Lecture 16
Unit Root Tests

1

Autoregressive Unit Root

• A shock is usually used to describe an unexpected change in a variable or in the value of the error terms at a particular time period.

• When we have a stationary system, effect of a shock will die out gradually. But, when we have a non-stationary system, effect of a shock is permanent.

• We have two types of non-stationarity. In an AR(1) model we have:
- Unit root: $|\phi_1| = 1$: homogeneous non-stationarity
- Explosive root: $|\phi_1| > 1$: explosive non-stationarity

• In the last case, a shock to the system become more influential as time goes on. It can never be seen in real life. We will not consider them.

Autoregressive Unit Root

• Consider the AR(p) process:

$\phi(L)\,y_t = \mu + \varepsilon_t$ where $\phi(L) = 1 - \phi_1 L^1 - L^2\phi_2 - .... - \phi_p L^p$

As we discussed before, if one of the r_j's equals 1, $\Phi(1)=0$, or

$$\phi_1 + \phi_2 + ... + \phi_p = 1$$

• We say y_t has a *unit root*. In this case, y_t is non-stationary.

Example: AR(1): $y_t = \mu + \phi_1 y_{t-1} + \varepsilon_t \Rightarrow$ Unit root: $\phi_1 = 1$.
H_0 (y_t non-stationarity): $\phi_1 = 1$ (or, $\phi_1 - 1 = 0$)
H_1 (y_t stationarity): $\phi_1 < 1$ (or, $\phi_1 - 1 < 0$)

• A *t-test* seems natural to test H_0. But, the ergodic theorem and MDS CLT do not apply: the *t*-statistic does not have the usual distributions.

Autoregressive Unit Root

• Now, let's reparameterize the AR(1) process. Subtract y_{t-1} from y_t:

$$\Delta y_t = y_t - y_{t-1} = \mu + (\phi_1 - 1)\,y_{t-1} + \varepsilon_t$$
$$= \mu + \alpha_0 y_{t-1} + \varepsilon_t$$

• Unit root test: H_0: $\alpha_0 = \phi_1 - 1 = 0$
H_1: $\alpha_0 < 0$.

• Natural test for H_0: *t-test* . We call this test the *Dickey-Fuller* (DF) test. But, what is its distribution?

• Back to the general, AR(p) process: $\phi(L)\,y_t = \mu + \varepsilon_t$
We rewrite the process using the *Dickey-Fuller reparameterization*::

$$\Delta y_t = \mu + \alpha_0 y_{t-1} + \alpha_1 \Delta y_{t-1} + \alpha_2 \Delta y_{t-2} + ... + \alpha_{p-1} \Delta y_{t-(p-1)} + \varepsilon_t$$

• Both AR(p) formulations are equivalent.

Autoregressive Unit Root – Testing

• AR(*p*) lag *Φ(L)*: $\phi(L) = 1 - \phi_1 L^1 - L^2\phi_2 - \dots - \phi_p L^p$

• DF reparameterization:

$$(1 - L) - \alpha_0 - \alpha_1 (L - L^2) + \alpha_2 (L^2 - L^3) - \dots - \alpha_{p-1} (L^{p-1} - L^p)$$

• Both parameterizations should be equal. Then, $\Phi(1)=-\alpha_0$.
$\Rightarrow$ unit root hypothesis can be stated as H_0: $\alpha_0=0$.

<u>Note</u>: The model is stationary if $\alpha_0 < 0 \Rightarrow$ natural H_1: $\alpha_0 < 0$.

• Under H_0: $\alpha_0=0$, the model is AR(*p*-1) stationary in Δy_t. Then, if y_t has a (single) unit root, then Δy_t is a stationary AR process.

• We have a linear regression framework. A *t-test* for H_0 is the *Augmented Dickey-Fuller* (ADF) test.

Autoregressive Unit Root – Testing: DF

• The *Dickey-Fuller* (DF) test is a special case of the ADF: No lags are included in the regression. It is easier to derive. We gain intuition from its derivation.

• From our previous example, we have:

$$\Delta y_t = \mu + (\phi - 1) y_{t-1} + \varepsilon_t = \mu + \alpha_0 y_{t-1} + \varepsilon_t$$

• If $\alpha_0 = 0$, system has a unit root:: $H_0 : \alpha_0 = 0$

$$H_1 : \alpha_0 < 0 \quad (|\alpha_0| < 0)$$

• We can test H_0 with a *t-test:* $t_{\phi=1} = \dfrac{\hat{\phi} - 1}{SE(\hat{\phi})}$

• There is another associated test with H_0, the ϱ-test:. $(T\text{-}1)(\hat{\phi} - 1)$.

Review: Stochastic Calculus

• **Kolmogorov Continuity Theorem**
• If for all $T > 0$, there exist a, b, $\delta > 0$ such that:
• $E(|X(t_1, \omega) - X(t_2, \omega)|^a) \leq \delta\ |t_1 - t_2|^{(1 + b)}$
• Then $X(t, \omega)$ can be considered as a continuous stochastic process.

– Brownian motion is a continuous stochastic process.

– Brownian motion (*Wiener* process): $X(t, \omega)$ is almost surely continuous, has independent normal distributed (N(0,t-s)) increments and $X(t=0, \omega) = 0$ ("a continuous random walk").

Review: Stochastic Calculus – Wiener process

• Let the variable $z(t)$ be almost surely continuous, with $z(t=0)=0$.
• Define $N(\mu,v)$ as a normal distribution with mean μ and variance v.
• The change in a small interval of time Δt is Δz

• Definition: The variable $z(t)$ follows a Wiener process if
– $z(0) = 0$
– $\Delta z = \varepsilon\sqrt{\Delta t}$, where $\varepsilon \sim N(0,1)$
– It has continuous paths.
– The values of Δz for any 2 different (non-overlapping) periods of time are independent.

Notation: W(t), W(t, ω), B(t).

Example: $W_T(r) = \frac{1}{\sqrt{T}}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \ldots + \varepsilon_{[Tr]});\quad r \in [0,1]$

8

Review: Stochastic Process: Wiener process

• What is the distribution of the change in z over the next 2 time units?
The change over the next 2 units equals the sum of:
- The change over the next 1 unit (distributed as N(0,1)) plus
- The change over the following time unit --also distributed as N(0,1).
- The changes are independent.
- The sum of 2 normal distributions is also normally distributed.

Thus, the change over 2 time units is distributed as N(0,2).

• Properties of Wiener processes:
– Mean of Δz is 0
– Variance of Δz is Δt
– Standard deviation of Δz is $\sqrt{\Delta t}$
– Let $N=T/\Delta t$, then $z(T) - z(0) = \sum_{i=1}^{n} \varepsilon_i \sqrt{\Delta t}$

9

Review: Stochastic Calculus – Wiener process

Example: $W_T(r) = \frac{1}{\sqrt{T}}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \ldots + \varepsilon_{[Tr]}) = \frac{1}{\sqrt{T}} S_{[Tr]};\quad r \in [0,1]$

• If T is large, $W_T(.)$ is a good approximation to $W(r)$; $r \in [0,1]$, defined:

$$W(r) = \lim_{T\to\infty} W_T(r) \quad \Rightarrow E[W(r)] = 0$$
$$\Rightarrow Var[W(r)] = t$$

• Check Billingsley (1986) for the details behind the proof that $W_T(r)$ converges as a function to a continuous function $W(r)$.

• In a nutshell, we need
- ε_t satisfying some assumptions (stationarity, $E[|\varepsilon_t|^q < \infty$ for $q>2$, etc.)
- a FCLT (*Functional CLT*).
- a Continuous Mapping Theorem. (Similar to Slutzky's theorem).

10

Review: Stochastic Calculus – Wiener process

• **Functional CLT** (Donsker's FCLT)
If ε_t satisfies some assumptions, then

$$W_T(r) \xrightarrow{D} W(r),$$

where W(r) is a standard Brownian motion for $r \in [0, 1]$.

Note: That is, sample statistics, like $W_T(r)$, do not converge to constants, but to functions of Brownian motions.

• A CLT is a limit for one term of a sequence of partial sums $\{S_k\}$, Donsker's FCLT is a limit for the entire sequence $\{S_k\}$ instead of one term.

11

Review: Stochastic Calculus – Wiener process

• Example: $y_t = y_{t-1} + \varepsilon_t$ (Case 1). Get distribution of $(\mathbf{X'X}/T^2)^{-1}$ for y_t.

$$T^{-2}\sum_{t=1}^{T}(y_{t-1})^2 = T^{-2}\sum_{t=1}^{T}[\sum_{i=1}^{t-1}\varepsilon_{t-i} + y_0]^2 = T^{-2}\sum_{t=1}^{T}[S_{t-1} + y_0]^2$$

$$= T^{-2}\sum_{t=1}^{T}[(S_{t-1})^2 + 2y_0 S_{t-1} + y_0^2]$$

$$= \sigma^2\sum_{t=1}^{T}\left(\frac{S_{t-1}}{\sigma\sqrt{T}}\right)^2 T^{-1} + 2y_0\sigma T^{-1/2}\sum_{t=1}^{T}\left(\frac{S_{t-1}}{\sigma\sqrt{T}}\right)T^{-1} + T^{-1}y_0^2$$

$$= \sigma^2\sum_{t=1}^{T}\int_{(t-1)T}^{t/T}\left(\frac{1}{\sigma\sqrt{T}}S_{[Tr]}\right)^2 dr + 2y_0\sigma T^{-1/2}\sum_{t=1}^{T}\int_{(t-1)T}^{t/T}\left(\frac{1}{\sigma\sqrt{T}}S_{[Tr]}\right)dr + T^{-1}y_0^2$$

$$= \sigma^2\int_0^1 X_T(r)^2 dr + 2y_0\sigma T^{-1/2}\int_0^1 X_T(r)\; dr + T^{-1}y_0^2$$

$$\xrightarrow{d} \quad \sigma^2\int_0^1 W(r)^2 dr, \qquad T \to \infty.$$

12

Review: Stochastic Calculus – Ito's Theorem

• The integral w.r.t a Brownian motion, given by Ito's theorem (integral):
$\int f(t,\omega)\, dB = \sum f(t_{\underline{k}},\omega)\, \Delta B_k$ where $t_{k^*} \in [t_k,t_{k+1})$ as $t_{k+1} - t_k \to 0$.

As we increase the partitions of $[0,T]$, the sum $\to^p$ to the integral.

• But, this is a probability statement: We can find a sample path where the sum can be arbitrarily far from the integral for arbitrarily large partitions (small intervals of integration).

• You may recall that for a Rienman integral, the choice of t_{k^*} (at the start or at the end of the partition) is not important. But, for Ito's integral, it is important (at the start of the partition).

• Ito's Theorem result: $\int B(t,\omega)\, dB(t,\omega) = B^2(t,\omega)/2 - t/2$.

13

Autoregressive Unit Root – Testing: Intuition

• We continue with $y_t = y_{t-1} + \varepsilon_t$ (Case 1). Using OLS, we estimate ϕ:

$$\hat{\phi} = \frac{\sum_{t=1}^{T} y_t y_{t-1}}{\sum_{t=1}^{T} y_{t-1}^2} = \frac{\sum_{t=1}^{T} (y_{t-1} + \Delta y_{t-1}) y_{t-1}}{\sum_{t=1}^{T} y_{t-1}^2} = 1 + \frac{\sum_{t=1}^{T} y_{t-1} \Delta y_{t-1}}{\sum_{t=1}^{T} y_{t-1}^2}$$

• This implies:

$$T(\hat{\phi} - 1) = T\frac{\sum_{t=1}^{T} y_{t-1} \Delta y_{t-1}}{\sum_{t=1}^{T} y_{t-1}^2} = \frac{\sum_{t=1}^{T} (y_{t-1}/\sqrt{T})(\varepsilon_t/\sqrt{T})}{\frac{1}{T}\sum_{t=1}^{T} (y_{t-1}/\sqrt{T})^2}$$

• From the way we defined $W_T(.)$, we can see that y_t/sqrt(T) converges to a Brownian motion. Under H_0, y_t is a sum of white noise errors.

Autoregressive Unit Root – Testing: Intuition

• Intuition for distribution under H_0:

- Think of y_t as a sum of white noise errors.
- Think of ε_t as $dW(t)$.

Then, using Billingley (1986), we guess that $T(\hat{\phi}$-1) converges to

$$T(\hat{\phi}-1) \xrightarrow{d} \frac{\int_0^1 W(t)dW(t)}{\int_0^1 W(t)^2 dt}$$

• We think of ε_t as $dW(t)$. Then, $\Sigma_{k=0 \text{ to } t}\, \varepsilon_k$, which corresponds to $\int_{0\text{to}(t/T)} dW(s)=W(s/T)$ (for $W(0)=0$). Using Ito's integral, we have

$$T(\hat{\phi}-1) \xrightarrow{d} \frac{1}{2}\frac{W(1)^2-1}{\int_0^1 W(t)^2 dt}$$

Autoregressive Unit Root – Testing: Intuition

• $$T(\hat{\phi}-1) \xrightarrow{d} \frac{1}{2}\frac{W(1)^2-1}{\int_0^1 W(t)^2 dt}$$

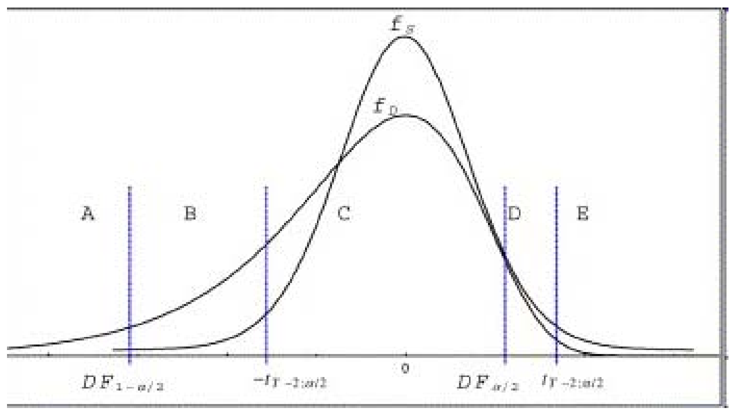
<u>Note</u>: $W(1)$ is a N(0,1). Then, $W(1)^2$ is just a $\chi^2(1)$ RV.

• Contrary to the stable model the denominator of the expression for the OLS estimator –i.e., $(1/T)\Sigma_t x_t^2$-- does not converge to a constant *a.s.*, but to a RV strongly correlated with the numerator.

• Then, the asymptotic distribution is not normal. It turns out that the limiting distribution of the OLS estimator is highly skewed, with a long tail to the left.

Autoregressive Unit Root – Testing: Intuition

• DF distribution relative to a Normal. It is skewed, with a long tail to the left.



Autoregressive Unit Root – Testing: DF

• Back to the AR(1) model. The *t*-test statistic for H_0: α_0=0 is given by

$$t_{\phi=1} = \frac{\hat{\phi}-1}{SE(\hat{\phi})} = \frac{\hat{\phi}-1}{\sqrt{s^2(\sum_{t=2}^{T} y_{t-1}^2)^2}}$$

• The test is a one-sided left tail test. If $\{y_t\}$ is stationary (i.e., $|\varphi| < 1$) then it can be shown

$$\sqrt{T}(\hat{\phi}-\phi) \xrightarrow{d} N(0,(1-\phi^2)).$$

• This means that under H_0, the asymptotic distribution of $t_{\phi=1}$ is N(0,1). That is, under H_0:

$$\hat{\phi} \xrightarrow{d} N(1,0)$$

which we know is not correct, since y_t is not stationary and ergodic.

Autoregressive Unit Root – Testing: DF

• Under H_0, y_t is not stationary and ergodic. The usual sample moments do not converge to fixed constants. Using the results discussed above, Phillips (1987) showed that the sample moments of y_t converge to random functions of Brownian motions. Under H_0:

$$(T-1)(\hat{\phi}-1) \xrightarrow{d} \frac{\int_0^1 W(r)\,dW(r)}{\int_0^1 W(r)^2\,dr}$$

$$t_{\phi=1} \xrightarrow{d} \frac{\int_0^1 W(r)\,dW(r)}{\left(\int_0^1 W(r)^2\,dr\right)^{1/2}}$$

where *W(r)* denotes a standard Brownian motion (Wiener process) defined on the unit interval.

Autoregressive Unit Root – Testing: DF

• $\hat{\phi}$ is not asymptotically normally distributed and $t_{\Phi=1}$ is not asymptotically standard normal.

• The limiting distribution of $t_{\Phi=1}$ is the DF distribution, which does not have a closed form representation. Then, quantiles of the distribution must be numerically approximated or simulated.

• The distribution of the DF test is non-standard. It has been tabulated under different scenarios.

1) with a constant: $y_t = \mu + \Phi\, y_{t-1} + \varepsilon_t$.

2) with a constant and a trend: $y_t = \mu + \delta\, t + \Phi\, y_{t-1} + \varepsilon_t$.

3) no constant: $y_t = \Phi\, y_{t-1} + \varepsilon_t$.

• The tests with no constant are not used in practice.

Autoregressive Unit Root – Testing: DF

• Critical values of the DF_t test under different scenarios.

Table 1: Selected Critical Values of Unit-Root Test Statistics

sample size	Probability			
T	0.01	0.025	0.05	0.10
	Model without constant			
100	−2.60	−2.24	−1.95	−1.61
250	−2.58	−2.23	−1.95	−1.62
500	−2.58	−2.23	−1.95	−1.62
∞	−2.58	−2.23	−1.95	−1.62
	Model with constant			
100	−3.51	−3.17	−2.89	−2.58
250	−3.46	−3.14	−2.88	−2.57
500	−3.44	−3.13	−2.87	−2.57
∞	−3.43	−3.12	−2.86	−2.57
	Model with time trend			
100	−4.04	−3.73	−3.45	−3.15
250	−3.99	−3.69	−3.43	−3.13
500	−3.98	−3.68	−3.42	−3.13
∞	−3.96	−3.66	−3.41	−3.12

Autoregressive Unit Root – DF: Case 2

• Case 2. DF with a constant term in DGP: $y_t = \mu + \phi y_{t-1} + \varepsilon_t$

The hypotheses to be tested:

$$H_0: \phi = 1, \mu = 0 \Rightarrow Y_t \sim I(1) \text{ without drift}$$
$$\text{H}_1: |\phi| < 1 \Rightarrow Y_t \sim I(0) \text{ with zero mean}$$

This formulation is appropriate for non-trending economic and financial series like interest rates, exchange rates and spreads.

• The test statistics $t_{\Phi=1}$ and $(T\text{-}1)(\hat{\phi} - 1)$ are computed from the estimation of the AR(1) model with a constant.

Autoregressive Unit Root – DF: Case 2

• Under $H_0 : \Phi = 1, \mu = 0,$ the asymptotic distributions of these test statistics are influenced by the presence, but not the coefficient value, of the constant in the test regression:

$$(T-1)(\hat{\phi}-1) \xrightarrow{d} \frac{\int_0^1 W^{\mu}(r)\,dW(r)}{\int_0^1 W^{\mu}(r)^2\,dr}$$

$$t_{\phi=1} \xrightarrow{d} \frac{\int_0^1 W^{\mu}(r)\,dW(r)}{\left(\int_0^1 W^{\mu}(r)^2\,dr\right)^{1/2}}$$

where $W^{\mu}(r) = W(r) - \int_0^1 W(r)\,dr$ is a de-meaned Wiener process, i.e., $\int_0^1 W^{\mu}(r)\,dr = 0$

• Inclusion of a constant pushes the tests' distributions to the left.

Autoregressive Unit Root – DF: Case 3

• Case 3. With constant and trend term in the DGP.
The test regression is $y_t = \mu + \delta t + \phi y_{t-1} + \varepsilon_t$

and includes a constant and deterministic time trend to capture the deterministic trend under the alternative. The hypotheses to be tested:

$$H_0 : \phi = 1, \delta = 0 \Rightarrow Y_t \sim I(1) \textit{ with drift}$$
$$H_1 : |\phi| < 1 \Rightarrow Y_t \sim I(0) \textit{ with determinis tic time trend}$$

• This formulation is appropriate for trending time series like asset prices or the levels of macroeconomic aggregates like real GDP. The test statistics $t_{\Phi=1}$ and $(T-1)\,(\hat{\phi} - 1)$ are computed from the above regression.

Autoregressive Unit Root – DF: Case 3

• Again, under $H_0 : \Phi = 1, \delta = 0,$ the asymptotic distributions of both test statistics are influenced by the presence of the constant and time trend in the test regression. Now, we have:

$$(T-1)\left(\hat{\phi}-1\right) \xrightarrow{d} \frac{\int_0^1 W^{\mu}(r)\,dW(r)}{\int_0^1 W^{\mu}(r)^2\,dr}$$

$$t_{\phi=1} \xrightarrow{d} \frac{\int_0^1 W^{\mu}(r)\,dW(r)}{\left(\int_0^1 W^{\mu}(r)^2\,dr\right)^{1/2}}$$

where $W^{\tau}(r) = W_{\mu}(r) - 12\left(r - \frac{1}{2}\right)\int_0^1 \left(s - \frac{1}{2}\right) W(r)\,dr$ is a de - meaned and de - trended Wiener process.

Autoregressive Unit Root – DF: Case 3

• Again, the inclusion of a constant and trend in the test regression further shifts the distributions of $t_{\Phi=1}$ and $(T-1)(\hat{\phi}-1)$ to the left.

Autoregressive Unit Root – Testing: DF

• Which version of the three main variations of the test should be used is not a minor issue. The decision has implications for the size and the power of the unit root test.

• For example, an incorrect exclusion of the time trend term leads to bias in the coefficient estimate for Φ, leading to size distortions and reductions in power.

• Since the normalized bias $(T-1)(\hat{\phi}-1)$ has a well defined limiting distribution that does not depend on nuisance parameters it can also be used as a test statistic for the null hypothesis $H_0: \Phi = 1$.

Autoregressive Unit Root – Testing: ADF

• Back to the general, AR(*p*) process. We can rewrite the equation as the *Dickey-Fuller reparameterization*:

$$\Delta y_t = \mu + \alpha_0 y_{t-1} + \alpha_1 \Delta y_{t-1} + \alpha_2 \Delta y_{t-2} + ... + \alpha_{p-1} \Delta y_{t-(p-1)} + \varepsilon_t$$

• The model is stationary if $\alpha_0 < 0$ => natural H_1: $\alpha_0 < 0$.

• Under H_0: $\alpha_0=0$, the model is AR(*p*-1) stationary in Δy_t. Then, if y_t has a (single) unit root, then Δy_t is a stationary AR process.

• The *t*-test for H_0 from OLS estimation is the Augmented Dickey-Fuller (ADF) test.

• Similar situation as the DF test, we have a non-normal distribution.

Autoregressive Unit Root – Testing: ADF

• The asymptotic distribution is:

$$T\hat{\alpha}_0 \xrightarrow{d} (1-\alpha_1-\alpha_2-\cdots-\alpha_{k-1})\,DF_\alpha$$

$$ADF = \frac{\hat{\alpha}_0}{s(\hat{\alpha}_0)} \rightarrow DF_t.$$

The limit distributions DF_α and DF_t are non-normal. They are skewed to the left, and have negative means.

• First result: $\hat{\alpha}_0$ converges to its true value (of zero) at rate T; rather than the conventional rate of sqrt(T) => *superconsistency*.

• Second result: The t-statistic for $\hat{\alpha}_0$ converges to a non-normal limit distribution, but does not depend on α.

Autoregressive Unit Root – Testing: ADF

• The ADF distribution has been extensively tabulated under the usual scenarios: 1) with a constant; 2) with a constant and a trend; and 3) no constant. This last scenario is seldom used in practice.

• Like in the DF case, which version of the three main versions of the test should be used is not a minor issue. A wrong decision has potential size and power implications.

• One-sided H_1: the ADF test rejects H_0 when ADF < *c*; where c is the critical value from the ADF table.

Note: The SE($\hat{\alpha}_0$)= s *sqrt*[Σy_{t-1}^2], the usual (homoscedastic) SE. But, we could be more general. Homoskedasticity is not required.

Autoregressive Unit Root – Testing: ADF

• We described the test with an intercept. Another setting includes a linear time trend:

$$\Delta y_t = \mu_1 + \mu_2 t + \alpha_0 y_{t-1} + \alpha_1 \Delta y_{t-1} + ... + \alpha_{p-1} \Delta y_{t-(p-1)} + \varepsilon_t$$

• Natural framework when the alternative hypothesis is that the series is stationary about a linear time trend.

• If t is included, the test procedure is the same, but different critical values are needed. The ADF test has a different distribution when t is included.

Autoregressive Unit Root – DF: Example 1

• Monthly USD/GBP exchange rate, S_t, (1800-2013), T=2534.

- Case 1 (no constant in DGP):

Variable	DF	Parameter Estimate	Standard Error	t Value	Pr > |t|
x1	1	**0.99934**	0.00061935	1613.52	<.0001

$(T\text{-}1)(\hat{\phi}\text{-}1)$=2533*(1-.99934)=-1.67178

Critical values at 5% level: −8.0 for T=500

−8.1 for $T=\infty$

• Cannot reject H_0 => Take 1st differences (changes in S_t) to model the series.

- With a constant, $\hat{\phi}$ =0.99631. Similar conclusion (Critical values at 5% level: −14.0 for T=500 and−14.1 for $T=\infty$): Model changes in S_t.

Autoregressive Unit Root – DF: Example 2

• Monthly US Stock Index (1800-2013), T=2534.

- No constant in DGP (unusual case, called Case 1): $y_t = \phi y_{t-1} + \varepsilon_t$

Variable	DF	Parameter Estimate	Standard Error	t Value	Pr > |t|
x1	1	**1.00298**	0.00088376	1134.90	<.0001

$(T\text{-}1)(\hat{\phi}\text{-}1)$=2533*(.00298)=7.5483 (positive, not very interesting)

Critical values at 5% level: −8.0 for T=500

−8.1 for T=∞

• Cannot reject H_0 => Take 1st differences (returns) to model the series.

- With a constant, $\hat{\phi}$ =1.00269. Same conclusion.

Autoregressive Unit Root – DF-GLS

• Elliott, Rothenberg and Stock (1992) (ERS) study *point optimal invariant tests* (POI) for unit roots. An invariant test is a test invariant to nuisance parameters.

• In the unit root case, we consider invariance to the parameters that capture the stationary movements around the unit roots -i.e., the parameters to AR(p) parameters.

• Consider: $y_t = \mu + \delta t + u_t, \quad u_t = \rho u_{t-1} + \varepsilon_t.$

• ERS show that the POI test for a unit root against $\varrho = \varrho^*$ is:

$$M_T = \frac{s^2_{\rho=1}}{s^2_{\rho=\rho^*}}$$

Autoregressive Unit Root – DF-GLS

•
$$M_T = \frac{s^2_{\rho=1}}{s^2_{\rho=\rho*}}$$

where s^2_{ϱ} is the variances residuals from the GLS estimation under both scenarios for ϱ, ϱ = 1 and ϱ = ϱ*, respectively:

• The critical value for the test will depend on c where ϱ* = 1 – c/T.

Note: When dynamics are introduced in the u_t equation, Δ u_t lags, the critical values have to be adjusted.

• In practice ϱ* is unknown. ERS suggest different values for different cases. Say, c=-13.5, for the case with a trend, gives a power of 50%.

Autoregressive Unit Root – DF-GLS

• It turns out that if we instead do the GLS-adjustment and then perform the ADF-test (without allowing for a mean or trend) we get approximately the POI-test. ERS call this test the DF-GLS$_t$ test.

• The critical values depend on T.

T	1%	5%
50	-3.77	-3.19
100	-3.58	-3.03
200	-3.46	-2.93
500	-3.47	-2.89
∞	-3.48	-2.89

• Check ERS for critical values for other scenarios.

Autoregressive Unit Root – Testing: ADF

• Important issue: lag *p*. :

- Check the specification of the lag length *p*. If *p* is too small, then the remaining serial correlation in the errors will bias the test. If *p* is too large, then the power of the test will suffer.

- Ng and Perron (1995) suggestion:
(1) Set an upper bound p_{max} for *p*.
(2) Estimate the ADF test regression with $p = p_{max}$.
If $|t_{\alpha(p)}| > 1.6$ set $p = p_{max}$ and perform the ADF test.
Otherwise, reduce the lag length by one. Go back to (1)

- Schwert's (1989) rule of tumb for determining p_{max}:

$$p_{\max} = \left[12 \left(\frac{T}{100} \right)^{1/4} \right]$$

Autoregressive Unit Root – Testing: PP Test

• The Phillips-Perron (PP) unit root tests differ from the ADF tests mainly in how they deal with serial correlation and heteroskedasticity in the errors.

• The ADF tests use a parametric autoregression to approximate the ARMA structure of the errors in the test regression. The PP tests correct the DF tests by the bias induced by the omitted autocorrelation.

• These modified statistics, denoted Z_t and Z_δ, are given by

.

$$Z_t = \sqrt{\frac{\hat{\sigma}^2}{\hat{\lambda}^2}}\, t_{\hat{\alpha}_0} - \frac{1}{2}\left(\frac{\hat{\lambda}^2 - \hat{\sigma}^2}{\hat{\lambda}^2}\right)\left(\frac{T\left(SE\left(\hat{\alpha}_0\right)\right)}{\hat{\sigma}^2}\right)$$

$$Z_\delta = T\hat{\alpha}_0 - \frac{1}{2}\frac{T^2\left(SE\left(\hat{\alpha}_0\right)\right)}{\hat{\sigma}^2}\left(\hat{\lambda}^2 - \hat{\sigma}^2\right)$$

Autoregressive Unit Root – Testing: PP Test

• The terms $\hat{\sigma}^2$ and $\hat{\lambda}$ are consistent estimates of the variance parameters:

$$\sigma^2 = \lim_{T \leftarrow \infty} T^{-1} \sum_{t=1}^{T} E\left(\varepsilon_t^2\right) \quad \lambda^2 = \lim_{T \to \infty} \sum_{t=1}^{T} E\left(\frac{1}{T} \sum_{t=1}^{T} \varepsilon_t^2\right)$$

• Under $H_0{:}\alpha_0 = 0$, the PP Z_t and $Z_{\alpha 0}$ statistics have the same asymptotic distributions as the DF t-statistic and normalized bias statistics.

• PP tests tend to be more powerful than the ADF tests. But, they can severe size distortions (when autocorrelations of ε_t are negative) and they are more sensitive to model misspecification (order of ARMA model).

Autoregressive Unit Root – Testing: PP Test

• Advantage of the PP tests over the ADF tests:

- Robust to general forms of heteroskedasticity in the error term ε_t.
- No need to specify a lag length for the ADF test regression.

Autoregressive Unit Root – Testing: Criticisms

• The ADF and PP unit root tests are very popular. They have been, however, widely criticized.

• Main criticism: Power of tests is low if the process is stationary but with a root close to the non-stationary boundary.

• For example, the tests are poor at distinguishing between $\varphi=1$ or $\varphi=0.976$, especially with small sample sizes.

• Suppose the true DGP is $y_t = 0.976\, y_{t-1} + \varepsilon_t$
=> H_0:$\alpha_0 = 0$ should be rejected.

• One way to get around this is to use a stationarity test (like KPSS test) as well as the unit root ADF or PP tests.

Autoregressive Unit Root – Testing: Criticisms

• The ADF and PP unit root tests are known (from simulations) to suffer potentially severe finite sample power and size problems.

1. Power – Both tests are known to have low power against the alternative hypothesis that the series is stationary (or TS) with a large autoregressive root. (See, DeJong, et al, *J. of Econometrics,* 1992.)

2. Size – Both tests are known to have severe size distortion (in the direction of over-rejecting H_0) when the series has a large negative MA root. (See, Schwert, *JBES,* 1989*:* MA = -0.8 =>size = 100%!)

Autoregressive Unit Root – Testing: KPSS

• A different test is the KPSS (Kwiatkowski, Phillips, Schmidt and Shin) Test (1992). It can be used to test whether we have a deterministic trend vs. stochastic trend:

$$H_0 : Y_t \sim I(0) \qquad \rightarrow \text{level (or trend) stationary}$$
$$H_1 : Y_t \sim I(1) \qquad \rightarrow \text{difference stationary}$$

• Setup

$$y_t = \mu + \delta t + r_t + u_t$$
$$r_t = r_{t-1} + \varepsilon_t$$

where $\varepsilon_t \sim WN(0,\sigma^2)$, uncorrelated with $u_t \sim WN$. Then,

H_0(trend stationary): $\sigma^2=0$

H_0(y_t (level) stationary): $\sigma^2=0$ & $\delta=0$.

Under H_1: $\sigma^2 \neq 0$, there is a RW in y_t .

Autoregressive Unit Root – Testing: KPSS

• Under some assumptions (normality, *i.i.d.* for u_t & ε_t), a one-sided LM test of the null that there is no random walk ($\varepsilon_t=0$, for all t) can be constructed with:

$$KPSS = T^{-2} \sum_{t=1}^{T} \frac{S_t}{s_u^{\;2}}$$

where s_u^2 is the variance of u_t, ("long run" variance) estimated as

$$s_u^2(l) = \frac{1}{T}\sum_{t=1}^{T} \hat{u}_t^2 + \frac{2}{T}\sum_{s=1}^{l} w(s,l) \sum_{t=s+1}^{T} \hat{u}_t \hat{u}_{t-s}$$

where $w(s,l)$ is a kernel function, for example, the Bartlett kernel. We also need to specify the number of lags, which should grow with T.

• Under H_0, $\hat{u}_t$ can be estimated by OLS.

Autoregressive Unit Root – Testing: KPSS

• Easy to construct. Steps:

1. Regress y_t on a constant and time trend. Get OLS residuals, **u^**.
2. Calculate the partial sum of the residuals: $S_t = \sum_{i=1}^{t} \hat{u}_i$
3: Compute the KPSS test statistic

$$KPSS = T^{-2} \sum_{t=1}^{T} \frac{S_t}{s_u^{\,2}}$$

where s_u^2 is the estimate of the long-run variance of the residuals.

4. Reject H_0 when KPSS is large (the series wander from its mean).

• Asymptotic distribution of the test statistic is non-standard –it can be derived using Brownian motions, appealing to FCLT and CMT.

Autoregressive Unit Root – Testing: KPSS

• KPSS converges to three different distribution, depending on whether the model is trend-stationary ($\delta \neq 0$),, level-stationary (δ=0), or zero-mean stationary (δ=0, μ=0).

• For example, if a constant is included (δ=0) KPSS converges to

$$KPSS \xrightarrow{d} \int_0^1 [W(r) - W(1)]\, dr$$

Note: $V = W(r) - rW(1)$ is called a standard Brownian bridge. It satisfies $V(0)=V(1)=0$.

• It is a very powerful unit root test, but if there is a volatility shift it cannot catch this type non-stationarity.

Autoregressive Unit Root – Structural Breaks

• A stationary time-series may look like non-stationary when there are structural breaks in the intercept or trend.

• The unit root tests lead to false non-rejection of the null when we do not consider the structural breaks. A low power problem.

• A single known breakpoint was studied by Perron (*Econometrica,* 1989). Perron (1997) extended it to a case of unknown breakpoint.

• Perron considers the null and alternative hypotheses

H_0: $y_t = a_0 + y_{t-1} + \mu_1 D_P + \varepsilon_t$ ($y_t \sim ST$ with a jump)

H_1: $y_t = a_0 + a_2 t + \mu_2 D_L + \varepsilon_t$ ($y_t \sim TS$ with a jump)

Pulse break: $D_P = 1$ if $t = T_B + 1$ and zero otherwise,
Level break: $D_L = 0$ for $t = 1, \ldots, T_B$ and one otherwise.

Unit Root – Single Structural Break: Perron

•. Power of ADF tests: Rejection frequencies of ADF–tests

	Model: $a_0 = a_2 = 0.5$ and $\mu_2 = 10$		
	1% level	5% level	10% level
ADF–tests	0.004	0.344	0.714
	Model: $a_0 = a_2 = 0.5$ and $\mu_2 = 12$		
ADF–tests	0.000	0.028	0.264

• Observations:
- ADF tests are biased toward non-rejection of the non-stationary H_0.
- Rejection frequency is inversely related to the magnitude of the shift.

• Perron estimated values of the AR coefficient in the DF regression. They were biased toward unity and that this bias increased as the magnitude of the break increased.

Unit Root – Single Structural Break: Perron

• Perron's suggestion: Running the following OLS regression:

$$y_t = a_0 + a_1 y_{t-1} + a_2 t + \mu_2 D_L + \sum_{i=1}^{p} \beta_i \Delta y_{t-i} + \varepsilon_t$$

H_0: a_1=1; $\Rightarrow$ use t–ratio, DF unit root test.

• Perron shows that the asymptotic distribution of the t-statistic depends on the location of the structural break, $\lambda = T_B/T$.

• Perron (1989) derives critical values for different cases. For example:
H_0: $y_t = a_0 + y_{t-1} + \mu_1 D_P + \gamma_1 D_L + \varepsilon_t$ ($y_t \sim ST$ with a jump & break)
H_1: $y_t = a_0 + a_2 t + \mu_2 D_L + \gamma_2 D_L + \varepsilon_t$ ($y_t \sim TS$ with a jump & break)

• Main problem with this test procedure: structural breaks are not known, they need to be estimated from data.

Unit Root – Single Structural Break: ZA

• Main problem with this test: Structural breaks are not known, they need to be estimated. Many papers dealing with *endogenous* structural breaks: Zivot and Andrews (ZA, 1992), Lumsdaine and Papell (1998), Lee and Strazicich (2003).

• ZA's test is a sequential ADF test, using a different dummy variable for each possible break date. The break date is selected where the t-statistic from the ADF test is a minimum (most negative) –break date is chosen where the evidence is least favorable for the unit root null.

• ZA's critical values are different from Perron's (1989). In general, ZA provide more evidence for unit roots than under Perron's.

Unit Root – Multiple Structural Breaks

• Lumsdaine and Papell (1998) and Lee and Strazicich (2003) allow for multiple breaks in their tests.

• Lumsdaine and Papell extend ZA, by allowing two structural breaks under the alternative hypothesis of the unit root test and additionally allow for breaks in level and trend.

• The derivation of critical values on ZA and Lumsdaine and Papell (1998) assumes no breaks under the null hypothesis. This assumption may lead to conclude incorrectly (*spuriously*) reject H_0 (unit root) when, in fact, the series is difference-stationary with breaks.

Unit Root – Multiple Structural Breaks

• The derivation of critical values on ZA and Lumsdaine and Papell (1998) assumes no breaks under the null hypothesis. This assumption may lead to conclude incorrectly (*spuriously*) reject H_0 (unit root) when, in fact, the series is difference-stationary with breaks .

• To deal with this issue, Lee and Strazicich (2003) propose a LM unit-root test, incorporating structural breaks under H_0 (& H_1), with DGPs (augmenting with p first-difference AR lags works well):

$$H_0\text{:}\; y_t = a_0 + y_{t-1} + \mu_1 D_{P,1} + \mu_1 D_{P,2} + \gamma_1 D_{L,1} + \gamma_2 D_{L,2} + \varepsilon_t$$
$$H_1\text{:}\; y_t = a_0 + (1\text{-}a_1)\, y_{t-1} + a_2 t + \mu_1 D_{P,1} + \mu_1 D_{P,2} + \gamma_1 D_{L,1} + \gamma_2 D_{L,2} + \varepsilon_t$$

• In general, using Lee and Strazicich (2003), we tend to reject more H_0 (unit root).

Autoregressive Unit Root - Relevance

• We can always decompose a unit root process into the sum of a random walk and a stable process. This is known as the *Beveridge-Nelson* (1981) (BN) composition.

• Let $y_t \sim I(1)$, $r_t \sim RW$ and $c_t \sim I(0)$.

$$y_t = r_t + c_t.$$

Since c_t is stable it has a Wold decomposition:

$$(1 - L) y_t = \psi(L) \varepsilon_t$$

Then,

$$(1-L)y_t = \psi(L)\varepsilon_t = \psi(1)\varepsilon_t + (\psi(L) - \psi(1))\varepsilon_t$$
$$= \psi(1)\varepsilon_t + \psi(L)^* \varepsilon_t$$

where $\psi(1)$=0. Then,

$$y_t = \psi(1)(1-L)^{-1}\varepsilon_t + \psi(L)^*(1-L)^{-1}\varepsilon_t = r_t + c_t$$

Autoregressive Unit Root - Relevance

• Usual finding in economics: Many time series seem to have unit roots. But, there is debate over power of unit root tests and the effect of structural breaks.

Example: Consumption, output, stock prices, interest rates, unemployment, size, compensation are usually I(1).

• Sometimes a linear combination of I(1) series produces an I(0). For example, (log consumption– log output) is stationary. This situation is called *cointegration*.

• Practical problems with cointegration:
- Asymptotics change completely.
- Not enough data to definitively say we have cointegration.

Autoregressive Unit Root – Structural Breaks 2

• Nelson and Plosser (1982) tested using ADF 14 macroeconomic series (GNP, IP, employment, etc) for unit roots: Rejected H_0 for only one. Summary of results from tests allowing for structural breaks, from Glyn et al. (2007):

Table 1: Unit Root Tests with the Nelson and Plosser's Data (1982) Set

Empirical Studies by:	Model	Unit Root (with possible breaks)	Stationary (with possible breaks)
Nelson and Plosser (1982)	ADF test with no break	13	1
Perron (1989)**	Exogenous with one break	3	11
Zivot and Andrews (1992)*	Endogenous with one break	10	3
Lumsdaine and Papell (1997)*	Endogenous with two breaks	8	5
Lee and Strazicich (2003)**	Endogenous with two breaks	10	4

* Assume no break(s) under the null hypothesis of unit root.
** Assume break(s) under both the null and the alternative hypothesis.