Chapter 9
Optimization: One Choice Variable

9.1 Optimum Values and Extreme Values

- Goal vs. non-goal equilibrium
- In the optimization process, we need to identify the objective function to optimize.
- In the objective function the dependent variable represents the object of maximization or minimization

Example:
- Define profit function: \( \pi = PQ - C(Q) \)
- Objective: Maximize \( \pi \)
- Tool: Q
9.2 Relative Maximum and Minimum: First-Derivative Test

*Critical Value*

The critical value of $x$ is the value $x_0$ if $f'(x_0) = 0$.

- A stationary value of $y$ is $f(x_0)$.
- A stationary point is the point with coordinates $x_0$ and $f(x_0)$.
- A stationary point is coordinate of the extremum.

*Theorem* (Weierstrass)

Let $f : S \rightarrow \mathbb{R}$ be a real-valued function defined on a compact (bounded and closed) set $S \subset \mathbb{R}^n$. If $f$ is continuous on $S$, then $f$ attains its maximum and minimum values on $S$. That is, there exists a point $c_1$ and $c_2$ such that

$$f(c_1) \leq f(x) \leq f(c_2) \quad \forall x \in S.$$

### 9.2 First-derivative test

- The *first-order condition* (f.o.c.) or necessary condition for extrema is that $f'(x^*) = 0$ and the value of $f(x^*)$ is:

  - A relative minimum if $f'(x^*)$ changes its sign from negative to positive from the immediate left of $x_0$ to its immediate right. (first derivative test of min.) 😊

  ![Graph of a relative minimum](image)

  - A relative maximum if the derivative $f'(x)$ changes its sign from positive to negative from the immediate left of the point $x^*$ to its immediate right. (first derivative test for a max.) 😊

  ![Graph of a relative maximum](image)
9.2 First-derivative test

- The first-order condition or necessary condition for extrema is that \( f'(x^*) = 0 \) and the value of \( f(x^*) \) is:
- Neither a relative maxima nor a relative minima if \( f'(x) \) has the same sign on both the immediate left and right of point \( x_0 \) (first derivative test for point of inflection).

\[
\lim_{x \to x_0^-} f'(x) = \lim_{x \to x_0^+} f'(x)
\]

\[
f'(x) = 0
\]

\[
x^*
\]

9.2 Example: Average Cost Function

\[
AC = Q^2 - 5Q + 8 \quad \text{Objective function}
\]

\[
f'(Q) = 2Q - 5 \quad \text{1st derivative function}
\]

\[
f''(Q) = 2Q - 5 = 0 \quad \text{f.o.c.}
\]

\[
Q' = 5/2 = 2.5 \quad \text{extrema}
\]

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9.3 The smile test

- Convexity and Concavity: The smile test for maximum/minimum

- If $f''(x) < 0$ for all $x$, then strictly concave.
  \[ \Rightarrow \text{critical points are global maxima} \]

- If $f''(x) > 0$ for all $x$, then strictly convex.
  \[ \Rightarrow \text{critical points are global minima} \]

- If a concave utility function (typical for risk aversion) is assumed for a utility maximizing representative agent, there is no need to check for s.o.c. Similar situation for a concave production function.

9.3 The smile test: Examples:

**Example 1: Revenue Function**

1) $TR = 1200Q - 2Q^2$  Revenue function
2) $MR = 1200 - 4Q = 0$  f.o.c.
3) $Q' = 300$  extrema
4) $MR' = -4$  2nd derivative
5) $MR' < 0$  \[ \Rightarrow Q' = 300 \text{ is a maximum} \]

**Example 2: Average Cost Function**

$AC = Q^2 - 5Q + 8$  Objective function

$f'(Q) = 2Q - 5 = 0$  f.o.c.

$Q' = 5/2 = 2.5$  extrema

$f''(Q) = 2 > 0$  \[ \Rightarrow Q' = 5/2 = 2.5 \text{ is a minimum} \]
9.3 Inflection Point

- **Definition**
  A twice differentiable function \( f(x) \) has an inflection point at \( x \) iff the second derivative of \( f(x) \) changes from negative (positive) in some interval \((m, \bar{x})\) to positive (negative) in some interval \((\bar{x}, n)\), where \( \bar{x} \in (m, n) \).

- **Alternative Definition**
  An inflection point is a point \((x, y)\) on a function, \( f(x) \), at which the first derivative, \( f'(x) \), is at an extremum, i.e. a minimum or maximum. (Note: \( f''(x)=0 \) is necessary, but not sufficient condition.)

**Example:** \( U(w) = w - 2w^2 + w^3 \)

\[
U'(w) = 4w + 3w^2
\]

\[
U''(w) = 4 + 6w
\]

\[
\Rightarrow 4 + 6w = 0
\]

\( w = 2/3 \) is an inflection point

9.4 Formal Second-Derivative Test: Necessary and Sufficient Conditions

- The zero slope condition is a necessary condition and since it is found with the first derivative, we refer to it as a 1st order condition.

- The sign of the second derivative is sufficient to establish the stationary value in question as a relative minimum if \( f''(x_0) > 0 \), the 2nd order condition or relative maximum if \( f''(x_0) < 0 \).
9.4 Example 1: Optimal Seignorage

\[
\frac{M}{P} = e^{-\lambda(x + \pi + a)} \quad \text{Demand for money}
\]

\[
S = \pi \frac{M}{P} = \pi e^{-\lambda(x + \pi + a)} \quad \text{Seignorage}
\]

F.o.c. (assume \( \pi = \pi^* \)):

\[
\frac{dS}{d\pi} = e^{-\lambda(x + \pi + a)} + \pi(-\lambda)e^{-\lambda(x + \pi + a)}
\]

\[
= e^{-\lambda(x + \pi + a)} + \pi(-\lambda)e^{-\lambda(x + \pi + a)} = e^{-\lambda(x + \pi + a)} (1 - \pi\lambda)
\]

\((1 - \pi^* \lambda) = 0 \Rightarrow \pi^* = \frac{1}{\lambda} \quad \text{(Critical point)}\)

S.o.c.:

\[
\frac{d^2S}{d\pi^2} = -\lambda e^{-\lambda(x + \pi + a)} (1 - \pi\lambda) + (-\lambda)e^{-\lambda(x + \pi + a)} = -\lambda e^{-\lambda(x + \pi + a)} (2 - \pi\lambda)
\]

\[
\frac{d^2S}{d\pi^2}(\pi^* = \frac{1}{\lambda}) = -\lambda e^{-\lambda(x + \pi + a)} < 0 \Rightarrow \pi^* = \frac{1}{\lambda} \text{ is a maximum}
\]

9.4 Example 2: Profit function (Two solutions)

Revenue and Cost functions
1) \(TR = 1200Q - 2Q^2\)
2) \(TC = Q^3 - 61.25Q^2 + 1528.5Q + 2000\)

Profit function
3) \(\pi = TR - TC = -Q^3 + 59.25Q^2 - 328.5Q - 2000\)

1st derivative of profit function
4) \(\pi' = -3Q^2 + 118.5Q - 328.5 = 0\)
5) \(Q_1' = 3 \quad Q_2' = 36.5\)

2nd derivative of profit function
6) \(\pi'' = -6Q + 118.5\)
7) \(\pi''(3) = 100.5 \quad \pi''(36.5) = -100.5\)

applying the smile test
8) \(\pi''(Q_1') > 0 \rightarrow \min \quad \pi''(Q_2') < 0 \rightarrow \max\)
9.4 Example 3: Optimal Timing - wine storage

\[ A(t) = Ve^{-rt} \]  \hspace{1cm} \text{Present value}
\[ V = ke^{\frac{r}{2}} \]  \hspace{1cm} \text{Growth in value}

\[ A(t) = ke^{\frac{r}{2}} e^{-rt} = ke^{\frac{r}{2} - rt} \]

\[ \ln A(t) = \ln k + \frac{r}{2} - rt = \ln k + \left( \frac{r}{2} - rt \right) \]

Monotonic transformation of objective function

\[ \frac{dA}{dt} = \frac{1}{2} t^{-\frac{3}{2}} - r \]

F.o.c.: \[ \frac{dA}{dt} = A \left( \frac{1}{2} t^{-\frac{3}{2}} - r \right) = 0 \]
\[ \Rightarrow \frac{1}{2} t^{-\frac{3}{2}} = r \]
\[ \Rightarrow t^* = \frac{1}{4r^2} \]

9.4 Example 3: Optimal Timing - wine storage

Optimal time: \[ t^* = \frac{1}{4r^2} \]

Let \( r = 10\% \)

\[ t^* = \frac{1}{4(0.10)^2} = 25 \text{ years} \]

Determine optimal values for \( A(t) \) and \( V \):

\[ A(t) = ke^{\frac{r}{2} - rt} \]

let \( k = \$1 \)/bottle

\[ A(t) = e^{\frac{r}{2} - (1/25)} = e^{2.5} = \$12.18 \)/bottle

\[ V = Ae^{rt} \]

\[ V = (\$12.18/bottle) \cdot e^{(1/25)} = \$148.38/bottle \]

\[ V = \$148.38/bottle \]
9.4 Example 3: Optimal Timing - wine storage

Plot of Optimal Time with $r = 0.10$ \(\Rightarrow t = 25\)

9.4 Example 4: Least Squares

- In the CLM, we assume a linear model, relating $y$ and $X$, which we call the DGP:
  $$y = X\beta + \varepsilon$$

- The relation is not exact, there is an error term, $\varepsilon$. We want to find the $\beta$ that minimizes the sum of square errors, $\varepsilon'\varepsilon$.

- Assume there is only one explanatory variable, $x$. Then,
  $$Min_{\beta} \ S(\beta \mid y, x) = \sum_{t=1}^{T} (y_t - x_t \beta)^2$$

Then, we write the likelihood function, $L$, as:

$$\frac{\partial S}{\partial \beta} = \sum_{t=1}^{T} 2(y_t - x_t \beta)(-x_t) = -\sum_{t=1}^{T} (y_t x_t - x_t^2 \beta) = 0$$

In the general, multivariate case, these f.o.c. are called normal equations.
9.4 Example 4: Least Squares

• The LS solution, \( b \), is
\[
    b = \frac{\sum_{t=1}^{T} y_t x_t}{\sum_{t=1}^{T} x_t^2} = (x'x)^{-1} x' y
\]

• Second order condition
\[
    \frac{\partial^2 S}{\partial \beta^2} = -\sum_{t=1}^{T} (-x_t^2) > 0 \implies b \text{ a minimum}
\]
(It’s a globally convex function.)

9.4 Example 5: Maximum Likelihood

• Now, in the CLM, we assume \( \varepsilon \) follow a normal distribution:
\[
    \varepsilon | X \sim N(0, \sigma^2 I_t)
\]

Then, we write the likelihood function, \( L \), as:
\[
    L = f(y_1, y_2, ..., y_T | \beta, \sigma^2) = \prod_{t=1}^{T} \left( \frac{1}{2 \pi \sigma^2} \right)^{1/2} \exp\left[-\frac{1}{2 \sigma^2} (y_t - x_t' \beta)^2 \right]
\]

• We want to find the \( \beta \) that maximizes the likelihood of the occurrence of the data. It is easier to do the maximization after taking logs –i.e, a monotonic increasing transformation. That is, we maximize the log likelihood function w.r.t. \( \beta \):
\[
    \ln L = \sum_{t=1}^{T} \ln \left( \frac{1}{2 \pi \sigma^2} \right)^{1/2} + \sum_{t=1}^{T} \left( \frac{1}{2 \sigma^2} (y_t - x_t' \beta)^2 \right)
    = -\frac{T}{2} \ln 2\pi - \frac{T}{2} \ln \sigma^2 - \frac{1}{2 \sigma^2} \varepsilon' \varepsilon
\]
9.4 Example 5: Maximum Likelihood

• Assume $\sigma$ known and only one explanatory variable, $x$. Then,

$$\text{Max } \beta \ln L(\beta \mid y, x) = -\frac{T}{2} \ln 2\pi - \frac{T}{2} \sigma^2 - \frac{1}{2\sigma^2} \sum_{t=1}^{T} (y_t - x_t \beta)^2$$

• Then, the f.o.c.:

$$\frac{\partial \ln L}{\partial \beta} = -\frac{1}{2\sigma^2} \sum_{t=1}^{T} 2(y_t - x_t \hat{\beta}_{MLE})(-x_t) = \frac{1}{\sigma^2} \sum_{t=1}^{T} (y_t x_t - x_t^2 \hat{\beta}_{MLE}) = 0$$

Note: The f.o.c. delivers the normal equations for $\beta$ (same condition as in Least Squares) $\Rightarrow$ MLE solution = LS estimator, $b$. That is,

$$\hat{\beta}_{MLE} = b = \frac{\sum_{t=1}^{T} y_t x_t}{\sum_{t=1}^{T} x_t^2} = (x'x)^{-1} x'y$$

• Nice result for OLS $b$: ML estimators have very good properties!

9.4 Example 5: Maximum Likelihood 🤔

• Second order condition

$$\frac{\partial \ln^2 L}{\partial \beta^2} = \sum_{t=1}^{T} -x_t^2 < 0 \quad \Rightarrow \hat{\beta}_{MLE} \text{ a maximum}$$

(It’s a globally concave function.) …
9.5 Taylor Series of a polynomial function: Revisited

Taylor series for an arbitrary function: Any function can be approximated by the weighted sum of its derivatives. Then the change is given by

\[ f(x) - f(x_0) = \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + R_{n+1} \]

Where \( R_{n+1} = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1} \)

If \( n = 2 \), then

\[ f(x) - f(x_0) = \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + R_3 \]

At \( x_0, f''(x_0) = 0 \) - i.e., \( x_0 \) is a max., min., or inflection.

Then,

\[ f(x) - f(x_0) \approx \frac{f''(x_0)}{2!}(x-x_0)^2 \Rightarrow \text{the sign of } f''(x_0) \text{ determines what } x_0 \text{ is.} \]

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9.5 Taylor expansion and relative extremum

A function \( f(x) \) attains a relative max (min) value at \( x_0 \) if \( f(x) - f(x_0) \) is neg. (pos.) for values of \( x \) in the immediate neighborhood of \( x_0 \) (the critical value) both to its left and right Taylor series approximation for a small change in \( x \):

\[ f(x) - f(x_0) = \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + R_{n+1} \]

At the max., min., or inflection, \( f''(x_0) = 0 \)

and if \( f''(x_0) = 0 \), and if \( f^{(3)}(x_0) = 0, \cdots \)

then \( f(x) - f(x_0) = R_{n+1} \)

What is the sign of \( R \) for the first nonzero derivative?
### 9.5 N\textsuperscript{th}-derivative test

If the first derivative of a function $f(x)$ at $x_0$ is $f^{(1)}(x_0) = 0$ and if the first nonzero derivative value at $x_0$ encountered in successive derivation is that of the $N\text{th}$ derivative, $f^{(n)}(x_0) \neq 0$, then the stationary value $f(x_0)$ will be:

- a relative max if $N$ is even and $f^{(n)}(x_0) < 0$
- a relative min if $N$ is even and $f^{(n)}(x_0) > 0$
- an inflection point if $N$ is odd

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### Example

**Function:**

$$Y = (7-x)^4$$

- **Primitive function**

**1\text{st} derivative:**

$$Y' = -4(7-x)^3$$

**0 at** $x^* = 7$ **the critical value**

**2\text{nd} derivative:**

$$Y''(7) = 12(7-x)^2 = 0$$

**3\text{rd} derivative:**

$$Y'''(7) = -24(7-x) = 0$$

**4\text{th} derivative:**

$$Y^{(4)}(7) = 24$$

Because first nonzero derivative $Y^{(n)}$ is even (4) and $Y^{(4)} > 0$ (24), critical value is a min.
9.5 \(N^{th}\)-derivative test

\[ Y = (7-x)^3 \]

primitive function

M

\[ Y^{(4)} = 24 \]

4\(^{th}\) derivative

decision rule:
n is even (4) and > 0

therefore a minimum