Lecture 16
Unit Root Tests

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Autoregressive Unit Root

• A shock is usually used to describe an unexpected change in a variable or in the value of the error terms at a particular time period.

• When we have a stationary system, effect of a shock will die out gradually. But, when we have a non-stationary system, effect of a shock is permanent.

• We have two types of non-stationarity. In an AR(1) model we have:
- Unit root: $|\Phi_1| = 1$: homogeneous non-stationarity
- Explosive root: $|\Phi_1| > 1$: explosive non-stationarity

• In the last case, a shock to the system become more influential as time goes on. It can never be seen in real life. We will not consider them.

Autoregressive Unit Root

• Consider the AR(p) process:

$\phi(L) y_t = \mu + \varepsilon_t$ where $\phi(L) = 1 - \phi_1 L^1 - L^2\phi_2 - .... - \phi_p L^p$

As we discussed before, if one of the r_j's equals 1, $\Phi(1)=0$, or

$$\phi_1 + \phi_2 + ... + \phi_p = 1$$

• We say y_t has a *unit root*. In this case, y_t is non-stationary.

Example: AR(1): $y_t = \mu + \phi_1 y_{t-1} + \varepsilon_t$ =>Unit root: Φ_1=1.

H_0 (y_t non-stationarity): Φ_1=1 (or, Φ_1-1 = 0)

H_1 (y_t stationarity): $\Phi_1 < 1$ (or, Φ_1-1 < 0)

• A *t-test* seems natural to test H_0. But, the ergodic theorem and MDS CLT do not apply: the *t*-statistic does not have the usual distributions.

Autoregressive Unit Root

• Now, let's reparameterize the AR(1) process. Subtract y_{t-1} from y_t:

$$\Delta y_t = y_t - y_{t-1} = \mu + (\phi_1 - 1) y_{t-1} + \varepsilon_t$$
$$= \mu + \alpha_0 y_{t-1} + \varepsilon_t$$

• Unit root test: H_0: $\alpha_0 = \Phi_1 - 1 = 0$ against H_1: $\alpha_0 < 0$.

• Natural test for H_0: *t-test* . We call this test the *Dickey-Fuller* (DF) test. But, what is its distribution?

• Back to the general, AR(p) process: $\phi(L) y_t = \mu + \varepsilon_t$

We rewrite the process using the *Dickey-Fuller reparameterization*::

$$\Delta y_t = \mu + \alpha_0 y_{t-1} + \alpha_1 \Delta y_{t-1} + \alpha_2 \Delta y_{t-2} + ... + \alpha_{p-1} \Delta y_{t-(p-1)} + \varepsilon_t$$

• Both AR(p) formulations are equivalent.

Autoregressive Unit Root – Testing

• AR(*p*) lag *Φ(L)*: $\phi(L) = 1 - \phi_1 L^1 - L^2\phi_2 - .... - \phi_p L^p$

• DF reparameterization:

$$(1-L) - \alpha_0 - \alpha_1(L - L^2) + \alpha_2(L^2 - L^3) - .... - \alpha_{p-1}(L^{p-1} - L^p)$$

• Both parameterizations should be equal. Then, $\Phi(1)=-\alpha_0$.
=> unit root hypothesis can be stated as H_0: $\alpha_0=0$.

Note: The model is stationary if $\alpha_0 < 0$ => natural H_1: $\alpha_0 < 0$.

• Under H_0: $\alpha_0=0$, the model is AR(*p*-1) stationary in Δy_t. Then, if y_t has a (single) unit root, then Δy_t is a stationary AR process.

• We have a linear regression framework. A *t-test* for H_0 is the *Augmented Dickey-Fuller* (ADF) test.

Autoregressive Unit Root – Testing: DF

• The *Dickey-Fuller* (DF) test is a special case of the ADF: No lags are included in the regression. It is easier to derive. We gain intuition from its derivation.

• From our previous example, we have:

$$\Delta y_t = \mu + (\phi - 1) y_{t-1} + \varepsilon_t = \mu + \alpha_0 y_{t-1} + \varepsilon_t$$

• If $\alpha_0 = 0$, system has a unit root:: $H_0: \alpha_0 = 0$

$$H_1: \alpha_0 < 0 \quad (|\alpha_0| < 0)$$

• We can test H_0 with a *t-test:* $t_{\phi=1} = \dfrac{\hat{\phi} - 1}{SE(\hat{\phi})}$

• There is another associated test with H_0, the ϱ-test:. $(T\text{-}1)(\hat{\phi} - 1)$.

Autoregressive Unit Root – Testing: DF

• Note that an OLS estimation of $\hat{\phi}$, under a unit root, involves taking averages of the integrated series, y_t. Recall that, by backward substitution, y_t:

$$y_t = y_{t-1} + \varepsilon_t = y_0 + \sum_{j=1}^{t} \varepsilon_j$$

⇒ each y_t is a result of a constant (y_0), plus a (partial) sum of errors.

• The mean of y_t (assuming y_0=0) involves a sum of partial sums:

$$\bar{y} = \frac{1}{T}\sum_{t=1}^{T} y_t = \frac{1}{T}\sum_{t=1}^{T} (\sum_{j=1}^{t} \varepsilon_j)$$

Then, the mean shows a different probabilistic behavior than the sum of stationary and ergodic errors. It turns out that the limit behavior of y when $\phi = 1$ is described by simple functionals of Brownian motion.

Review: Stochastic Calculus

• **Brownian motion** (*Wiener* process)

Preliminaries:

– Let the variable $W(t)$ be almost surely continuous, with $W(t=0)=0$.

– The change in a small interval of time Δt is ΔW.

– $N(\mu,v)$: usual normal distribution, mean=μ & variance= v.

Definition: The variable $W(t)$ follows a Wiener process if

– $W(0) = 0$

– $W(t)$ has continuous paths at t.

– $\Delta W = \varepsilon\sqrt{\Delta t}$, where $\varepsilon \sim N(0,1)$

– The values of ΔW for any 2 different (non-overlapping) periods of time are independent.

Review: Stochastic Calculus – Wiener process

• A Brownian motion is a continuous stochastic process, but not differentiable ("*jagged path*") and with unbounded variation. ("A *continuous random walk*").

Notation: W(t), W(t, ω), B(t).

Example: A partial sum: $W_T(r) = \frac{1}{\sqrt{T}}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + ... + \varepsilon_{[Tr]}); \quad r \in [0,1]$

• Summary: Properties of the Wiener processes $\{W(t)\}$ for $t \geq 0$.

– Let $N = T/\Delta t$, then $W(T) - W(0) = \Sigma_i \varepsilon_i \sqrt{\Delta t}$

Thus, $W(T) - W(0) = W(T) \sim N(0, T)$

$\Rightarrow E[W(T)]=0$ & $Var[W(T)]=T$

– $E[\Delta W] = 0$

– $Var[\Delta W] = \Delta t$ $\Rightarrow$ Standard deviation of ΔW is $\sqrt{\Delta t}$

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Review: Stochastic Process – Partial Sums

• **Partial Sum Process**

Let $\varepsilon_t \sim iid\ N(0, \sigma^2)$. For $r \in [0, 1]$, define the partial sum process:

$$X_T(r) = \frac{1}{T}\sum_{t=1}^{Tr} \varepsilon_t = \frac{1}{T} S_{[Tr]}; \quad r \in [0,1]$$

Example: Let T=20, r = {0.01,..., 0.1,...}

$r = 0.01$; $[Tr] = [20*0.01] = 0$ $\Rightarrow$ $X_{20}(0.01) = \frac{1}{20}\sum_{t=1}^{[20*.01]} \varepsilon_t = 0.$

$r = 0.1$; $[Tr] = [20*0.1] = 2$ $\Rightarrow$ $X_{20}(0.1) = \frac{1}{20}\sum_{t=1}^{[20*0.1]} \varepsilon_t = \frac{1}{T}(\varepsilon_1 + \varepsilon_2).$

=> $X_T(r)$ is a random step function. As T grows, the steps get smaller. $X_T(r)$ looks more like a Wiener process.

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Review: Stochastic Calculus – FCLT

• **Functional CLT** (Donsker's FCLT)

If ε_t satisfies some assumptions (stationarity, $E[|\varepsilon_t|^q < \infty$ for $q>2$), then

$$W_T(r) \xrightarrow{D} W(r),$$

where $W(r)$ is a standard Brownian motion for $r \in [0, 1]$.

Note: That is, sample statistics, like $W_T(r)$, do not converge to constants, but to functions of Brownian motions. Check Billingsley (1986).

• A CLT is a limit for one term of a sequence of partial sums $\{S_k\}$, Donsker's FCLT is a limit for the entire sequence $\{S_k\}$, instead of one term.

• Donsker's FCLT is an important extension to the CLT because it is tool to obtain additional limits, such as statistics with I(1) variables. 11

Review: Stochastic Calculus – CMT

• These extensions primarily follow from the continuous mapping theorem (CMT) and generalizations of the CMT.

• **Continuous Mapping Theorem** (CMT)

If $W_T(r) \xrightarrow{D} W(r)$ and $h(.)$ is a continuous functional on D[0,1], the space of all real valued functions on [0,1] that are right continuous at each point on [0,1] and have finite left limits, then

$$h(W_T(r)) \xrightarrow{D} h(W(r)) \quad \text{as } T \to \infty$$

Example: Let $h(Y_T(.)) = \int_0^1 Y_T(r)dr$

& $\sqrt{T}\, X_T(r) \xrightarrow{D} W(r)$.

Then, $h(\sqrt{T}\, X_T(r)) \xrightarrow{D} \int_0^1 W_T(r)dr$ 12

Review: Stochastic Calculus – FCLT (2)

• FCLT: If T is large, $W_T(.)$ is a good approximation to W(r); $r \in [0,1]$:

$$W(r) = \lim_{T\to\infty} \{W_T(r) = \sqrt{T}\, X_T(r)\}$$

Sketch of proof:

For a fixed $r\in[0,1]$: $W_T(r) = \sqrt{T} X_T(r) = \frac{1}{\sqrt{T}}\sum_{t=1}^{[Tr]} \varepsilon_t = \frac{\sqrt{[Tr]}}{\sqrt{T}}\left(\frac{1}{\sqrt{[Tr]}}\sum_{t=1}^{[Tr]}\varepsilon_t\right)$

Since, as $T\to\infty$, $\frac{\sqrt{[Tr]}}{\sqrt{T}} \to \sqrt{r}$ & $\left(\frac{1}{\sqrt{[Tr]}}\sum_{t=1}^{[Tr]}\varepsilon_t\right) \xrightarrow{d} N(0,\sigma^2)$.

Then, from Slutsky's theorem, we get (again, for a fixed r):

$W_T(r)/\sigma \xrightarrow{d} N(0,r) \equiv W(r)$ (r =1, $W_T(r)/\sigma \xrightarrow{d}$ N(0,1)≡ W(1).)

This result holds for any $r \in$ [0, 1]. We suspect it holds uniformly for $r\in$[0, 1]. In fact, the pdf of the sequence of random step functions $W_T(.)/\sigma$, defined on [0,1], $\xrightarrow{D}$ $W(.)$ (This is the FCLT.)

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Review: Stochastic Calculus – Example

• **Example**: $y_t = y_{t-1} + \varepsilon_t$. Get distribution of $(\mathbf{X'X}/T^2)^{-1}$ for y_t.

$$T^{-2}\sum_{t=1}^{T}(y_{t-1})^2 = T^{-2}\sum_{t=1}^{T}[\sum_{i=1}^{t-1}\varepsilon_{t-i} + y_0]^2 = T^{-2}\sum_{t=1}^{T}[S_{t-1}+y_0]^2$$

$$= T^{-2}\sum_{t=1}^{T}[(S_{t-1})^2 + 2y_0 S_{t-1} + y_0^2]$$

$$= \sigma^2\sum_{t=1}^{T}\left(\frac{S_{t-1}}{\sigma\sqrt{T}}\right)^2 T^{-1} + 2y_0\sigma T^{-1/2}\sum_{t=1}^{T}\left(\frac{S_{t-1}}{\sigma\sqrt{T}}\right)T^{-1} + T^{-1}y_0^2$$

$$= \sigma^2\sum_{t=1}^{T}\int_{(t-1)/T}^{t/T}\left(\frac{1}{\sigma\sqrt{T}}S_{[Tr]}\right)^2 dr + 2y_0\sigma T^{-1/2}\sum_{t=1}^{T}\int_{(t-1)/T}^{t/T}\left(\frac{1}{\sigma\sqrt{T}}S_{[Tr]}\right)dr + T^{-1}y_0^2$$

$$= \sigma^2\int_0^1 W_T(r)^2 dr + 2y_0\sigma T^{-1/2}\int_0^1 W_T(r)\; dr + T^{-1}y_0^2$$

$$\xrightarrow{d} \quad \sigma^2\int_0^1 W(r)^2 dr, \qquad T\to\infty.$$

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Review: Stochastic Calculus – Example

• We can also derive the additional results for $y_t = y_{t-1} + \varepsilon_t$.

$$T^{-3/2}\sum_{t=1}^{T} y_{t-1} = T^{-3/2}\sum_{t=1}^{T}[S_{t-1}+y_0]$$

$$= \sigma\sum_{t=1}^{T}\frac{S_{t-1}}{\sigma\sqrt{T}}T^{-1}+T^{-3/2}y_0$$

$$= \sigma\sum_{t=1}^{T}\int_{(t-1)/T}^{t/T}\left(\frac{1}{\sigma\sqrt{T}}S_{[Tr]}\right)dr+T^{-3/2}y_0^2$$

$$= \sigma\int_0^1 X_T(r)dr+T^{-3/2}y_0^2 \xrightarrow{d} \sigma\int_0^1 W(r)dr \qquad (\text{as } T\to\infty).$$

Similarly,

$$T^{-1}\sum_{t=1}^{T} y_{t-1}\varepsilon_{t-1} \xrightarrow{d} \sigma^2\int_0^1 W(r)dW$$

(Intuition: think of ε_t as $dW(t)$.)

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Review: Stochastic Calculus – Ito's Theorem

• The integral w.r.t a Brownian motion, given by Ito's theorem (integral):
$\int f(t,\omega)\, dB = \sum f(t_{\underline{k}},\omega)\, \Delta B_k$ where $t_{\underline{k}} \in [t_k, t_{k+1})$ as $t_{k+1} - t_k \to 0$.

As we increase the partitions of $[0,T]$, the sum $\xrightarrow{p}$ to the integral.

• Ito's Theorem result: $\int B(t,\omega)\, dB(t,\omega) = B^2(t,\omega)/2 - t/2$.

Example: $$\int_0^1 W(t)dW(t) = \frac{1}{2}[W(1)^2 - W(0)^2 - 1] = \frac{1}{2}[W(1)^2 - 1]$$

Intuition: We think of ε_t as $dW(t)$. Then, $\Sigma_{k=0 \text{ to } t}\, \varepsilon_k$, which corresponds to $\int_{0\text{to}(t/T)} dW(s)=W(s/T)$ (for $W(0)=0$).

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Autoregressive Unit Root – DF Test

• **Case 1.** We continue with $y_t = y_{t-1} + \varepsilon_t$. Using OLS, we estimate ϕ:

$$\hat{\phi} = \frac{\sum_{t=1}^{T} y_t y_{t-1}}{\sum_{t=1}^{T} y_{t-1}^2} = \frac{\sum_{t=1}^{T} (y_{t-1} + \Delta y_{t-1}) y_{t-1}}{\sum_{t=1}^{T} y_{t-1}^2} = 1 + \frac{\sum_{t=1}^{T} y_{t-1} \Delta y_{t-1}}{\sum_{t=1}^{T} y_{t-1}^2}$$

• This implies for the normalized bias:

$$T(\hat{\phi} - 1) = T \frac{\sum_{t=1}^{T} y_{t-1} \Delta y_{t-1}}{\sum_{t=1}^{T} y_{t-1}^2} = \frac{\sum_{t=1}^{T} (y_{t-1} / \sqrt{T})(\varepsilon_t / \sqrt{T})}{\frac{1}{T} \sum_{t=1}^{T} (y_{t-1} / \sqrt{T})^2}$$

• From the way we defined $W_T(.)$, we can see that y_t/sqrt(T) converges to a Brownian motion. Under H_0, y_t is a sum of white noise errors.

Autoregressive Unit Root – DF: Distribution

• Using previous results, we get the DF's asymptotic distribution under H_0 (see Billingley (1986) and Phillips (1987), for details):

$$T(\hat{\phi} - 1) \xrightarrow{d} \frac{\int_0^1 W(t) dW(t)}{\int_0^1 W(t)^2 dt}$$

• Using Ito's integral, we have

$$T(\hat{\phi} - 1) \xrightarrow{d} \frac{1}{2} \frac{W(1)^2 - 1}{\int_0^1 W(t)^2 dt}$$

<u>Note</u>: $W(1)$ is a N(0,1). Then, $W(1)^2$ is just a $\chi^2(1)$ RV.

Autoregressive Unit Root – DF: Intuition

$$T(\hat{\phi}-1)\xrightarrow{d}\frac{1}{2}\frac{W(1)^2-1}{\int_0^1 W(t)^2\,dt}$$

• Contrary to the stationary model the denominator of the expression for the OLS estimator –i.e., $(1/T)\Sigma_t x_t^2$– does not converge to a constant *a.s.*, but to a RV strongly correlated with the numerator.

• Then, the asymptotic distribution is not normal. It turns out that the limiting distribution of the OLS estimator is highly skewed, with a long tail to the left.

• The normalized bias has a well defined limiting distribution. It can be used as a test of H_0.

Autoregressive Unit Root – Consistency

• In the AR(1) model, $y_t = \phi\, y_{t-1} + \varepsilon_t$, we show the consitency of the OLS estimator under H_0: ϕ=1.

$$\hat{\phi}-1=\frac{\sum_{t=1}^{T} y_{t-1}\varepsilon_t}{\sum_{t=1}^{T} y_{t-1}^2}=\frac{T^{-2}\sum_{t=1}^{T} y_{t-1}\varepsilon_t}{T^{-2}\sum_{t=1}^{T} y_{t-1}^2}$$

$$\xrightarrow{p}\left(\sigma^2\int_0^1 W(r)^2\,dr\right)^{-1}\times\left(T^{-1}\sigma^2\int_0^1 W(r)\,dW\right)=0$$

Thus, $\hat{\phi}\xrightarrow{p}1$.

Autoregressive Unit Root – Testing: DF

• Back to the AR(1) model. The *t*-test statistic for H_0: α_0=0 is given by

$$t_{\phi=1} = \frac{\hat{\phi} - 1}{SE(\hat{\phi})} = \frac{\hat{\phi} - 1}{\sqrt{s^2 (\sum_{t=2}^{T} y_{t-1}^2)^{-1}}}$$

• The $_{\phi=1}$ test is a one-sided left tail test. Q: What is its distribution ?

<u>Note</u>: If $\{y_t\}$ is stationary (i.e., $|\varphi| < 1$) then it can be shown

$$\sqrt{T}(\hat{\phi} - \phi) \xrightarrow{d} N(0, (1 - \phi^2)).$$

• Then, under H_0, the asymptotic distribution of $t_{\phi=1}$ is N(0,1). That is, under H_0:

$$\hat{\phi} \xrightarrow{d} N(1,0)$$

which we know is not correct, since y_t is not stationary and ergodic.

Autoregressive Unit Root – Testing: DF

• Recall that under H_0 we got.

$$T(\hat{\phi} - 1) \xrightarrow{D} \frac{\int_0^1 W(r) dW(r)}{\int_0^1 W(r)^2 dr}$$

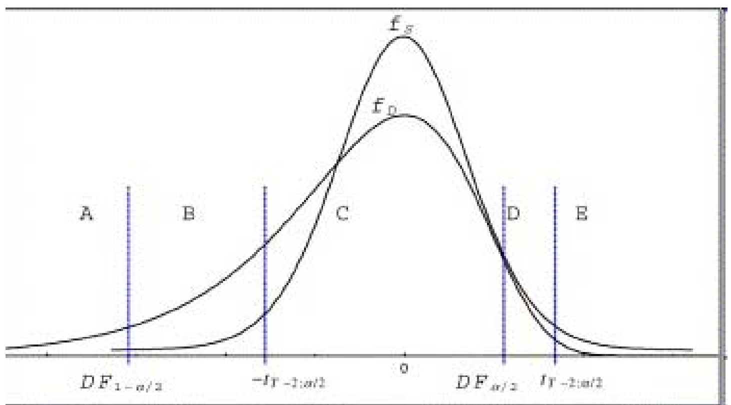
• A little bit of algebra delivers the distribution for the t-test, t_ϕ.

$$t_{\phi=1} = \frac{(\hat{\phi} - 1)}{SE(\hat{\phi})} = \frac{T(\hat{\phi} - 1)}{(s_T^2)^{1/2}} \{T^{-2} \sum_{t=1}^{T} y_{t-1}^2\}^{1/2} \xrightarrow{D} \frac{\int_0^1 W(r) dW(r)}{\left(\int_0^1 W(r)^2 dr\right)^{1/2}}$$

where *W(r)* denotes a standard Brownian motion (Wiener process) defined on [0,1].

Autoregressive Unit Root – DF: Intuition

• The limiting distribution of $t_{\Phi=1}$ is the DF distribution,. Below, we graph the DF distribution relative to a Normal. It is skewed, with a long tail to the left.



Autoregressive Unit Root – Testing: DF

• $\hat{\phi}$ is not asymptotically normally distributed and $t_{\Phi=1}$ is not asymptotically standard normal.

• The DF distribution, which does not have a closed form representation, is *non-standard*. The quantiles of the DF distribution can be numerically approximated or simulated.

• The DF distribution has been tabulated under different scenarios.

1) no constant: $y_t = \Phi\, y_{t-1} + \varepsilon_t$.

2) with a constant: $y_t = \mu + \Phi\, y_{t-1} + \varepsilon_t$.

3) with a constant and a trend: $y_t = \mu + \delta\, t + \Phi\, y_{t-1} + \varepsilon_t$.

• The tests with no constant are not used in practice.

Autoregressive Unit Root – Testing: DF

• Critical values of the DF_t test under different scenarios.

Table 1: Selected Critical Values of Unit-Root Test Statistics

sample size	Probability			
T	0.01	0.025	0.05	0.10
	Model without constant			
100	−2.60	−2.24	−1.95	−1.61
250	−2.58	−2.23	−1.95	−1.62
500	−2.58	−2.23	−1.95	−1.62
∞	−2.58	−2.23	−1.95	−1.62
	Model with constant			
100	−3.51	−3.17	−2.89	−2.58
250	−3.46	−3.14	−2.88	−2.57
500	−3.44	−3.13	−2.87	−2.57
∞	−3.43	−3.12	−2.86	−2.57
	Model with time trend			
100	−4.04	−3.73	−3.45	−3.15
250	−3.99	−3.69	−3.43	−3.13
500	−3.98	−3.68	−3.42	−3.13
∞	−3.96	−3.66	−3.41	−3.12

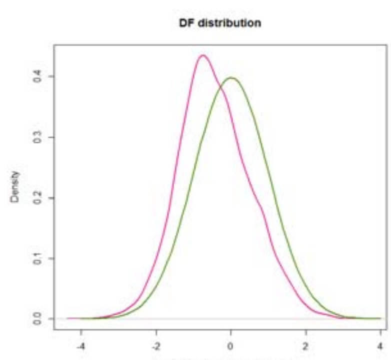
Autoregressive Unit Root – DF: Case 1

Example: We want to simulate the DF distribution in R:

$$t_{\phi=1} \xrightarrow{d} \frac{W(1)^2 - 1}{2\{\int_0^1 W(t)^2 dt\}^{1/2}}$$

```
T <- 500
reps <- 10000
DFstats <- rep(NA,reps)
for (i in 1:reps){
  u <- rnorm(T)
  W <- 1/sqrt(T)*cumsum(u)
  DFstats[i] <- (W[T]^2-1)/(2*sqrt(mean(W^2)))
}
plot(density(DFstats),lwd=2,col=c("deeppink2"))
xax <- seq(-4,4,by=.1)
lines(xax,dnorm(xax),lwd=2,col=c("chartreuse4"))

CriticalValues <- sort(DFstats)[c(0.01,0.05,0.1)*reps]
> CriticalValues
[1] -2.563463 -1.919828 -1.581882
```



Autoregressive Unit Root – DF: Case 2

• **Case 2.** DF with a constant term in DGP: $y_t = \mu + \phi y_{t-1} + \varepsilon_t$

Hypotheses to be tested:

$$H_0: \phi = 1, \mu = 0 \Rightarrow Y_t \sim I(1) \text{ without drift}$$
$$H_1: |\phi| < 1 \qquad \Rightarrow Y_t \sim I(0) \text{ with zero mean}$$

• This formulation is appropriate for non-trending economic and financial series like interest rates, exchange rates and spreads.

• The test statistics $t_{\Phi=1}$ and $(T\text{-}1)(\hat{\phi} - 1)$ are computed from the estimation of the AR(1) model with a constant. That is,

$$(\hat{\beta} - \beta) = \begin{pmatrix} \hat{\mu} - 0 \\ \hat{\phi} - 1 \end{pmatrix} = \begin{pmatrix} T & \sum_{t=1}^{T} y_{t-1} \\ \sum_{t=1}^{T} y_{t-1} & \sum_{t=1}^{T} y_{t-1}^2 \end{pmatrix}^{-1} \begin{pmatrix} \sum_{t=1}^{T} \varepsilon_t \\ \sum_{t=1}^{T} y_{t-1}\varepsilon_t \end{pmatrix}.$$

Autoregressive Unit Root – DF: Case 2

• Under $H_0: \Phi = 1, \mu = 0,$ the asymptotic distributions of these test statistics are influenced by the presence, but not the coefficient value, of the constant in the test regression:

$$T(\hat{\phi} - 1) \xrightarrow{D} \frac{\int_0^1 W^{\mu}(r)dW(r)}{\int_0^1 W^{\mu}(r)^2 dr}$$

$$t_{\phi=1} \xrightarrow{D} \frac{\int_0^1 W^{\mu}(r)dW(r)}{\left(\int_0^1 W^{\mu}(r)^2 dr\right)^{1/2}} = \frac{W^{\mu}(1)^2 - W^{\mu}(0)^2 - 1}{2\left(\int_0^1 W^{\mu}(r)^2 dr\right)^{1/2}}$$

where $W^{\mu}(r) = W(r) - \int_0^1 W(r)dr$ is a de-meaned Wiener process, i.e., $\int_0^1 W^{\mu}(r)dr = 0$

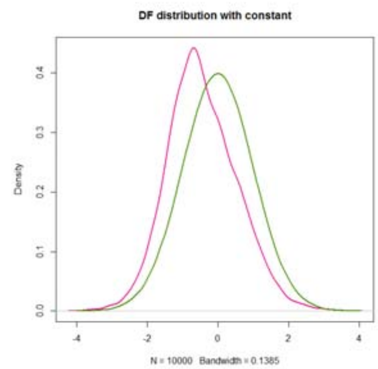
• Inclusion of a constant pushes the tests' distributions to the left.

Autoregressive Unit Root – DF: Case 2

Example: Now, we modify the DF test.

```
for (i in 1:reps){
  u <- rnorm(T)
  W <- 1/sqrt(T)*cumsum(u)
  W_mu <- W - mean(W)                              #de-mean W
  DFstats[i] <- (W_mu[T]^2-W_mu[1]^2-1)/(2*sqrt(mean(W_mu^2)))
}

plot(density(DFstats),lwd=2,col=c("deeppink2"),
main="DF distribution with constant")
xax <- seq(-4,4,by=.1)
lines(xax,dnorm(xax),lwd=2,col=c("chartreuse4"))
```



Autoregressive Unit Root – DF: Case 3

• **Case 3.** With constant and trend term in the DGP.
The test regression is $y_t = \mu + \delta t + \phi y_{t-1} + \varepsilon_t$

• It includes a constant and deterministic time trend to capture the deterministic trend under the alternative. The hypotheses to be tested:

$$H_0: \phi = 1, \delta = 0 \quad \Rightarrow Y_t \sim I(1) \text{ with drift}$$
$$H_1: |\phi| < 1 \quad \Rightarrow Y_t \sim I(0) \text{ with deterministic time trend}$$

• This formulation is appropriate for trending time series like asset prices or the levels of macroeconomic aggregates like real GDP. The test statistics $t_{\Phi=1}$ and $T(\hat{\phi} - 1)$ are computed from the above regression.

Autoregressive Unit Root – DF: Case 3

• Again, under $H_0 : \Phi = 1, \delta = 0,$ the asymptotic distributions of both test statistics are influenced by the presence of the constant and time trend in the test regression. Now, we have:

$$(T-1)(\hat{\phi}-1) \xrightarrow{d} \frac{\int_0^1 W^{\tau}(r)\,dW(r)}{\int_0^1 W^{\tau}(r)^2\,dr}$$

$$t_{\phi=1} \xrightarrow{d} \frac{\int_0^1 W^{\tau}(r)\,dW(r)}{\left(\int_0^1 W^{\tau}(r)^2\,dr\right)^{1/2}} = \frac{W^{\tau}(1)^2 - W^{\tau}(0)^2 - 1}{2\left(\int_0^1 W^{\tau}(r)^2\,dr\right)^{1/2}}$$

where $W^{\tau}(r) = W(r) - (4-6r)\int_0^1 W(s)\,ds - (12r-6)\int_0^1 sW(s)\,ds$ is

a de - meaned & de - trended Wiener process.

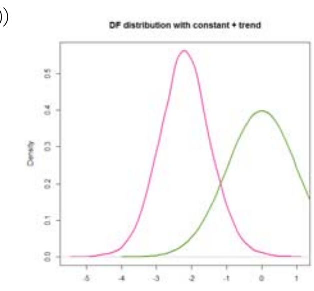
Autoregressive Unit Root – DF: Case 3

• Again, the inclusion of a constant and trend in the test regression further shifts the distributions of $t_{\Phi=1}$ and $T(\hat{\phi}-1)$ to the left.

Example: Now, we modify the DF test.

```
for (i in 1:reps){
  u <- rnorm(T)
  W <- 1/sqrt(T)*cumsum(u)
  W_tau <- W - (4-6*s)*mean(W) - (12*s-6)*mean(s*W)    #de-mean & de-trend W
DFstats[i] <- (W_tau[T]^2-W_tau[1]^2-1)/(2*sqrt(mean(W_tau^2)))
}

CriticalValues <- sort(DFstats)[c(0.01,0.05,0.1)*reps]
>CriticalValues
[1] -3.930855 -3.390544 -3.118117
```



Autoregressive Unit Root – DF: Remarks

• Which version of the three main variations of the test should be used is not a minor issue. The decision has implications for the size and the power of the unit root test.

• For example, an incorrect exclusion of the time trend term leads to bias in the coefficient estimate for Φ, leading to size distortions and reductions in power.

• Since the normalized bias $(T-1)(\hat{\phi}-1)$ has a well defined limiting distribution that does not depend on nuisance parameters it can also be used as a test statistic for the null hypothesis $H_0 : \Phi = 1$.

Autoregressive Unit Root – Testing: ADF

• Back to the general, AR(*p*) process. We can rewrite the equation as the *Dickey-Fuller reparameterization*:

$$\Delta y_t = \mu + \alpha_0 y_{t-1} + \alpha_1 \Delta y_{t-1} + \alpha_2 \Delta y_{t-2} + ... + \alpha_{p-1} \Delta y_{t-(p-1)} + \varepsilon_t$$

• The model is stationary if $\alpha_0 < 0$ => natural H_1: $\alpha_0 < 0$.

• Under H_0: $\alpha_0 = 0$, the model is AR(*p*-1) stationary in Δy_t. Then, if y_t has a (single) unit root, then Δy_t is a stationary AR process.

• The *t*-test for H_0 from OLS estimation is the Augmented Dickey-Fuller (ADF) test.

• Similar situation as the DF test, we have a non-normal distribution.

Autoregressive Unit Root – Testing: ADF

• The asymptotic distribution is:

$$T\hat{\alpha}_0 \xrightarrow{d} (1-\alpha_1-\alpha_2-\cdots-\alpha_{k-1})\,DF_\alpha$$

$$ADF = \frac{\hat{\alpha}_0}{s(\hat{\alpha}_0)} \to DF_t.$$

The limit distributions DF_α and DF_t are non-normal. They are skewed to the left, and have negative means.

• First result: $\hat{\alpha}_0$ converges to its true value (of zero) at rate T; rather than the conventional rate of sqrt(T) => *superconsistency*.

• Second result: The t-statistic for $\hat{\alpha}_0$ converges to a non-normal limit distribution, but does not depend on α.

Autoregressive Unit Root – Testing: ADF

• The ADF distribution has been extensively tabulated under the usual scenarios: 1) no constant; 2) with a constant; and 3) with a constant and a trend. The first scenario is seldom used in practice.

• Like in the DF case, which version of the three main versions of the test should be used is not a minor issue. A wrong decision has potential size and power implications.

• One-sided H_1: the ADF test rejects H_0 when ADF < c; where c is the critical value from the ADF table.

Note: The SE($\hat{\alpha}_0$)= s $sqrt[\Sigma y_{t-1}^2]$, the usual (homoscedastic) SE. But, we could be more general. Homoskedasticity is not required.

Autoregressive Unit Root – Testing: ADF

• We described the test with an intercept. Another setting includes a linear time trend:

$$\Delta y_t = \mu_1 + \mu_2 t + \alpha_0 y_{t-1} + \alpha_1 \Delta y_{t-1} + ... + \alpha_{p-1} \Delta y_{t-(p-1)} + \varepsilon_t$$

• Natural framework when the alternative hypothesis is that the series is stationary about a linear time trend.

• If t is included, the test procedure is the same, but different critical values are needed. The ADF test has a different distribution when t is included.

Autoregressive Unit Root – DF: Example 1

• Monthly USD/GBP exchange rate, S_t, (1800-2013), T=2534.

- **Case 1** (no constant in DGP):

Variable	DF	Parameter Estimate	Standard Error	t Value	Pr > |t|
x1	1	**0.99934**	0.00061935	1613.52	<.0001

$(T\text{-}1)(\hat{\phi}\text{-}1)$=2533*(1-.99934)=-1.67178

Critical values at 5% level: −8.0 for T=500

−8.1 for T=∞

• Cannot reject H_0 => Take 1st differences (changes in S_t) to model the series.

- With a constant, $\hat{\phi}$ = 0.99631. Similar conclusion (Critical values at 5% level: −14.0 for T=500 and−14.1 for T=∞): Model changes in S_t.

Autoregressive Unit Root – DF: Example 2

• Monthly US Stock Index (1800-2013), T=2534.

- No constant in DGP (unusual case, called Case 1): $y_t = \phi y_{t-1} + \varepsilon_t$

Variable	DF	Parameter Estimate	Standard Error	t Value	Pr > |t|
x1	1	**1.00298**	0.00088376	1134.90	<.0001

$(T\text{-}1)(\hat{\phi}\text{-}1)$=2533*(.00298)=7.5483 (positive, not very interesting)

Critical values at 5% level: −8.0 for T=500

−8.1 for T=∞

• Cannot reject H_0 => Take 1st differences (returns) to model the series.

- With a constant, $\hat{\phi}$ = 1.00269. Same conclusion.

Autoregressive Unit Root – DF-GLS

• Elliott, Rothenberg and Stock (1992) (ERS) study *point optimal invariant tests* (POI) for unit roots. An invariant test is a test invariant to nuisance parameters.

• In the unit root case, we consider invariance to the parameters that capture the stationary movements around the unit roots -i.e., the parameters to AR(*p*) parameters.

• Consider: $y_t = \mu + \delta t + u_t, \quad u_t = \rho u_{t-1} + \varepsilon_t.$

• ERS show that the POI test for a unit root against ρ = ρ* is:

$$M_T = \frac{s^2_{\rho=1}}{s^2_{\rho=\rho^*}}$$

Autoregressive Unit Root – DF-GLS

- $$M_T = \frac{s^2_{\rho=1}}{s^2_{\rho=\rho^*}}$$

where s^2_ρ is the variances residuals from the GLS estimation under both scenarios for ρ, ρ = 1 and ρ = ρ*, respectively:

• The critical value for the test will depend on c where $\rho^* = 1 - c/T$.

Note: When dynamics are introduced in the u_t equation, Δu_t lags, the critical values have to be adjusted.

• In practice ρ* is unknown. ERS suggest different values for different cases. Say, c=-13.5, for the case with a trend, gives a power of 50%.

Autoregressive Unit Root – DF-GLS

• It turns out that if we instead do the GLS-adjustment and then perform the ADF-test (without allowing for a mean or trend) we get approximately the POI-test. ERS call this test the DF-GLS$_t$ test.

• The critical values depend on T.

T	1%	5%
50	-3.77	-3.19
100	-3.58	-3.03
200	-3.46	-2.93
500	-3.47	-2.89
∞	-3.48	-2.89

• Check ERS for critical values for other scenarios.

Autoregressive Unit Root – Testing: ADF

• Important issue: lag p

- Check the specification of the lag length p. If p is too small, then the remaining serial correlation in the errors will bias the test. If p is too large, then the power of the test will suffer.

- Ng and Perron (1995) suggestion:
(1) Set an upper bound p_{max} for p.
(2) Estimate the ADF test regression with $p = p_{max}$.
If $|t_{\alpha(p)}| > 1.6$ set $p = p_{max}$ and perform the ADF test.
Otherwise, reduce the lag length by one. Go back to (1)

- Schwert's (1989) rule of tumb for determining p_{max}:

$$p_{\max} = \left[12 \left(\frac{T}{100} \right)^{1/4} \right]$$

Autoregressive Unit Root – Testing: PP Test

• The Phillips-Perron (PP) unit root tests differ from the ADF tests mainly in how they deal with serial correlation and heteroskedasticity in the errors.

• The ADF tests use a parametric autoregression to approximate the ARMA structure of the errors in the test regression. The PP tests correct the DF tests by the bias induced by the omitted autocorrelation.

• These modified statistics, denoted Z_t and Z_δ, are given by

.

$$Z_t = \sqrt{\frac{\hat{\sigma}^2}{\hat{\lambda}^2}} t_{\hat{\alpha}_0} - \frac{1}{2}\left(\frac{\hat{\lambda}^2 - \hat{\sigma}^2}{\hat{\lambda}^2}\right)\left(\frac{T\left(SE\left(\hat{\alpha}_0\right)\right)}{\hat{\sigma}^2}\right)$$

$$Z_\delta = T\hat{\alpha}_0 - \frac{1}{2}\frac{T^2\left(SE\left(\hat{\alpha}_0\right)\right)}{\hat{\sigma}^2}\left(\hat{\lambda}^2 - \hat{\sigma}^2\right)$$

Autoregressive Unit Root – Testing: PP Test

• The terms $\hat{\sigma}^2$ and $\hat{\lambda}$ are consistent estimates of the variance parameters:

$$\sigma^2 = \lim_{T \leftarrow \infty} T^{-1} \sum_{t=1}^{T} E\left(\varepsilon_t^2\right)$$

$$\lambda^2 = \lim_{T \to \infty} \sum_{t=1}^{T} E\left(\frac{1}{T} \sum_{t=1}^{T} \varepsilon_t^2\right)$$

• Under H_0: $\alpha_0 = 0$, the PP Z_t and $Z_{\alpha 0}$ statistics have the same asymptotic distributions as the DF t-statistic and normalized bias statistics.

• PP tests tend to be more powerful than the ADF tests. But, they can severe size distortions (when autocorrelations of ε_t are negative) and they are more sensitive to model misspecification (order of ARMA model).

Autoregressive Unit Root – Testing: PP Test

• Advantage of the PP tests over the ADF tests:

- Robust to general forms of heteroskedasticity in the error term ε_t.
- No need to specify a lag length for the ADF test regression.

Autoregressive Unit Root – Testing: Criticisms

• The ADF and PP unit root tests are very popular. They have been, however, widely criticized.

• Main criticism: Power of tests is low if the process is stationary but with a root close to the non-stationary boundary.

• For example, the tests are poor at distinguishing between $\phi=1$ or $\phi=0.976$, especially with small sample sizes.

• Suppose the true DGP is $y_t = 0.976\, y_{t-1} + \varepsilon_t$
=> H_0: $\alpha_0 = 0$ should be rejected.

Autoregressive Unit Root – Testing: Criticisms

• The ADF and PP unit root tests are known (from simulations) to suffer potentially severe finite sample power and size problems.

1. Power – Both tests are known to have low power against the alternative hypothesis that the series is stationary (or TS) with a large autoregressive root. (See, DeJong, et al, *J. of Econometrics,* 1992.)

2. Size – Both tests are known to have severe size distortion (in the direction of over-rejecting H_0) when the series has a large negative MA root. (See, Schwert, *JBES,* 1989*:* MA = -0.8 =>size = 100%!)

• One potential solution to these issues: Use a stationarity test (like KPSS test) under H_0.

Autoregressive Unit Root – Testing: KPSS

• A different test is the KPSS (Kwiatkowski, Phillips, Schmidt and Shin) Test (1992): It is stationary under H_0. It can be used to test whether we have a deterministic trend vs. stochastic trend:

$$H_0 : Y_t \sim I(0) \qquad \rightarrow \text{level (or trend) stationary}$$
$$H_1 : Y_t \sim I(1) \qquad \rightarrow \text{difference stationary}$$

• Setup

$$y_t = \mu + \delta t + r_t + u_t$$
$$r_t = r_{t-1} + \varepsilon_t$$

where $\varepsilon_t \sim WN(0,\sigma^2)$, uncorrelated with $u_t \sim WN$. Then,

H_0 (trend stationary): $\sigma^2=0$

H_0 (y_t (level) stationary): $\sigma^2=0$ & $\delta=0$.

Under H_1: $\sigma^2 \neq 0$, there is a RW in y_t.

Autoregressive Unit Root – Testing: KPSS

• Under some assumptions (normality, *i.i.d.* for u_t & ε_t), a one-sided LM test of the null that there is no random walk ($\varepsilon_t=0$, for all t) can be constructed with:

$$KPSS = T^{-2} \sum_{t=1}^{T} \frac{S_t^2}{s_u^2}$$

where s_u^2 is the variance of u_t, ("long run" variance) estimated as

$$s_u^2(l) = \frac{1}{T}\sum_{t=1}^{T} \hat{u}_t^2 + \frac{2}{T}\sum_{s=1}^{l} w(s,l) \sum_{t=s+1}^{T} \hat{u}_t \hat{u}_{t-s}$$

where $w(s,l)$ is a kernel function, for example, the Bartlett kernel. We also need to specify the number of lags, which should grow with T.

• Under H_0, $\hat{u}_t$ can be estimated by OLS.

Autoregressive Unit Root – Testing: KPSS

• Easy to construct. Steps:

1. Regress y_t on a constant and time trend. Get OLS residuals, $\hat{u}_t$.
2. Calculate the partial sum of the residuals: $S_t = \sum_{i=1}^{t} \hat{u}_i$
3. Compute the KPSS test statistic:

$$KPSS = T^{-2} \sum_{t=1}^{T} \frac{S_t^2}{s_u^2}$$

where s_u^2 is the estimate of the long-run variance of the residuals.

4. Reject H_0 when KPSS is large (the series wander from its mean).

• Asymptotic distribution of the test statistic is non-standard –it can be derived using Brownian motions, appealing to FCLT and CMT.

Autoregressive Unit Root – Testing: KPSS

• KPSS converges to different distribution, depending on whether the model is trend-stationary ($\delta \neq 0$), level-stationary ($\delta=0$), or zero-mean stationary ($\delta=0$, $\mu=0$).

• For example, if a constant is included ($\delta=0$): $KPSS \xrightarrow{D} \int_0^1 V(r)^2 \, dr$

where $V=W(r)-rW(1)$ is a standard Brownian bridge ($V(0)=V(1)=0$).

• If there is a constant and trend:

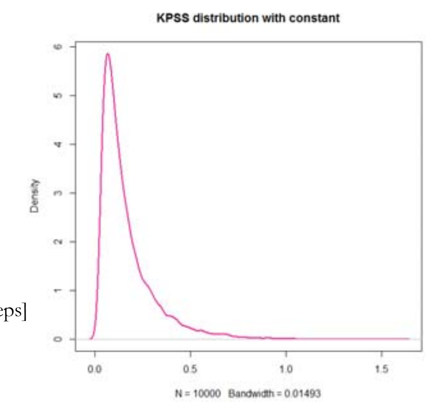
$$KPSS \xrightarrow{D} \int_0^1 [W(r) + r(2-3r)W(1) + 6r(r^2-1)\int_0^1 W(s)ds]^2 \, dr$$

• Very powerful test, but it has problems with structural breaks (say, volatility shifts).

Autoregressive Unit Root – KPSS Distribution

• For the δ=0 case, the KPSS distribution was approximated by MacNeill (1978). It can easily be simulated under H_0. For the case with only a constant:

```
T <- 500
reps <- 10000
KPSSstats <- rep(NA,reps)
s <- seq(0,1,length.out = T)
for (i in 1:reps){
    u <- rnorm(T)
    u_mu <- u - mean(u)
   s1 <- cumsum(u_mu)
   KPSSstats[i] <- (1/T^2)*sum(s1^2)
}
plot(density(KPSSstats),lwd=2,col=c("deeppink2"),
main="KPSS distribution with constant")
CriticalValues <- sort(KPSSstats)[c(0.90,0.95,0.99)*reps]
> CriticalValues
[1] 0.3426913 0.4525498 0.7229417
```



Autoregressive Unit Root – Structural Breaks

• A stationary time-series may look like non-stationary when there are structural breaks in the intercept or trend. The persistence is due to rare structural breaks.

• The unit root tests lead to false non-rejection of the null when we do not consider the structural breaks. A low power problem.

• A single known breakpoint was studied by Perron (*Econometrica,* 1989). Perron (1997) extended it to a case of unknown breakpoint.

• Perron considers the null and alternative hypotheses

H_0: $y_t = a_0 + y_{t-1} + \mu_1 D_P + \varepsilon_t$ ($y_t \sim ST$ with a jump)

H_1: $y_t = a_0 + a_2 t + \mu_2 D_L + \varepsilon_t$ ($y_t \sim TS$ with a jump)

Pulse break: $D_P = 1$ if $t = T_B + 1$ and zero otherwise,
Level break: $D_L = 0$ for $t = 1, \ldots, T_B$ and one otherwise.

Autoregressive Unit Root – Structural Breaks

•. Power of ADF tests: Rejection frequencies of ADF–tests

	Model: $a_0 = a_2 = 0.5$ and $\mu_2 = 10$		
	1% level	5% level	10% level
ADF–tests	0.004	0.344	0.714
	Model: $a_0 = a_2 = 0.5$ and $\mu_2 = 12$		
ADF–tests	0.000	0.028	0.264

• Observations:
- ADF tests are biased toward non-rejection of the non-stationary H_0.
- Rejection frequency is inversely related to the magnitude of the shift.

• Perron estimated values of the AR coefficient in the DF regression. They were biased toward unity and that this bias increased as the magnitude of the break increased.

Autoregressive Unit Root – Structural Breaks

• Perron suggestion: Running the following OLS regression:

$$y_t = a_0 + a_1 y_{t-1} + a_2 t + \mu_2 D_L + \sum_{i=1}^{p} \beta_i \Delta y_{t-i} + \varepsilon_t$$

H_0: a_1=1 => use t–ratio, DF unit root test.

• Perron shows that the asymptotic distribution of the t-statistic depends on the location of the structural break, $\lambda = T_B/T$. Critical values are supplied in Perron (1989) for different cases.

• Criticism: The breakpoint is known ("data mining") and only one (misspecification). Many tests incorporating endogenously determined breakpoints -see, Zivot and Andrews (1992), & Lumsdaine and Papell (1997)- and multiple breakpoints -see, Bai (1997). Check Perron (2005, *Handbook of Econometrics*) for a survey of literature.

Autoregressive Unit Root - Relevance

• We can always decompose a unit root process into the sum of a random walk and a stable process. This is known as the *Beveridge-Nelson* (1981) (BN) composition.

• Let y_t ~ I(1), r_t ~ RW and c_t ~I(0).

$$y_t = r_t + c_t.$$

Since c_t is stable it has a Wold decomposition:

$$(1 - L)y_t = \psi(L)\varepsilon_t$$

Then,

$$\begin{aligned}(1-L)y_t &= \psi(L)\varepsilon_t = \psi(1)\varepsilon_t + (\psi(L) - \psi(1))\varepsilon_t \\ &= \psi(1)\varepsilon_t + \psi(L)^*\varepsilon_t\end{aligned}$$

where $\psi(1)$=0. Then,

$$y_t = \psi(1)(1-L)^{-1}\varepsilon_t + \psi(L)^*(1-L)^{-1}\varepsilon_t = r_t + c_t$$

Autoregressive Unit Root - Relevance

• Usual finding in economics: Many time series have unit roots, see Nelson and Plosser (1982).

Example: Consumption, output, stock prices, interest rates, size, etc.

• But, sturcural breaks matter.

Table 1: Unit Root Tests and the Nelson and Plosser Data Set

	Model	# of Breaks	Unit Root	Stationary (with possible breaks)
Nelson and Plosser (1982)	ADF	0	13	1
Perron (1989)	Exogenous breaks	1	3	10
Zivot and Andrews (1992)	Endogenous Breaks	1	10	3
Lumsdaine and Papell (1997)	Endogenous Breaks	2	8	5

Autoregressive Unit Root - Relevance

• Usual finding in economics: Many time series have unit roots, see Nelson and Plosser (1982).

Example: Consumption, output, stock prices, interest rates, unemployment, size, compensation are usually I(1).

• Sometimes a linear combination of I(1) series produces an I(0). For example, (log consumption– log output) is stationary. This situation is called *cointegration*.

• Practical problems with cointegration:
- Asymptotics change completely.
- Not enough data to definitively say we have cointegration.