Lecture 9
Models for Censored and Truncated Data – Truncated Regression and Sample Selection

Censored and Truncated Data: Definitions

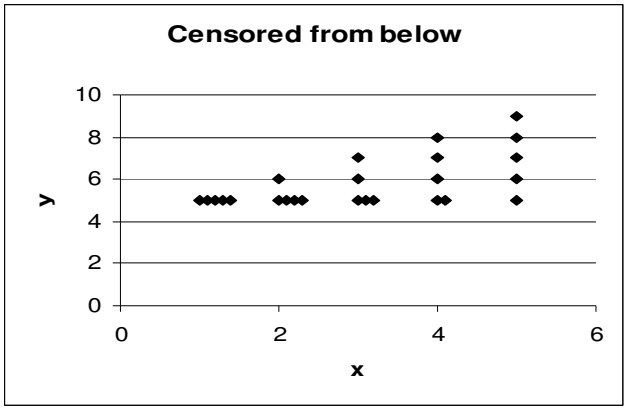
• Y is *censored* when we observe X for all observations, but we only know the true value of Y for a restricted range of observations. Values of Y in a certain range are reported as a single value or there is significant clustering around a value, say 0.

- If $Y = k$ or $Y > k$ for all Y => Y is *censored from below* or *left-censored.*
- If $Y = k$ or $Y < k$ for all Y => Y is *censored from above* or *right-censored.*

We usually think of an uncensored Y, Y^*, the true value of Y when the censoring mechanism is not applied. We typically have all the observations for $\{Y,X\}$, but not $\{Y^*,X\}$.

• Y is *truncated* when we only observe X for observations where Y would not be censored. We do not have a full sample for $\{Y,X\}$, we exclude observations based on characteristics of Y.

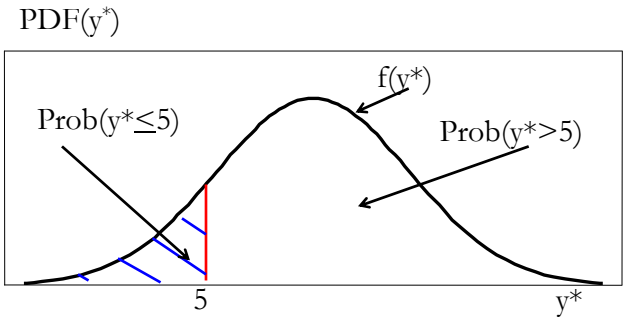
Censored from below: Example



• If $Y \leq 5$, we do not know its exact value.

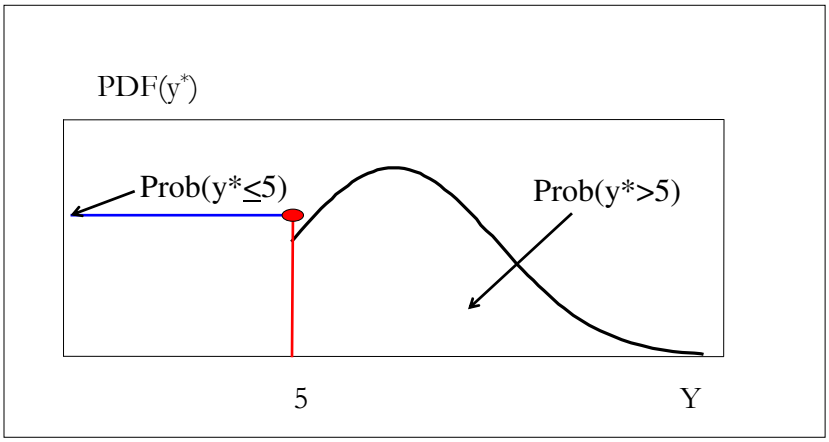
Example: A Central Bank intervenes if the exchange rate hits the band's lower limit. => If $S_t \leq \bar{E}$ => $S_t = \bar{E}$.

Censored from below: Example



• The pdf of the observable variable, y, is a mixture of discrete (prob. mass at Y=5) and continuous (Prob[Y*>5]) distributions.

Censored from below: Example



• Under censoring we assign the full probability in the censored region to the censoring point, 5.

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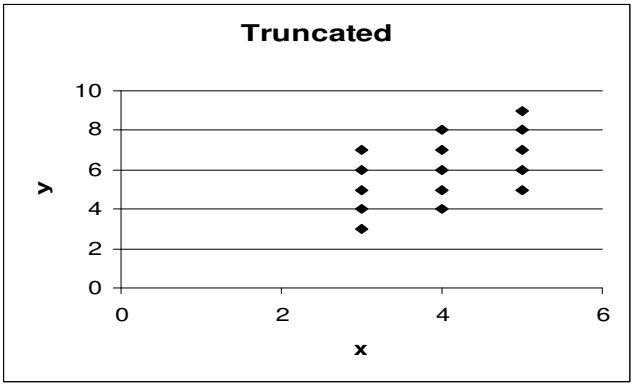
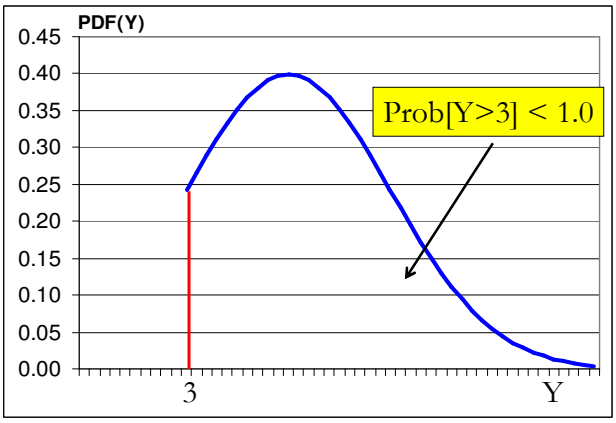
Truncated Data: Example



• If $Y < 3$, the value of X (or Y) is unknown. *(Truncation from below.)*

Example: If a family's income is below certain level, we have no information about the family's characteristics.

Truncated Data: Example



• Under data censoring, the censored distribution is a combination of a pmf plus a pdf. They add up to 1. We have a different situation under truncation. To create a pdf for Y we will use a conditional pdf.

Truncated regression

• Truncated regression is different from censored regression in the following way:

Censored regressions: The dependent variable may be censored, but you can include the censored observations in the regression

Truncated regressions: A subset of observations are dropped, thus, only the truncated data are available for the regression.

• Q: Why do we have truncation?

(1) *Truncation by survey design*: Studies of poverty. By survey's design, families whose incomes are greater than that threshold are dropped from the sample.

(2) *Incidental Truncation*: Wage offer married women. Only those who are working has wage information. It is the people's decision, not the survey's design, that determines the sample. selection.

Truncation and OLS

Q: What happens when we apply OLS to a truncated data?

- Suppose that you consider the following regression:

$$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i,$$

- We have a random sample of size N. All CLM assumptions are satisfied. (The most important assumption is (**A2**) $E(\varepsilon_i | x_i)=0$.)
- Instead of using all the N observations, we use a subsample. Then, run OLS using this sub-sample (truncated sample) only.

• Q: Under what conditions, does *sample selection* matter to OLS?

(A) OLS is Unbiased

(A-1) Sample selection is randomly done.

(A-2) Sample selection is determined solely by the value of **x-variable**. For example, suppose that x is age. Then if you select sample if age is greater than 20 years old, this OLS is unbiased.

Truncation and OLS

(B) OLS is Biased

(B-1) Sample selection is determined by the value of **y-variable**.

Example: *Y* is family income. We select the sample if y is greater than certain threshold. Then this OLS is biased.

(B-2) Sample selection is correlated with ε_i.

Example: We run a wage regression $w_i = \beta_0 + \beta_1 educ_i + \varepsilon_i$, where ε_i contains unobserved ability. If sample is selected based on the unobserved ability, this OLS is biased.

- In practice, this situation happens when the selection is based on the survey participant's decision. Since the decision to participate is likely to be based on unobserved factors which are contained in ε, the selection is likely to be correlated with ε_i.

Truncation and OLS: When does (A2) hold?

• Consider the previous regression:

$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$,

- All CLM assumptions are satisfied.

- Instead of using all the N observations, we use a subsample. Let s_i be a selection indicator: If $s_i=1$, then person *i* is included in the regression. If $s_i=0$, then person *i* is dropped from the data.

• If we run OLS using the selected subsample, we use only the observation with $s_i=1$. That is, we run the following regression:

$s_i y_i = \beta_0 s_i + \beta_1 s_i x_i + s_i \varepsilon_i$

• Now, $s_i x_i$ is the explanatory variable, and $u_i = s_i \varepsilon_i$ is the error term.

• OLS is unbiased if $E(u_i = s_i \varepsilon_i \mid s_i x_i) = 0$.

=> we need check under what conditions the new (**A2**) is satisfied.

Truncation and OLS: When does (A2) hold?

Q: When does $E(u_i = s_i \varepsilon_i \mid s_i x_i)=0$ hold?

It is sufficient to check: $E(u_i \mid s_i x_i)=0$. (If this is zero, then new (A2) is also zero.)

• $E(u_i \mid x_i, s_i) = s_i E(\varepsilon_i \mid x_i, s_i)$ - s_i is in the conditional set.

• It is sufficient to check the condition which ensures $E(u_i \mid x_i, s_i)=0$.

• CASES:

(A-1) Sample selection is done randomly.

s is independent of ε and x. => $E(\varepsilon \mid x,s)=E(\varepsilon \mid x)$.

Since the CLM assumptions are satisfied => we have $E(\varepsilon \mid x)=0$.

=> OLS is unbiased.

Truncation and OLS: When does (A2) hold?

(A-2) Sample is selected based solely on the value of x-variable.

Example: We study trading in stocks, y_i. One of the dependent variables, x_i, is wealth, and we select person *i* if wealth is greater than 50K. Then,

$s_i=1$ if $x_i \geq 50K$,

$s_i=0$ if $x_i < 50K$.

-Now, s_i is a deterministic function of x_i.

• Since s is a deterministic function of x, it drops out from the conditioning set. Then,

$E(\varepsilon | x, s) = E(\varepsilon | x, s(x))$ - s is a deterministic function of x.

$= E(\varepsilon | x) = 0$ - CLM assumptions satisfied.

=> OLS is unbiased.

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Truncation and OLS: When does (A2) hold?

(B-1) Sample selection is based on the value of y-variable.

Example: We study determinants of wealth, *Y*. We select individuals whose wealth is smaller than 150K. Then, $s_i=1$ if $y_i < 150K$.

-Now, s_i depends on y_i (and ε_i). It cannot be dropped out from the conditioning set like we did before. Then, $E(\varepsilon | x, s) \neq E(\varepsilon | x) = 0$.

• For example,

$E(\varepsilon | x, s=1) = E(\varepsilon | x, y \leq 150K)$

$= E(\varepsilon | x, \beta_0+\beta_1 x+ \varepsilon \leq 150K)$

$= E(\varepsilon | x, \varepsilon \text{ u } \leq 150K-\beta_0-\beta_1 x)$

$\neq E(\varepsilon | x) = 0.$ => OLS is biased.

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Truncation and OLS: When does (A2) hold?

(B-2) Sample selection is correlated with u_i.

The inclusion of a person in the sample depends on the person's decision, not the surveyor's decision. This type of truncation is called the *incidental truncation*. The bias that arises from this type of sample selection is called the *Sample Selection Bias*.

Example: wage offer regression of married women:

$$wage_i = \beta_0 + \beta_1 edu_i + \varepsilon_i.$$

Since it is the woman's decision to participate, this sample selection is likely to be based on some unobservable factors which are contained in ε_i. Like in (B-1), s cannot be dropped out from the conditioning set:

$$E(\varepsilon \mid x, s) \neq E(\varepsilon \mid x) = 0 \quad \Rightarrow \text{OLS is biased.}$$

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Truncation and OLS: When does (A2) hold?

• CASE (A-2) can be more complicated, when the selection rule based on the **x-variable** may be correlated with ε_i.

Example: *X* is IQ. A survey participant responds if IQ > *v*.

Now, the sample selection is based on x-variable *and* a random error *v*.

Q: If we run OLS using only the truncated data, will it cause a bias?

Two cases:

- (1) If *v* is independent of ε, then it does not cause a bias.
- (2) If *v* is correlated with ε, then this is the same case as (B-2). Then, OLS will be biased.

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Estimation with Truncated Data.

• CASES

- Under cases (A-1) and (A-2), OLS is appropriate.
- Under case (B-1), we use *Truncated regression*.
- Under case (B-2) –i.e., incidental truncation-, we use the *Heckman Sample Selection Correction* method. This is also called the *Heckit model*.

Truncated Regression

• Data truncation is (B-1): the truncation is based on the **y-variable**.

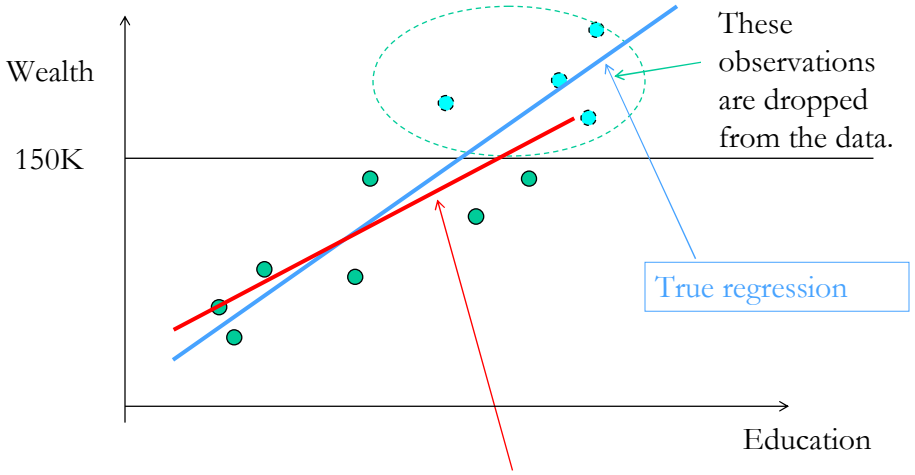
• We have the following regression satisfies all CLM assumptions:

$$y_i = \mathbf{x}_i' \beta + \varepsilon_i, \qquad \varepsilon_i \sim N(0, \sigma^2)$$

- We sample only if $y_i < c_i$ - Observations dropped if $y_i \geq c_i$ by design.
- We know the exact value of c_i for each person.

• We know that OLS on the truncated data will cause biases. The model that produces unbiased estimate is based on the ML Estimation.

Truncated Regression



Truncated Regression: Conditional Distribution

• Given the normality assumption for ε_i, ML is easy to apply.

- For each, $\varepsilon_i = y_i - \mathbf{x}_i'\boldsymbol{\beta}$, the likelihood contribution is $f(\varepsilon_i)$.
- But, we select sample only if $y_i < c_i$

=> we have to use the density function of ε_i conditional on $y_i < c_i$:

$$f(\varepsilon_i \mid y_i < c) = f(\varepsilon_i \mid \varepsilon_i < c_i - x_i'\beta) = \frac{f(\varepsilon_i)}{P(u_i < c_i - x_i'\beta)}$$

$$= \frac{f(\varepsilon_i)}{P(\frac{\varepsilon_i}{\sigma} < \frac{c_i - x_i'\beta}{\sigma})} = \frac{f(\varepsilon_i)}{\Phi(\frac{c_i - x_i'\beta}{\sigma})}$$

$$= \frac{1}{\Phi(\frac{c_i - x_i'\beta}{\sigma})} \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{\varepsilon_i^2}{2\sigma^2})$$

$$= \frac{\frac{1}{\sigma}\phi\left(\frac{\varepsilon_i}{\sigma}\right)}{\Phi(\frac{c_i - x_i'\beta_i}{\sigma})}$$

Truncated Normal

• Moments:

Let $y^* \sim N(\mu^*, \sigma^2)$ and $\alpha = (c - \mu^*)/\sigma$.

- First moment:

$E[y^* | y > c] = \mu^* + \sigma\,\lambda(\alpha)$ <= This is the truncated regression.

=> If $\mu^* > 0$ and the truncation is from below –i.e., $\lambda(\alpha) > 0$–, the mean of the truncated variable is greater than the original mean

Note: For the standard normal distribution $\lambda(\alpha)$ is the mean of the truncated distribution.

- Second moment:

- $Var[y^* | y > c] = \sigma^2[1 - \delta(\alpha)]$ where $\delta(\alpha) = \lambda(\alpha)\,[\lambda(\alpha) - \alpha]$

=> Truncation reduces variance! This result is general, it applies to upper or lower truncation given that $0 \leq \delta(\alpha) \leq 1$

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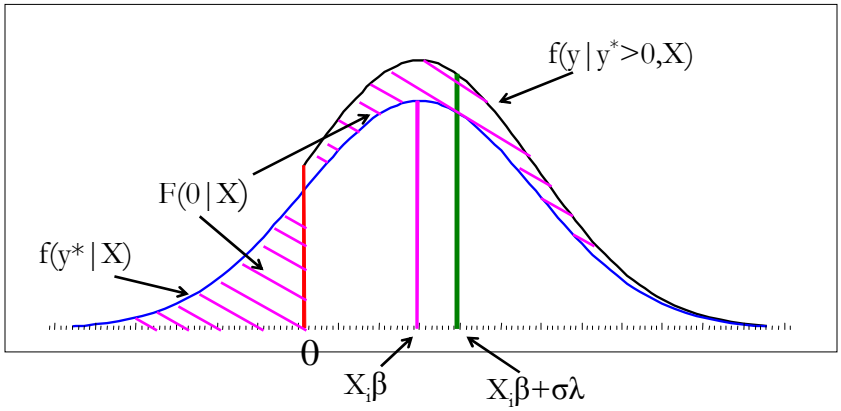
Truncated Normal

Model: $y_i^* = X_i\beta + \varepsilon_i$

Data: $y = y^* \mid y^* > 0$



• Truncated (from below –i.e., $y^* > 0$) regression model:

$$E(y_i \mid y_i^* > 0, X_i) = X_i\beta + \sigma\lambda_i > E(y_i \mid X_i)$$

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Truncated Regression: ML Estimation

• The likelihood contribution for i^{th} observation is given by

$$L_i = \frac{\frac{1}{\sigma}\phi\left(\frac{y_i - x_i'\beta}{\sigma}\right)}{\Phi\left(\frac{x_i'\beta}{\sigma}\right)}$$

• The likelihood function is given by

$$Log\ L(\beta,\sigma) = \sum_{i=1}^{N} \log L_i = \underbrace{-\frac{N}{2}[\log(2\pi) + \log(\sigma^2)] - \frac{1}{2\sigma^2}\sum_{i=1}^{N}\varepsilon_i^2}_{\text{ln(joint density of N values of y*)}} - \underbrace{\sum_{i=1}^{N}\log\left[\Phi\left(\frac{x_i'\beta}{\sigma}\right)\right]}_{\text{log(joint probability of y*> 0)}}$$

• The values of (β, σ) that maximizes Log L are the ML estimators of the *Truncated Regression*.

The partial effects

• The estimated β_k shows the effect of x_{ki} on y_i. Thus,

$$\frac{\partial E\left(y_i \mid X_i, y_i^* > 0\right)}{\partial X_{k,i}} = \beta_k + \frac{\partial E\left(\varepsilon_i \mid y_i^* > 0\right)}{\partial X_{k,i}}$$

$$= \beta_k + \sigma\frac{\partial \lambda}{\partial X_{k,i}} = \beta_k + \sigma\left(\lambda_i^2 - \alpha_i\lambda_i\right)\left(-\frac{\beta_k}{\sigma}\right)$$

$$= \beta_k\left(1 - \lambda_i^2 - \alpha_i\lambda_i\right) = \beta_k\left(1 - \delta_i\right)$$

where $\delta_i = \lambda(\alpha_i)\left[\lambda(\alpha_i) + \alpha_i\right],\ 0 < \delta_i < 1$

Truncated Regression: MLE - Example

• DATA: From a survey of family income in Japan (JPSC_familyinc.dta). The data is originally not truncated.

Model: $y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$

y_i = family income in JPY 10,000

x_i: husband's education

• Three cases:

EX1. Use all observations to estimate model

EX2. Truncate sample from above (y<800). Then run the OLS using on the truncated sample.

EXe. Run the truncated regression model for the data truncated from above.


```
. reg familyinc huseduc
```

Source	SS	df	MS
Model	38305900.9	1	38305900.9
Residual	318850122	7693	41446.7856
Total	357156023	7694	46420.0705

Number of obs = 7695
F(1, 7693) = 924.22
Prob > F = 0.0000
R-squared = 0.1073
Adj R-squared = 0.1071
Root MSE = 203.58

familyinc	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
huseduc	32.93413	1.083325	30.40	0.000	30.81052	35.05775
_cons	143.895	15.09181	9.53	0.000	114.3109	173.479

OLS using all the observations

```
. reg familyinc huseduc if familyinc<800
```

Source	SS	df	MS
Model	11593241.1	1	11593241.1
Residual	120645494	6272	19235.5699
Total	132238735	6273	21080.621

Number of obs = 6274
F(1, 6272) = 602.70
Prob > F = 0.0000
R-squared = 0.0877
Adj R-squared = 0.0875
Root MSE = 138.69

familyinc	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
huseduc	20.27929	.8260432	24.55	0.000	18.65996	21.89861
_cons	244.5233	11.33218	21.58	0.000	222.3084	266.7383

OLS on truncated sample.

The parameter on huseband's education is biased towards zero.

```
. truncreg familyinc huseduc, ul(800)
(note: 1421 obs. truncated)

Fitting full model:

Iteration 0:   log likelihood = -39676.782
Iteration 1:   log likelihood = -39618.757
Iteration 2:   log likelihood = -39618.629
Iteration 3:   log likelihood = -39618.629

Truncated regression
Limit:   lower =      -inf                  Number of obs =   6274
         upper =       800                  Wald chi2(1)  =  569.90
Log likelihood = -39618.629                 Prob > chi2   =  0.0000
```

familyinc	Coef.	Std. Err.	z	P>|z|	[95% Conf.	Interval]
huseduc	24.50276	1.0264	23.87	0.000	22.49105	26.51446
_cons	203.6856	13.75721	14.81	0.000	176.7219	230.6492
/sigma	153.1291	1.805717	84.80	0.000	149.59	156.6683

Truncated regression model on the truncated sample

Note: Bias seems to be corrected, but not perfect in this example.

Sample Selection Bias Correction Model

- The most common case of truncation is (B-2): Incidental truncation.

- This data truncation usually occurs because sample selection is determined by the people's decision, not the surveyor's decision.

- Back to the wage regression example. If person *i* has chosen to participate (work), person *i* has *self-selected into the sample*. If person *i* has decided not to participate, person *i* has self-selected out of the sample.

- The bias caused by this type of truncation is called *sample selection bias*.

- This model involves two decisions: (1) participation and (2) amount. It is a generalization of the Tobit Model.

Tobit Model – Type II

• Different ways of thinking about how the latent variable and the observed variable interact produce different Tobit Models.

• The Type I Tobit Model presents a simple relation:

- $y_i = 0$ if $y_i^* = \mathbf{x}_i'\beta + \varepsilon_i \leq 0$
  $= y_i^* = \mathbf{x}_i'\beta + \varepsilon_i$ if $y_i^* = \mathbf{x}_i'\beta + \varepsilon_i > 0$

The effect of the *X*'s on the probability that an observation is censored and the effect on the conditional mean of the non-censored observations are the same: β.

• The Type II Tobit Model presents a more complex relation:

- $y_i = 0$ if $y_i^* = \mathbf{x}_i'\alpha + \varepsilon_{1,i} \leq 0$, $\varepsilon_{1,i} \sim N(0,1)$
  $= \mathbf{x}_i'\beta + \varepsilon_{2,i}$ if $y_i^* = \mathbf{x}_i'\alpha + \varepsilon_{1,i} > 0$, $\varepsilon_{2,i} \sim N(0,\sigma_2^2)$

Now, we have different effects of the *X*'s.

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Tobit Model – Type II

• The Type II Tobit Model:

- $y_i = 0$ if $y_i^* = \mathbf{x}_i'\alpha + \varepsilon_{1,i} \leq 0$, $\varepsilon_{1,i} \sim N(0,\sigma_1^2=1)$
  $= \mathbf{x}_i'\beta + \varepsilon_{2,i}$ if $y_i^* = \mathbf{x}_i'\alpha + \varepsilon_{1,i} > 0$, $\varepsilon_{2,i} \sim N(0,\sigma_2^2)$

- A more flexible model. **X** can have an effect on the decision to participate (Probit part) and a different effect on the amount decision (truncated regression).

- Type I is a special case: $\varepsilon_{2,i} = \varepsilon_{1,i}$ and $\alpha=\beta$.

Example: Age affects the decision to donate to charity. But it can have a different effect on the amount donated. We may find that age has a positive effect on the decision to donate, but given a positive donation, younger individuals donate more than older individuals.

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Tobit Model – Type II

• The Tobit Model assumes a bivariate normal distribution for $(\varepsilon_{1,i};\varepsilon_{2,i})$; with covariance given by $\sigma_{12}(=\rho\sigma_1\sigma_2.)$.

- Conditional expectation:

$$E[y|y>0,x] = \mathbf{x}_i'\boldsymbol{\beta} + \sigma_{12}\,\lambda(\mathbf{x}_i'\boldsymbol{\alpha})$$

- Unconditional Expectation

$$E[y|x] = Prob(y>0|x) * E[y|y>0, x] + Prob(y=0|x)* 0$$
$$= Prob(y>0|x) * E[y|y>0, x]$$
$$= \Phi(\mathbf{x}_i'\boldsymbol{\alpha}) * [\mathbf{x}_i'\boldsymbol{\beta} + \sigma_{12}\,\lambda(\mathbf{x}_i'\boldsymbol{\alpha})]$$

Note: This model is known as the Heckman selection model, or the Type II Tobit model (Amemiya), or the probit selection model (Wooldridge).

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Tobit Model – Type II – Sample selection

• We generalized the model presented, making the decision to participate dependent on a different variable, **z**. Then,

$y_i = 0$ if $y_i^* = \mathbf{z}_i'\boldsymbol{\alpha} + \varepsilon_{1,i} \leq 0$, $\varepsilon_{1,i} \sim N(0,\sigma_1^2)$

$= \mathbf{x}_i'\boldsymbol{\beta} + \varepsilon_{2,i}$ if $y_i^* = \mathbf{z}_i'\boldsymbol{\alpha} + \varepsilon_{1,i} > 0$, $\varepsilon_{2,i} \sim N(0,\sigma_2^2=1)$

• This model is called the *Sample selection model*, due to Heckman.

Example (from Heckman (*Econometrica*, 1979): Structural Labor model:

• Labor Supply equation:

$$h_i^* = \delta_0 + \delta_1 w_i + \delta_2 Z_i + \varepsilon_i \quad (1)$$

- h_i^*: desired hours by i^{th} person (latent variable)
- w_i: wage that could be earned
- Z_i: non-labor income, taste variables (married, kids, etc.)
- ε_i (error term): unobserved taste for work.

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Tobit Model – Type II – Sample selection

Example (from Heckman) (continuation)

• Market wage equation: $w_i=\beta_0+\beta_1 X_i+ \mu_i$ (2)

- X_i: productivity, age, education, previous experience, etc.
- μ_i (error term): unobserved wage earning ability.

Goal: Estimation of wage offer equation for people of working age

Q: The sample is non longer random. How can we estimate (2) if we only observe wages for those who work?

• Problem: Selection bias. Non-participation is rarely random

- Not distributed equally across subgroups
- Agents decide to participate or not –i.e., self-select into a group.

Q: Can we test for selection bias?

Tobit Model – Type II – Sample selection

• Terminology:

- *Selection equation*:

$y_i^* = \mathbf{z}_i'\boldsymbol{\alpha} + \varepsilon_{1,i}$ (often, a latent variable equation, say market wage vs. value of home production)

- *Selection Rule*:

$y_i = 0$ if $y_i^* \leq 0$

$= \mathbf{x}_i'\boldsymbol{\beta} + \varepsilon_{2,i}$ if $y_i^* > 0$

- *Outcome equation*:

$y_i = \mathbf{x}_i'\boldsymbol{\beta} + \varepsilon_{2,i}$

- Expectations:

  - Conditional expectation:

$$E[y \mid y>0,x] = \mathbf{x}_i'\boldsymbol{\beta} + \sigma_{12}\lambda(\mathbf{z}_i'\boldsymbol{\alpha}/\sigma_1)$$

  - Unconditional Expectation:

$$E[y \mid x] = \Phi(\mathbf{z}_i'\boldsymbol{\alpha}/\sigma_1) * [\mathbf{x}_i'\boldsymbol{\beta} + \sigma_{12}\lambda(\mathbf{z}_i'\boldsymbol{\alpha}/\sigma_1)]$$

Tobit Model – Type II – Sample selection

• From the conditional expectation:

$$E[y \mid y>0,x] = \mathbf{x}_i'\boldsymbol{\beta} + \rho\sigma_2\lambda(\mathbf{z}_i'\boldsymbol{\alpha}/\sigma_1)$$

• From the conditional expectation we see that applying OLS to observed sample will produce biased (and inconsistent) estimators. This is called *sample selection bias* (an omitted variable problem). It depends on ρ (and $\mathbf{z}$).

• But OLS y on X and λ on the sub-sample with $y^*>0$ produces consistent estimates. But, we need an estimator for λ. This idea is the basis of Heckman's two-step estimation.

• Estimation
- ML –complicated, but efficient
- Two-step –easier, but not efficient. Not the usual standard errors.

Tobit Model – Type II – Sample selection

• The Marginal effects of changes in exogenous variables have two components:
- Direct effect on mean of y_i, β_i via (2)
- If a variable affects the probability that $y_i^* > 0$, then it will affect y_i via λ_i

• Marginal effect if regressor appears in both $\mathbf{z}_i$ and $\mathbf{x}_i$:

$$\frac{\partial E[y_i \mid y_i > 0]}{\partial w_{ik}} = \beta_k - \alpha_k(\rho\sigma_1)\delta_k = \beta_k - \alpha_k\left(\lambda_i^2 - \left(-\frac{z_i\alpha}{\sigma_1}\right)\lambda_i\right)$$

Conditional distribution: Bivariate Normal

• To derive the likelihood function for the Sample selection model, we will use results from the conditional distribution of two bivariate normal RVs.

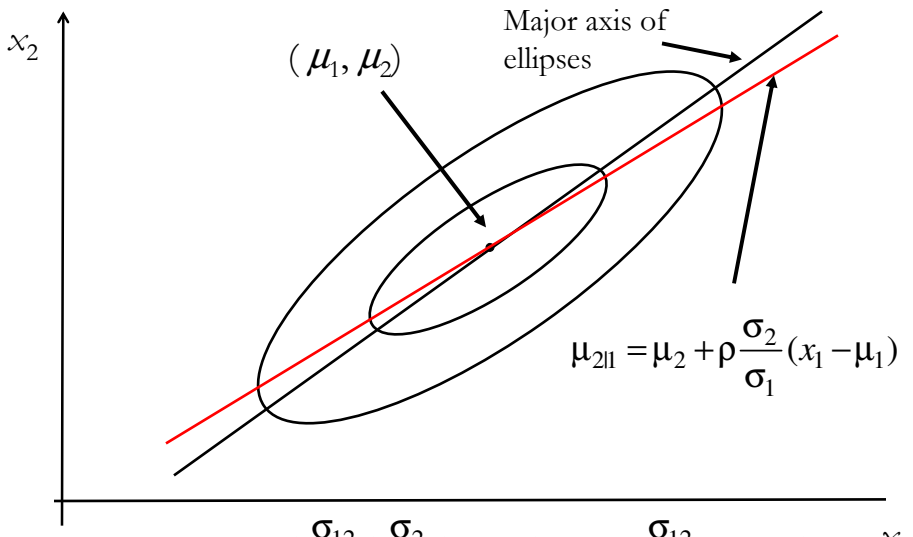
• Recall the definition of conditional distributions for continuous RVs:

$$f_{1|2}\left(x_1|x_2\right)=\frac{f\left(x_1,x_2\right)}{f_2\left(x_2\right)} \quad \text{and} \quad f_{2|1}\left(x_2|x_1\right)=\frac{f\left(x_1,x_2\right)}{f_1\left(x_1\right)}$$

• In the case of the bivariate normal distribution the conditional distribution of x_i given x_j is Normal with mean and standard deviation (using the standard notation):

$$\mu_{i|j}=\mu_i+\rho\frac{\sigma_i}{\sigma_j}\left(x_j-\mu_j\right) \quad \text{and} \quad \sigma_{i|j}=\sigma_i\sqrt{1-\rho^2}$$

Conditional distribution: Bivariate Normal



Note: $\mu_{2|1}=\mu_2+(\frac{\sigma_{12}}{\sigma_1\sigma_2})\frac{\sigma_2}{\sigma_1}(x_1-\mu_1)=\mu_2+\frac{\sigma_{12}}{\sigma_1^{\,2}}(x_1-\mu_1)$

Tobit Model – Type II – ML Estimation

• The model assumes a bivariate distribution for $(\varepsilon_{1,i};\varepsilon_{2,i})$, with covariance given by $\sigma_{12}(=\rho\sigma_1\sigma_2)$. We use a participation dummy variable: $D_i=0$ (No), $D_i=1$ (Yes).

• The likelihood reflects two contributions:

(1) Observations with y=0 –i.e., $y_i^* = \mathbf{z}_i'\boldsymbol{\alpha} + \varepsilon_{1,i} \leq 0 \Rightarrow D_i=0$.

- $\text{Prob}(D_i=0|\mathbf{x}) = P(y_i^* = \mathbf{z}_i'\boldsymbol{\alpha} + \varepsilon_{1,i} \leq 0|\mathbf{x}) = P(\varepsilon_{1,i} \leq -z_i'\boldsymbol{\alpha}|\mathbf{x}) = 1-\Phi(\mathbf{z}_i'\boldsymbol{\alpha})$

(2) Observations with y>0 –i.e., $y_i^* = \mathbf{z}_i'\boldsymbol{\alpha} + \varepsilon_{1,i} > 0 \Rightarrow D_i=1$.

- $f(y|D_i=1,\mathbf{x},\mathbf{z}) * \text{Prob}(D_i=1|\mathbf{x},\mathbf{z},\mathbf{y})$

(2.a) $f(y|D_i=1,\mathbf{x}) = P(D=1|\mathbf{y},\mathbf{x})\, f(y|\mathbf{x})]/P(D=1,\mathbf{x})$ (Bayes' Rule)

$= P(D=1|\mathbf{y},\mathbf{x})\, f(y|\mathbf{x})]/P(D=1,\mathbf{x})$

- $f(y|\mathbf{x}) = (1/\sigma_2)\, \varphi((y_i - \mathbf{x}_i'\boldsymbol{\beta})/\sigma_2)$

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Tobit Model – Type II – ML Estimation

(2.b) $- \text{Prob}(D_i=1|\mathbf{x},\mathbf{z},y_i) = P(\varepsilon_{1,i} > -\mathbf{z}_i'\boldsymbol{\alpha}|\mathbf{x},y_i)$

$= P[\{\varepsilon_{1,i} - (\varrho/\sigma_2)\,(y_i - \mathbf{x}_i'\boldsymbol{\beta})\}/\text{sqrt}\{\sigma_1^2(1-\varrho)^2\}$

$> \{-\mathbf{z}_i'\boldsymbol{\alpha} - (\varrho/\sigma_2)\,(y_i - \mathbf{x}_i'\boldsymbol{\beta})\}/\text{sqrt}\{\sigma_1^2(1-\varrho)^2\}]$

$= 1-\Phi(\{-\mathbf{z}_i'\boldsymbol{\alpha} - (\varrho/\sigma_2)\,(y_i - \mathbf{x}_i'\boldsymbol{\beta})\}/\text{sqrt}\{\sigma_1^2(1-\varrho)^2\})$

$= \Phi(\{\mathbf{z}_i'\boldsymbol{\alpha} + (\varrho/\sigma_2)\,(y_i - \mathbf{x}_i'\boldsymbol{\beta})\}/\text{sqrt}(1-\varrho)^2)$

- Moments of the conditional distribution $(y_1|y_2)$ of a normal RV:
  - Mean for RV 1: $\mu_1 + (\sigma_{12}/\sigma_2^2)\,(y_2 - \mu_2) = (\varrho/\sigma_2)\,(y_i - \mathbf{x}_i'\boldsymbol{\beta})$
  - Variance for RV 1: $\sigma_1^2\,(1-\varrho^2) = 1 - \varrho^2$ (Recall: $\sigma_1=1$)

• Now, we can put all the contributions together:

$L(\boldsymbol{\beta}) = \Pi_{y=0}\, P(y=0)\ \ \Pi_{y>0}\, P(y>0,\mathbf{x})\, f(y|\mathbf{x},\mathbf{z}) =$

$= \Pi_{D=0}\, P(D=0)\ \ \Pi_{D=1}\, P(D=1|\mathbf{x})\ \{P(D=1|\mathbf{y},\mathbf{x}) f(y|\mathbf{x})/P(D=1,\mathbf{x})\}$

$= \Pi_{D=0}\, P(y=0)\ \Pi_{D=1}\, P(D=1|\mathbf{y},\mathbf{x})\, f(y|\mathbf{x})$

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Tobit Model – Type II – ML Estimation

• Then, combining all the previous parts:

$$L(\boldsymbol{\beta},\boldsymbol{\alpha},\sigma_1,\varrho) = \Sigma_i (1-D_i) \log(1-\Phi(\mathbf{z}_i'\boldsymbol{\alpha}) + \Sigma_i D_i \log[\Phi(\{\mathbf{z}_i'\boldsymbol{\alpha} + (\varrho/\sigma_2)(y_i - \mathbf{x}_i'\boldsymbol{\beta})\}/\text{sqrt}(1-\varrho^2)] + \Sigma_i \log[(1/\sigma_2)\,\varphi((y_i - \mathbf{x}_i'\boldsymbol{\beta})/\sigma_2)]$$

• Complicated likelihood. The computational problem tends to be somewhat badly behaved:

=> Iterative methods do not always converge to the MLE.

Tobit Model – Type II – Two-step estimator

• It is much easier two use Heckman's two-step (*Heckit*) estimator:

(1) Probit part: Estimate $\boldsymbol{\alpha}$ using ML => get $\mathbf{a}$

(2) Truncated regression:

- For each D_i=1 (participation), calculate $\lambda_i = \lambda(\mathbf{x}_i'\mathbf{a})$
- Regress y_i -against $\mathbf{x}_i$ and $\lambda(\mathbf{x}_i'\mathbf{a})$. => get $\mathbf{b}$ and $\mathbf{b}_\lambda$.

• Problems:

- Not efficient (relative to MLE)
- Getting Var[$\mathbf{b}$] is not easy (we are estimating $\boldsymbol{\alpha}$ too).

• In practice it is common to have close to perfect collinearity, between $\mathbf{z}_i$ and $\mathbf{x}_i$. Large standard errors are common.

• In general, it is difficult to justify different variables for $\mathbf{z}_i$ and $\mathbf{x}_i$. This is a problem for the estimates. It creates an identification problem.

Tobit Model – Type II – Identification

• Technically, the parameters of the model are identified, even when **z**=**x**. But, identification is based on the distributional assumptions.

• Estimates are very sensitive to assumption of bivariate normality - Winship and Mare (1992) and **z**=**x**.

• ϱ parameter very sensitive in some common applications. Sartori (2003) comes with 95% C.I. for ϱ = -.999999 to +0.99255!

• Identification is driven by the non-linearity in the selection equation, through λ_i (and, thus, we need variation in the **z**'s too!).

• We find that when **z**=**x**, identification tends to be tenuous unless there are many observations in the tails, where there is substantial nonlinearity in the λ_i. We need exclusion restrictions.

Tobit Model – Type II – Testing the model

• Q: Do we have a sample selection problem?
Based on the conditional expectation, a test is very simple. We need to test if there is an omitted variable. That is, we need to test if λ_i belongs in the conditional expectation of $y|y>0$.

• Easy test: H_0: $\beta_\lambda=0$.

We can do this test using the estimator for β_λ, $\mathbf{b}_\lambda$, from the second step of Heckman's two-step procedure.

• Usual problems with testing.
- The test assumes correct specification. If the selection equation is incorrect, we may be unable to reject H_0.
- Rejection of H_0 does not imply accepting the alternative –i.e., sample selection problem. We may have non-linearities in the data!

Tobit Model – Type II – Testing the model

• Rejection of H_0 does not imply accepting the alternative –i.e., sample selection problem. We may have non-linearities in the data!

Identification issue II

We are not sure about the functional form. We may not be comfortable interpreting nonlinearities as evidence for endogeneity of the covariates.

Tobit Model – Type II – Application

```
. ************************************************
. * Estimating heckit model manually          *
. ************************************************
.
. ****************************
. * First create selection  *
. * Variable              *
. ****************************
.     gen s=0 if wage==.
(428 missing values generated)

.     replace s=1 if wage~=.
(428 real changes made)

. *******************************
. *Next, estimate the probit    *
. *selection equation           *
. *******************************
. probit s educ exper expersq nwifeinc age kidslt6 kidsge6

Iteration 0:   log likelihood =  -514.8732
Iteration 1:   log likelihood = -405.78215
Iteration 2:   log likelihood = -401.32924
Iteration 3:   log likelihood = -401.30219
Iteration 4:   log likelihood = -401.30219

Probit regression                                 Number of obs   =        753
                                                  LR chi2(7)      =     227.14
                                                  Prob > chi2     =     0.0000
Log likelihood = -401.30219                       Pseudo R2       =     0.2206

------------------------------------------------------------------------------
           s |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        educ |   .1309047   .0252542     5.18   0.000     .0814074    .180402
       exper |   .1233476   .0187164     6.59   0.000     .0866641    .1600311
     expersq |  -.0018871      .0006    -3.15   0.002    -.003063   -.0007111
    nwifeinc |  -.0120237   .0048398    -2.48   0.013    -.0215096   -.0025378
         age |  -.0528527   .0084772    -6.23   0.000    -.0694678   -.0362376
     kidslt6 |  -.8683285   .1185223    -7.33   0.000    -1.100628    -.636029
     kidsge6 |    .036005   .0434768     0.83   0.408    -.049208    .1212179
       _cons |   .2700768    .508593     0.53   0.595    -.7267473    1.266901
------------------------------------------------------------------------------
```

Estimating Heckit Manually. (note: you will not get the correct standard errors.

First step:
Probit selection equation

Tobit Model – Type II – Application

```
. ********************************
. *Then create inverse lambda   *
. ********************************
.  predict xdelta, xb

.  gen lambda =normalden(xdelta)/normal(xdelta)

. ***************************************
. *Finally, estimate the Heckit model *
. ***************************************
. reg lwage educ exper expersq lambda
```

Source	SS	df	MS
Model	35.0479487	4	8.76198719
Residual	188.279492	423	.445105182
Total	223.327441	427	.523015084

Number of obs = 428
F(4, 423) = 19.69
Prob > F = 0.0000
R-squared = 0.1569
Adj R-squared = 0.1490
Root MSE = .66716

lwage	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
educ	.1090655	.0156096	6.99	0.000	.0783835	.1397476
exper	.0438873	.0163534	2.68	0.008	.0117434	.0760313
expersq	-.0008591	.0004414	-1.95	0.052	-.0017267	8.49e-06
lambda	.0322619	.1343877	0.24	0.810	-.2318889	.2964126
_cons	-.5781032	.306723	-1.88	0.060	-1.180994	.024788

Second step: Truncated regression

Note: The standard errors are not correct.

Tobit Model – Type II – Application

```
. heckman lwage educ exper expersq, select(s=educ exper expersq nwifeinc age kidslt6 kidsge6) twostep
```

Heckman selection model -- two-step estimates
(regression model with sample selection)

Number of obs = 753
Censored obs = 325
Uncensored obs = 428

Wald chi2(3) = 51.53
Prob > chi2 = 0.0000

	Coef.	Std. Err.	z	P>|z|	[95% Conf.	Interval]
lwage						
educ	.1090655	.015523	7.03	0.000	.0786411	.13949
exper	.0438873	.0162611	2.70	0.007	.0120163	.0757584
expersq	-.0008591	.0004389	-1.96	0.050	-.0017194	1.15e-06
_cons	-.5781032	.3050062	-1.90	0.058	-1.175904	.019698
s						
educ	.1309047	.0252542	5.18	0.000	.0814074	.180402
exper	.1233476	.0187164	6.59	0.000	.0866641	.1600311
expersq	-.0018871	.0006	-3.15	0.002	-.003063	-.0007111
nwifeinc	-.0120237	.0048398	-2.48	0.013	-.0215096	-.0025378
age	-.0528527	.0084772	-6.23	0.000	-.0694678	-.0362376
kidslt6	-.8683285	.1185223	-7.33	0.000	-1.100628	-.636029
kidsge6	.036005	.0434768	0.83	0.408	-.049208	.1212179
_cons	.2700768	.508593	0.53	0.595	-.7267473	1.266901
mills						
lambda	.0322619	.1336246	0.24	0.809	-.2296376	.2941613
rho	0.04861					
sigma	.66362875					
lambda	.03226186	.1336246				

Heckit Model estimated automatically.

Note H_0:ρ=0 cannot be rejected. There is little evidence that sample selection bias is present.