indeed true. We will show later that $E(X) = 1/\lambda$.

We say X has an *exponential distribution* with parameter λ.

Exponential distribution: Comments

• The exponential distribution is also used a distribution for waiting times between *events* occurring uniformly in time. Typical example is stock trading intervals.

• An interesting feature of the exponential distribution is its memoryless property –the distribution "forgets" the past.

The Weibull distribution

A model for the lifetime of objects that *do* age.

Waloddi Weibull (Sweden, 1887-1979)

The Weibulll distribution

Recall the properties of continuous RV, X, that lead to the Exponential distribution. namely

1. $P[X \geq 0] = 1$, and
2. $P[x \leq X \leq x + dx | X \geq x] = \lambda dx$ for all $x > 0$ and small dx.

Suppose that the second property is replaced by:

2. $P[x \leq X \leq x + dx | X \geq x] = \lambda(x)dx = \alpha x^{\beta - 1}dx$

for all $x > 0$ and small dx.

Again, let X= lifetime of an object. Now, (2) implies that the object *does* age. If it has lived up to time x, the chance that it dies in the small interval x to $x + dx$ depends both on the length of that interval, dx, and its age x.

Derivation of the Weibulll distribution

A continuous random variable, X satisfies the following properties:

1. $P[X \geq 0] = 1$, and
2. $P[x \leq X \leq x + dx \mid X \geq x] = \alpha x^{\beta -1} dx$ for all $x > 0$ and small dx.

Let $F(x) = P[X \leq x]$ be the CDF of the random variable, X.
Then $P[X \geq 0] = 1$ implies that $F(0) = 0$.
Also $P[x \leq X \leq x + dx \mid X \geq x] = \alpha x^{\beta -1} dx$ implies

$$P\left[x \leq X \leq x + dx \middle| X \geq x\right] = \frac{P\left[x \leq X \leq x + dx\right]}{P\left[X \geq x\right]}$$

$$= \frac{F\left(x + dx\right) - F\left(x\right)}{1 - F\left(x\right)} = \alpha x^{\beta - 1} dx$$

$$\text{or} \quad \frac{F\left(x + dx\right) - F\left(x\right)}{dx} = \frac{dF\left(x\right)}{dx} = \alpha x^{\beta - 1}\left[1 - F\left(x\right)\right]$$

Derivation of the Weibulll distribution

We can solve the ODE, for the unknown F: $\dfrac{dF}{dx} = \alpha x^{\beta - 1}\left[1 - F\right]$

$$\frac{1}{1 - F} dF = \alpha x^{\beta - 1} dx \qquad \text{or} \qquad \int \frac{1}{1 - F} dF = \alpha \int x^{\beta - 1} dx$$

$$\text{and} \quad -\ln\left(1 - F\right) = \alpha \frac{x^{\beta}}{\beta} + c \qquad \ln\left(1 - F\right) = -\frac{\alpha}{\beta} x^{\beta} - c$$

$$1 - F = e^{-\frac{\alpha}{\beta} x^{\beta} - c} \quad \text{or} \quad F\left(x\right) = 1 - e^{-\frac{\alpha}{\beta} x^{\beta} - c}$$

again $F(0) = 0$ implies $c = 0$.

Thus $\quad F\left(x\right) = 1 - e^{-\frac{\alpha}{\beta} x^{\beta}}$

and $\quad f\left(x\right) = F'\left(x\right) = \alpha x^{\beta - 1} e^{-\frac{\alpha}{\beta} x^{\beta}} \quad x \geq 0$

Derivation of the Weibulll distribution

and $f(x) = F'(x) = \alpha x^{\beta-1} e^{-\frac{\alpha}{\beta}x^{\beta}} \quad x \geq 0$

F(*X*) is called the *Weibull distribution* with parameters α, β.

Special cases:

When β = 1, we have the exponential distribution.

When β = 2, it mimics the Rayleigh distribution (used in telecoms)

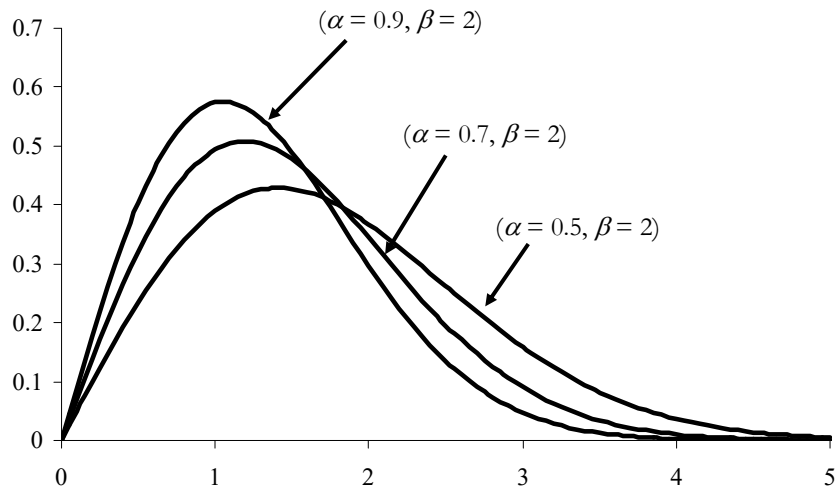
When β = 3.5, it is similar to the normal distribution.

Popular Application: Time to failure

Let *X* measure *time-to-failure* and follow the Weibull distribution. Then, the *failure rate* is proportional to a power of time. The *shape* parameter, β, is that power plus one: $\beta < 1$ ("infant mortality phenomenon"); $\beta=1$ (constant failure); and $\beta > 1$ (aging).

The Weibulll distribution: pdf

- The Weibull density, *f*(*x*)



The Gamma distribution

An important family of distributions

Karl Pearson (1857–1936)

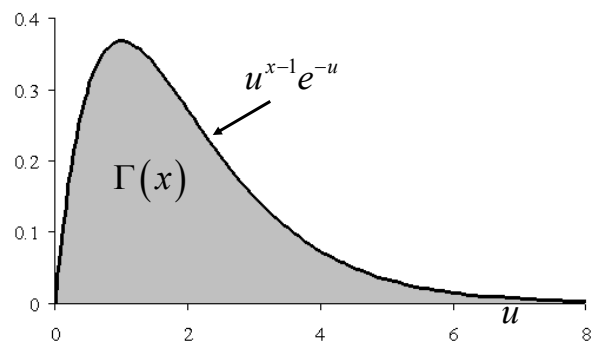
Pierre-Simon, m. de Laplace (1749–1827)

The Gamma Function, $\Gamma(x)$

An important function in mathematics, due to Euler (1729) to interpolate factorials. Current formulation and notation due to Legendre.

The Gamma function is defined for $x \geq 0$ by

$$\Gamma(x) = \int_0^\infty u^{x-1} e^{-u} du$$



Properties of the Gamma Function, $\Gamma(x)$

1. $\Gamma(1) = 1$

$$\Gamma(1) = \int_0^\infty e^{-u} du = \left[-e^{-u}\right]_0^\infty = -e^{-\infty} - \left(-e^{-0}\right) = 1$$

2. $\Gamma(x+1) = x\Gamma(x) \qquad x > 0.$

$$\Gamma(x+1) = \int_0^\infty u^x e^{-u} du = \int u^x e^{-u} du \Big|_0^\infty$$

We will use integration by parts to evaluate $\int u^x e^{-u} du$

$$\int u^x e^{-u} du = \int w dv = wv - \int v dw$$

where $w = u^x$ and $dv = e^{-u} du$, hence $dw = xu^{x-1} dx, v = -e^{-u}$

Thus $\int u^x e^{-u} du = -u^x e^{-u} + x \int u^{x-1} e^{-u} du$

Properties of the Gamma Function, $\Gamma(x)$

and $\int_0^\infty u^x e^{-u}\,du = -u^x e^{-u}\Big|_0^\infty + x\int_0^\infty u^{x-1}e^{-u}\,du$

or $\Gamma(x+1) = 0 + x\Gamma(x)$

3. $\Gamma(n) = (n-1)!$ for any positive integer n

using $\Gamma(x+1) = x\Gamma(x)$ and $\Gamma(1) = 1$

$$\Gamma(2) = 1\Gamma(1) = 1 = 1!$$
$$\Gamma(3) = 2\Gamma(2) = 2 = 2!$$
$$\Gamma(4) = 3\Gamma(3) = 3\cdot 2 = 3!$$
$$\vdots$$

Properties of the Gamma Function, $\Gamma(x)$

4. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$\Gamma(x) = \int_0^\infty u^{x-1}e^{-u}\,du$ thus $\Gamma\left(\frac{1}{2}\right) = \int_0^\infty u^{-\frac{1}{2}}e^{-u}\,du$

Recall the density for the normal distribution

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

If $\mu = 0$ and $\sigma = 1$ then

$$f(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

Now $1 = \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx = 2\int_0^\infty\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx = \int_0^\infty\frac{\sqrt{2}}{\sqrt{\pi}}e^{-\frac{x^2}{2}}dx$

Properties of the Gamma Function, $\Gamma(x)$

Thus $\int_0^\infty e^{-\frac{x^2}{2}}dx = \frac{\sqrt{\pi}}{\sqrt{2}}$ Make the substitution $u = \frac{x^2}{2}$.

Hence $x = \sqrt{2}u^{\frac{1}{2}}$ and $du = xdx$ or $dx = \frac{1}{x}du = \frac{u^{-\frac{1}{2}}}{\sqrt{2}}du$.

Also when $x = 0, \infty$ then $u = 0, \infty$

$$\frac{\sqrt{\pi}}{\sqrt{2}} = \int_0^\infty e^{-\frac{x^2}{2}}dx = \int_0^\infty e^{-u}\frac{u^{-\frac{1}{2}}}{\sqrt{2}}dx = \frac{1}{\sqrt{2}}\Gamma\left(\tfrac{1}{2}\right)$$

and $\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$.

Properties of the Gamma Function, $\Gamma(x)$

5. $\Gamma\left(n+\tfrac{1}{2}\right) = \frac{(2n)!}{n!2^{2n}}\sqrt{\pi}$

Using $\Gamma(x+1) = x\Gamma(x)$.

$$\Gamma\left(\tfrac{3}{2}\right) = \tfrac{1}{2}\Gamma\left(\tfrac{1}{2}\right) = \tfrac{1}{2}\sqrt{\pi}.$$

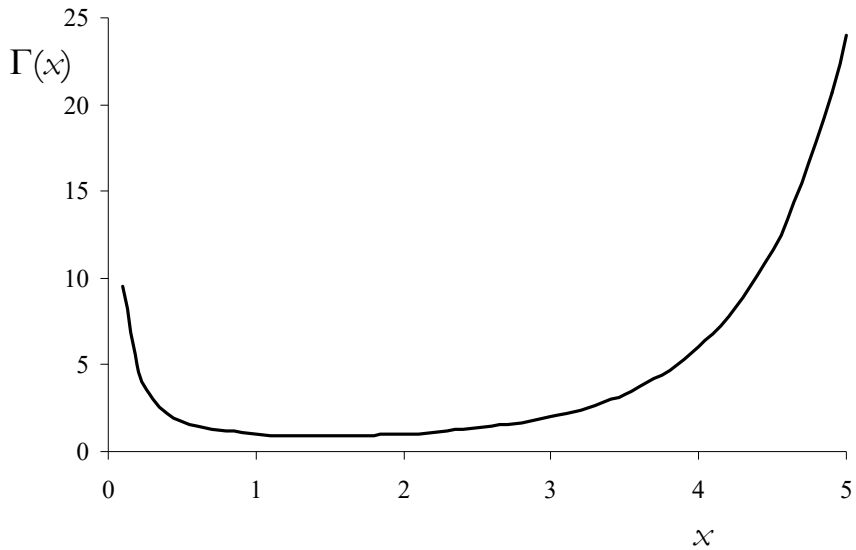
$$\Gamma\left(\tfrac{5}{2}\right) = \tfrac{3}{2}\Gamma\left(\tfrac{3}{2}\right) = \tfrac{3}{2}\tfrac{1}{2}\sqrt{\pi}.$$

$$\Gamma\left(\tfrac{7}{2}\right) = \tfrac{5}{2}\Gamma\left(\tfrac{5}{2}\right) = \tfrac{5}{2}\tfrac{3}{2}\tfrac{1}{2}\sqrt{\pi}.$$

$$\Gamma\left(\tfrac{2n+1}{2}\right) = \Gamma\left(n+\tfrac{1}{2}\right) = \tfrac{2n-1}{2}\cdots\tfrac{5}{2}\tfrac{3}{2}\tfrac{1}{2}\sqrt{\pi}$$

$$= \frac{(2n)!}{2^n n!2^n}\sqrt{\pi} = \frac{(2n)!}{n!2^{2n}}\sqrt{\pi}$$

Gamma Function, $\Gamma(x)$: Graph



The Gamma distribution

Let the continuous random variable X have density function:

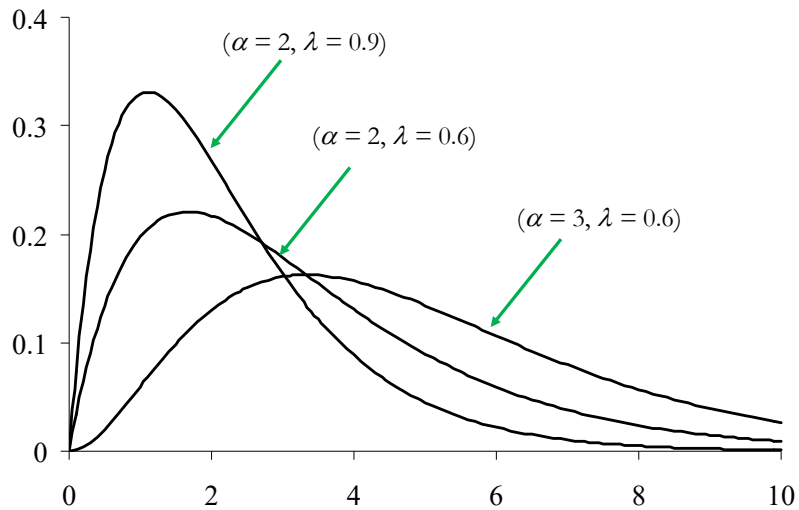
$$f(x) = \begin{cases} \dfrac{\lambda^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

where $\alpha, \lambda > 0$ and $\Gamma(\alpha)$ is the gamma function evaluated at α.

Then, X is said to have a *Gamma distribution* with parameters α and λ., denoted as $X \sim \text{Gamma}(\alpha, \lambda)$ or $\Gamma(\alpha, \lambda)$.

We can think of the Gamma distribution as a sum of α (independent) exponential distributions.

The Gamma distribution: Graph



The Gamma Distribution: Comments

1. The set of gamma distributions is a *family* of distributions (parameterized by α and λ).

$$f(x) = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} \qquad x \geq 0$$

2. Contained within this family are other distributions:

   <u>The Exponential distribution</u> – when $\alpha = 1$, the gamma distribution becomes the exponential distribution with parameter λ. It is used to model time to failure or waiting times between events occurring uniformly in time.

   <u>The Chi-square distribution</u> – when $\alpha = \nu/2$ and $\lambda = ½$, the gamma distribution becomes the chi-square (χ^2) distribution with ν degrees of freedom. Later, we will see that sum of squares of independent N(0,1) variates have a chi-square distribution, with degrees of freedom given by the number of independent terms in the sum of squares.

The Gamma Distribution: Comments

The Exponential distribution:

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

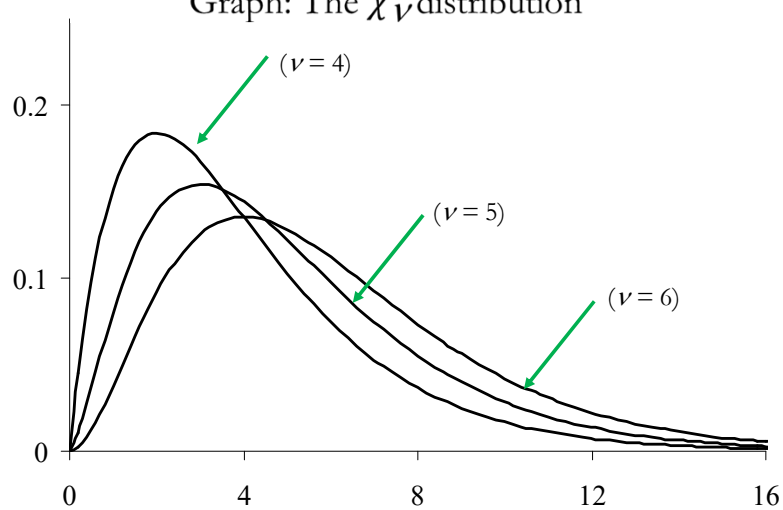
The Chi-square, χ^2, distribution with ν *degrees of freedom* (or just χ^2_ν):

$$f(x) = \begin{cases} \dfrac{\left(\frac{1}{2}\right)^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} x^{\frac{\nu}{2}-1} e^{-\frac{1}{2}x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$= \begin{cases} \dfrac{1}{2^{\frac{\nu}{2}}\Gamma\left(\frac{\nu}{2}\right)} x^{\frac{\nu-2}{2}} e^{-\frac{x}{2}} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

The Gamma Distribution: Comments

Graph: The χ^2_ν distribution



The Gamma Distribution: Comments

3. Convergence to a Normal distribution.
   When α is large, the gamma distribution converges to a normal distribution with $\mu = \alpha\lambda^{-1}$ and $\sigma^2 = \alpha\lambda^{-2}$.
4. Summation property. If $x_i \sim \Gamma(\alpha_\iota, \lambda)$ and independent
   $$\Rightarrow \Sigma_i x_i \sim \Gamma(\Sigma_i \alpha_\iota, \lambda)$$
5. Scaling. If $x \sim \Gamma(\alpha, \lambda) \quad \Rightarrow kx \sim \Gamma(\alpha, k\lambda)$
6. Typically used to model waiting times.
7. Other related distributions:
   a. **<u>Inverse Gamma distribution</u>**
      If X has a $\Gamma(\alpha, \lambda)$ distribution, then $1/X$ has an *inverse-gamma distribution* with parameters α and λ^{-1}.
   b. **<u>The Beta distribution</u>**
      If X and Y are independently distributed $\Gamma(\alpha_1, \lambda)$ and $\Gamma(\alpha_2, \lambda)$ respectively, then $X/(X + Y)$ has a *beta distribution* with parameters α_1 and α_2.

The Gamma Distribution: Application of properties - χ_N^2

• Suppose $x_i \sim \chi_1^2$ –or $x_i \sim \Gamma(½, ½)$. Suppose they are *i.i.d.* We want to know the distribution of the sum of $\Sigma_i x_i = x_1 + ... + x_N$.

Summation property. If $x_i \sim iid\, \Gamma(\alpha_\iota, \lambda) \qquad \Rightarrow \Sigma_i x_i \sim \Gamma(\Sigma_i \alpha_\iota, \lambda)$

Then, $\Sigma_i x_i \sim \Gamma(\Sigma_i ½, ½) = \Gamma(N/2, ½)$ or χ_N^2.

<u>Note</u>: The degrees of freedom tell us the number of independent χ_1^2 variables we are adding.

If interested in the mean of the x_i's, then using the scaling property:

$$\bar{x} = \sum_{i=1}^{N} \frac{x_i}{N} \sim \Gamma(\frac{N}{2}, \frac{1}{2N})$$

The Gamma Distribution: Application of properties - Exponential

Suppose $x_i \sim Exp(\lambda)$ –or $x_i \sim \Gamma(1, \lambda)$. Suppose they are *i.i.d.*

The parameter $\hat{\theta} = 1/\hat{\lambda}$ can be estimated by $\bar{x}$, where

$$\bar{x} = \sum_{i=1}^{N} \frac{x_i}{N}$$

We want to derive the distribution of $\hat{\theta}$.

$\hat{\theta}$ is a scaled sum of n independent $\Gamma(1, \lambda)$ RV. That is,

$$\hat{\theta} \sim \Gamma(N, \lambda/N)$$