REVIEW CHAPTER

CAPITAL MARKETS RETURNS IN EQUILIBRIUM

To explain how capital markets work, economists usually assume that markets are in equilibrium. Many financial economists believe that markets are efficient, which means that market prices reflect all available information. In this chapter, first, we will examine the efficient market hypothesis (EMH). Then, we will study the two major theories of capital market returns based on the equilibrium concept: the capital asset pricing model (CAPM) and arbitrage pricing theory (APT).

I. Efficient Market Hypothesis (EMH)

As long as stock markets have existed, "experts" have attempted to predict future movements in stock prices. Those experts, using the information that was available to investors, issued recommendations based on their forecasts. In the early days of stock markets, forecasters had very weak theories behind their forecasts. Early attempts to evaluate the effectiveness of the early forecasters –see the pioneering paper of Alfred Cowles III, published in 1933, in the first volume of Econometrica- suggested that those forecasts had no value to investors. Thus, even during those early days, stock price forecasting was an extremely difficult task. The fact that forecasting stock prices is very difficult highlights a usual finding in financial economics: asset prices seem to reflect all publicly available information. That is, stock markets are *efficient*.

Definition

Efficient markets are those that, at every moment in time, fully reflect all available information in the price of every asset.

If stock prices reflect all that is known about firms, then asset prices should be equal to their true value defined as discounted future cash flows. Hence, investors cannot earn abnormal returns based on public information. In practice, asset prices might differ from their true value due to transaction costs, technological and institutional consideration.

Efficiency can only be defined with reference to a specific type of information. There are three classes of information:

(a) Historical sequence of prices (weak form EMH).

(b) Public records of companies and public forecasts regarding the future performance and possible actions (a + b: => semi-strong form EMH).

(c) Private or inside information (a + b + c => strong form EMH).

1.A Weak Form EMH

In this version of the EMH, the information set consists of the history of price series. The weak form EMH states that there cannot be reliable time patterns to the random returns of any asset. There is a powerful intuition behind the weak EMH: if there were reliable patters, traders would use them. Traders would try to buy at the low points and sell at the high points. This process would lead to price pressures that would lower the high points and raise the low points, therefore, destroying the patterns.

Tests of the weak form EMH tend to have a strong support for the hypothesis that stock price changes are independent over time. Many researchers have found that asset returns follow a random walk, i.e., their behavior is consistent with the notion of an efficient market in its weak form. Sometimes autocorrelations are significantly different from zero, however, they are found to be too small to profit from them.

The most popular tests are tests based on the autocorrelation coefficients: t-test and Q(K) test (see Chapter IV).

Example RC.1: Table RC.1 present autocorrelations for four international stock markets.

TABLE RC.1
Sample Autocorrelations for weekly stock returns (1980-1990)
Number of Observations (T): 522

	U.S.	U.K.	JAPAN	AUSTRALIA
ρ_1	.054	.001	.098	.184
ρ_2	-.008	.036	.005	.080
ρ_3	-.030	-.077	.073	.013
ρ_4	-.017	.068	-.030	.030
ρ_5	-.058	-.051	.033	-.065
ρ_6	.128	-.057	.010	-.057
ρ_7	.034	-.036	-.005	-.040
ρ_8	.001	-.059	-.031	-.108
Q(4)	2.80	6.22	8.38	22.03
Q(8)	13.31	11.59	9.54	32.96

The evidence for the U.S., the U.K. and Japan is consistent with the random walk hypothesis. For Australia, there is evidence in favor of serial correlation, however the autocorrelations are small. ¶

Other tests of the weak form EMH are based on testing if trading rules can deliver a higher return than simple buy-and-hold strategies. These tests tend to find support for the weak form EMH.

1.B Semi-strong form EMH

This version of the EMH states that prices reflect all information available to the public. The information set consists of the history of prices plus other exogenous and relevant information, such as dividend announcements, changes in Treasury-bill rates, etc. The reason behind this theory is also simple: traders will respond quickly and rationally to public announcements of relevant information. Therefore, trading rules based on public information cannot earn abnormal profits. Tests of the semi-strong form EMH check if asset prices incorporate immediately this exogenous information. The most popular method used to test the semi-strong EMH is the so-called *event study* method. This technique is based on measuring abnormal returns before and after the `event' -for example, a dividend announcement- and then compare the performance of abnormal returns (using a t-test).

There are several measures of abnormal returns for the asset i. A popular measure is:

$$(r_i - r_m),$$

r_i = return for asset i.
r_m = return for an appropriate market portfolio.

Example RC.2: Suppose Bank One announces an "unexpected" stock repurchase program, which is usually seen as a good sign. In this case, the test will consist in measuring the stock price reaction at different points in time. If for example, the Bank One stocks are earning abnormal returns (measured against a banking market portfolio) two weeks after the announcement, it will be taken as a violation of the semi-strong EMH. ¶

Example RC.3: From 1993-August 1994, 22 banks announced stock repurchases. Around the announcement date, the average stock of the sample earned an excess (abnormal) return close to 3%. Three days after the announcement date, the average stock of the sample did not show any excess return. ¶

For the U.S. market, using event studies there is strong support for the semi-strong form EMH. Stock prices seem to incorporate very quickly new information about stock splits, IPOs, world events, accounting changes, and a variety of corporate finance events. There is however, conflicting results from event studies investigating the effect of changing the exchange listing of a firm.

1.C Strong form EMH

The strong version of the EMH states that prices reflect all available information. This is a very strict definition. For example, under this view, it is impossible to profit from inside information.

For the NYSE, this form of the EMH is rejected. For example, there is evidence indicating that corporate insiders tend to purchase in the months before a "good" event (an increase in dividends) and to sell before a "bad" event (an unexpected decrease in profits).

The widely held view in international markets is that insider trading is frequent, especially in emergent markets (see Chapter XI).

An alternative test of the strong form of market efficiency, however, is to examine the performance of professionally managed portfolios. Professional managers should have better information than the general public and, therefore, should be able to profit from it. Several studies have shown that U.S. professional managers are not able to consistently beat the market.

1.D Application: Anomalies in Stock Returns and the EMH

Researchers have found anomalies in the behavior of stock returns. Two types of anomalies have been documented in international markets: (1) seasonalities (day-of-the-week, month-of-the-month) and (2) size effects. These anomalies cast a doubt on the EMH. For example, a typical critic of EMH is: "if markets are efficient, how can markets have on average positive returns in January?"

The EMH, however, is still valid unless we can show that the higher average returns earned exploiting the above mentioned anomalies do not disappear once returns are adjusted for transaction costs and incremental risk (if any).

1.D.1 Day-of-the-week effect

A-day-of-the-week effect is inconsistent with the weak form of the EHM: a pattern based on the past history can consistently be used to earn higher average returns.

In Table RC.2, we can observe the average daily stock returns of six market indices for the first five days of the week.

TABLE RC.2
Daily Stock Returns (%) 1969-1984

	Monday	Tuesday	Wednesday	Thursday	Friday
Canada	-.016*	-.003	.073*	.075*	.094*
France	-.050	-.157*	.100*	.152*	.082*
Japan	.090*	-.095*	.139*	.025	.039
Singapore	-.036	-.107*	.079*	.121*	.100*
UK	-.095*	.106*	.090*	.011	.044
US	-.134*	.013	.057	.021	.058

Note: * significantly different from zero at the 5% level (t-test).

There are several patterns worth noticing:

i. There is a significant `day-of-the-week' effect in all countries. Average stock market returns on different days of the week are not the same.
ii. Average daily returns on Wednesday, Thursday and Fridays are positive, while the first two days are often negative.
iii. There is a negative Monday effect in the U.S., Canada and the U.K.
iv. There is a negative Tuesday effect on the Far East markets.

A partial explanation has been suggested to explain the day-of-the-week effect. The Monday effect is based on the settlement-delay hypothesis: for example, there exist a delay of several days between the day a stock is traded and the day the funds are actually transferred (before 1995, in the U.S., it used to be five business days). This means that for stock purchases on a business day other than Friday, the buyer will have eight calendar days before losing funds. For Friday purchases, the buyer will have ten calendar days and hence two more days of interest earning. In efficient markets, buyers would be willing to pay extra for stocks bought on Fridays.

Similarly, the Tuesday effect may be the result of a carry over effect of the Monday effect from NYSE. It is partially explained as a time-zone phenomenon. Recall that the Far East is one day ahead of New York.

It is doubtful that a trader can profit from a strategy of buying on Thursday and selling on Friday, once transaction costs are included.

1.D.2 Month-of-the-year effect

A Month-of-the-year effect is also inconsistent with the weak form of the EMH. Table RC.3 shows evidence of seasonality in monthly stock returns in five international markets.

TABLE RC.3
JAN-JUN STOCK RETURNS (1959-1979)

	January	February	March	April	May	June
Canada	2.90	0.07	.79	0.41	-0.96	-.30
France	3.72	-.18	1.98	0.94	-0.66	-1.90
Japan	3.53	1.13	1.88	0.94	0.96	2.06
UK	3.40	0.69	1.25	3.13	-1.00	-.85
US	1.04	-.41	1.27	0.96	-1.38	-.56
US-EW	5.08	0.55	1.55	0.44	-1.42	-1.00

Note: US-EW (equally weighted index)
Source of data: MSCI.

Note:
i. Average returns on January are always positive and generally significantly higher than during the rest of the year.
ii. The two indices for the US give us an idea of the source of the January effect: when we look at an index where small stocks are equally weighted January returns are significantly higher, but for the rest of the months, returns are very similar.
iii. There seems to be an April effect on the U.K. ("sell in May and ran away").

Possible explanation for the January effect: tax laws.

1.D.3 Size effect

A size effect is evidence against the semi-strong EMH. (Why?) In Table RC.4, we present evidence for a small size premium in international stock markets.

TABLE RC.4
MONTHLY STOCK RETURNS

Country	Canada	France	Japan	U.K.	U.S.
Period	73-80	68-80	66-83	58-82	26-79
Return on smallest portfolio	1.67	1.62	2.03	1.33	1.77
Return on largest portfolio	1.23	0.97	1.14	0.93	0.93
Size premium (small - large)	0.44	0.65	0.89	0.40	0.84

The size premium is positive in all markets. It is highest in Japan (actually, the largest is Australia with a 5.73 % per month) and lowest in the U.K.

Is the size effect stable over time? No. For example, in the U.S., the size premium varied between -10.20 percent over 1926-30 and a high of 43.80 percent over the subperiod 1931-35.

Is the size effect the same across the year? No. It is particularly large and positive in January. For example, in Japan, the size premium is 6.1 percent in January and 0.7 percent the rest of the year.

Implications for investment management:

(1) If investors know there is a January size premium in Japan, why do they not take advantage of it by simply buying small firms in December and reselling them in January? If they did, the January size premium would eventually disappear. Possible reasons for its continued presence are transaction costs.

(2) There are years when the premium is negative. Therefore, any strategy should be a long-term strategy.

II. Preliminaries to Asset Pricing Models

If future prices are unknown, returns can be considered random variables. We usually describe possible future values of a financial asset and their likelihoods with a probability distribution (see Appendix RC). Probability distributions are described by its moments. The first two moments measure central location and dispersion, respectively. These two moments are crucial for the Capital Assets Pricing Model (CAPM).

The first moment, or expected value, provides a measure of the most likely value (mean). The second moment measures the dispersion of the data. The variance provides a measure of the deviation of a series around its mean. In general, it is preferred to work with the standard deviation, instead of the variance, as a measure of uncertainty or risk.

The standard deviation (σ_X) has three attractive properties as a measure of uncertainty:

i. it measures how spread out X_t is around $E[X_t]$.
ii. because differences around the mean are squared, large differences have much more impact than small ones.
iii. σ_X is in the same unit as $E[X_t]$ and the stock price itself.

The standard deviation of returns of financial asset is called *volatility*.

2.A Risk and Return

We want a theory that relates risk and expected return. We will assume that investors like positive returns and dislike risk. In this context, our typical investor wishes to maximize the expected rate of return and minimize risk. We will measure risk with a measure of dispersion, for example, the standard deviation.

Example RC.5: Table RC.5 shows the USD weekly dollar returns for seven indices.

TABLE RC.5
USD weekly returns for Pacific and N. American markets (1980-1990)

Portfolio	Return	SD	Number of Firms
Australia	.16	3.70	63
Hong Kong	.17	4.97	32
Japan	.46	2.73	265
Singapore	.20	3.48	53
Canada	.15	2.64	89
U.S.	.22	2.25	320
World Index	.28	1.91	1419

There is a negative relation between number of firms in the country index and the SD. Hong Kong has the highest SD and the smallest number of firms in the index. On the other hand the U.S. and Japan have the largest number of firms indices and the smallest country's SD. The number of firms gives us an idea of the *diversification* of the index. ¶

It turns out there is an important distinction between the risk of an *individual security* and the risk of a collection of securities, or *portfolio*. The reason is *diversification*, that is, by holding a portfolio of financial assets it is possible to wipe out risk.

The extent to which risk can be wiped out, depends upon whether the assets in a portfolio move with or against each other. A measure of this co-movement is the *correlation coefficient*, ρ. The correlation coefficient ρ is a number between -1 and +1. When equal to one, the two variables are said to have perfect positive correlation, that is, when one variable changes the other variable also changes by exactly the same amount. Similarly, when ρ=-1, the two variables are said to have perfect negative correlation, that is, when one variable changes, the other variable also changes by exactly the same amount but in an opposite direction.

Example RC.6: Table RC.6 reports the correlation for the Pacific Indices reported in Table 5.

TABLE RC.6
Correlation Coefficients for Pacific Markets Returns (1980-1990)

	AUST	HK	JAPAN	SING
AUSTRALIA	1.00	0.37	0.36	0.47
HONG KONG		1.00	0.21	0.48
JAPAN			1.00	0.26
SINGAPORE				1.00

All markets show a moderate to low correlation coefficient. ¶

? International Stock Market Big Co-movements does not Imply High Correlations
It is common to observe that international stock markets move together when there is a big movement in one of the major stock markets in the world, especially in the U.S. These episodes give the impression of highly correlated (*integrated*) international markets. Empirical analysis of correlations of international stock markets has repeatedly found that international stock markets are not highly correlated. ?

2.A.1 Technical note

Using the results from the Appendix, the variance of a portfolio of N assets, σ_p^2, which have a set of weights given by ω_j, for j=1,2,...N, is given by:

$$\sigma_w^2 = \omega_1^2\,\sigma_1^2 + \omega_2^2\,\sigma_2^2 + ... + \omega_N^2\,\sigma_N^2 + 2\,\omega_1\omega_2\,\sigma_{12} + 2\omega_1\omega_2\,\sigma_{12} + 2\omega_1\omega_3\,\sigma_{13} + ...$$
$$... + 2\,\omega_{N-1}\omega_N\,\sigma_{(N-1)N},$$

where σ_{ij} measures the covariance between securities i and j.

Recall the definition of the correlation coefficient, $\rho_{ij}=\sigma_{ij}/\sigma_i\sigma_j$. Now, we can rewrite σ_w^2 as:

$$\sigma_w^2 = \omega_1^2\,\sigma_1^2 + \omega_2^2\,\sigma_2^2 + \ldots\ldots + \omega_N^2\,\sigma_N^2 + 2\,\omega_1\omega_2\,\sigma_{12} + 2\omega_1\omega_2\rho_{12}\,\sigma_1\,\sigma_2 + 2\omega_1\omega_3\rho_{13}\,\sigma_1\,\sigma_3 + \ldots\ldots$$
$$\ldots + 2\omega_{N-1}\omega_N\rho_{(N-1)N}\,\sigma_{N-1}\,\sigma_N.$$

Example RC.7: Diversification with two stocks.
Suppose an investor is interested in investing in Singapore and Japan and decides to form an equally weighted ($\omega_s=\omega_j=.5$) portfolio. The correlation coefficient, ρ_{SJ}, is equal to .26.

The variance of the portfolio is by:

$$\begin{aligned}\sigma_w^2 &= \omega_S^2\,\sigma_S^2 + \omega_J^2\,\sigma_J^2 + 2x\omega_S\omega_J\,\rho_{SJ}\,\sigma_S\,\sigma_J \\ &= (.5)^2\,(3.48)^2 + (.5)^2\,(2.73)^2 + 2(.5)(.5)\,(.26)\,(3.48)\,(2.73) \\ &= 6.126.\end{aligned}$$

Therefore, $\sigma_w = (\sigma_w^2)^5 = 2.48$, which is smaller than σ_S and σ_J. ¶

2.B Measuring the Risk of Individual Assets

We will see that the standard deviation is not the relevant measure of risk for an individual security.

2.B.1 A Brief Review of Regression Analysis:

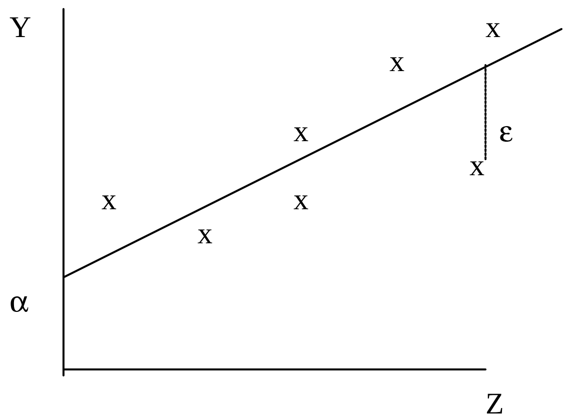
We want to fit a line through a set of points

$$Y = \alpha + ß\,Z + \varepsilon,$$

where:
Y = value of the dependent variable.
Z = value of the independent variable.
ε = a random error for the observation.
α = the point where the line crosses the Y axis (the interception).
ß = the slope of the line (change in Y given a change in Z).

The general idea is quite simple. There is a linear relation between Y and Z, which is not quite exact. Typically we do not the relationship, we need to estimate it. In general, we will assume that $\varepsilon\sim N(0,\sigma_\varepsilon^2)$, where σ_ε^2 measures how much of the variability of Y is unexplained by Z. A regression will estimate the unknown parameters α and ß, by fitting the line through the scatter plot that minimizes the sum of squared errors.

Graphically:



The formula that estimates ß is:

$$ß = (\rho_{yz}\ \sigma_y\ \sigma_z)/\sigma_z^2 = (\rho_{yz}\ \sigma_y)/\sigma_z.$$

Remember this formula:

i. It will a be crucial one.

ii. We have already discussed each of its components.

2.B.2 <u>Economy-Wide and Specific Risk</u>

Recall that conceptually we can divide the risk of an individual asset into economy-wide risk and specific risk, which in an international context might translate into worldwide (non-diversifiable, systematic) risk and country (diversifiable, non-systematic) risk.

Let:
r_i = the rate of return on asset i (for example, a country index).
Z = a measure of economy-wide risk (for example, a world index).
ε_i = the specific risk of asset i.

In a regression framework: $r_i = \alpha + ß_i\ Z + \varepsilon_i$.

The larger $ß_i$, the more asset i is subject to economy-wide risk.

Example RC.8: Suppose $ß_i$=1, r_i will change by the same amount as the economy-wide variable. If Z changes by 10%, then r_i will change by 10% too. ¶

The larger is σ_ε^2, the more important is specific risk. We usually think of specific risk as independent of economy-wide risk.

Recall from statistics that if two variables are independent, the variance of the sum is equal to the sum of the variances. Therefore, the variance of r_i is given by:

$\sigma_i^2 = ß_i^2\, \sigma_z^2 + \sigma_\varepsilon^2$.

? The Components of Risk The total risk of an asset is made up of the risk from economy-wide movements and the risk from specific movements. Two assets can have the same total risk, yet that risk can be made up of very different components. ?

Example RC.9: Decomposing the Total Risk of a Country Stock Market.
Let Y be the U.S. Index and Z be the World Index, used in Table RC.5. A regression estimation gives ß = .9736 and σ_ε^2 = 1.623.

The variance of the World Index (σ_z^2) is 3.648 (from Table RC.5). Therefore,

$\sigma_{US}^2 = ß_i^2\, \sigma_z^2 + \sigma_\varepsilon^2 = (.9736)^2\, (3.648) + 1.623 = 5.081$.

Now, let Y be the Singapore Index. A regression estimation gives ß=.8196 and σ_ε^2 = 9.6908

$\sigma_S^2 = ß_i^2\, \sigma_z^2 + \sigma_\varepsilon^2 = (.8196)^2\, (3.648) + 9.6908 = 12.142$.

This example illustrates two empirical findings. First, the U.S. Index has a ß almost equal to one (i.e., the U.S. market moves up and down with world markets). Second, a big market (the U.S.) has a larger component of worldwide risk, while a small market (Singapore) has a larger component of country risk. ¶

2.C Putting Portfolios Together

2.C.1 First a digression: The Law of Large Numbers

If we have a number of random variables $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$, (each with zero mean and not very correlated with each other) then their weighted average approaches 0 as n gets large.

$\omega_1\, \varepsilon_1 + \omega_2\, \varepsilon_2 + \ldots + \omega_n\, \varepsilon_n \rightarrow 0, \qquad \text{as } n \rightarrow \infty,$

where the weights: $\omega_1 + \omega_2 + \ldots + \omega_n = 1$, and each of ω_i becomes relatively small.

Now, let's assume that each security has a rate of return defined by:

$r_i = \alpha_i + ß_i\, Z + \varepsilon_i$.

A portfolio is defined by a set of portfolio weights, $\omega_1, \omega_2, \ldots, \omega_n$, where ω_i refers to the proportion of the portfolio in asset i. Now, the overall rate of return of the portfolio is:

$r_p = \omega_1\, r_1 + \omega_2\, r_2 + \ldots + \omega_n\, r_n$.

We can replace r_i by its definition and we get:

$r_p = \alpha_p + ß_p Z + \varepsilon_p$,

where α_p, $ß_p$, ε_p are weighted averages of the intercepts, slopes and specific risks (respectively) of the individual assets. (Show it!)

In our global framework, by the Law of Large Numbers ε_p will be closed to zero for portfolios with a large number of assets from many different countries. This type of portfolio is called *well-diversified*.

Major point: $r_p = \alpha_p + ß_p Z$, if r_p is a well-diversified portfolio.

2.D Market Model

Lets get specific about Z. Lets impose: α_m=0 and $ß_m$=1. Then, $Z = r_m$.

We are letting the Market Portfolio represent economy-wide risk (for example, in Example RC.9, the World Index represents worldwide risk). This is sensible because the market portfolio has no diversifiable risk and it certainly captures the idea of economy-wide risk.

We are using what is called the Market Model: $r_i = \alpha_i + ß_i r_m + \varepsilon_i$.

We can be more precise about Beta: $ß_i = (\rho_{im}\, \sigma_i\, \sigma_m)/\sigma_m^2 = (\rho_{im}\, \sigma_i)/\sigma_m$.

Example RC.10: Go back to Table RC.5. The correlation coefficient between the World Index and the Singapore Index, ρ_{sw}, is .4498. Then,

$ß_S = (\rho_{sw}\, \sigma_s)/\sigma_w = (.4498)x(3.48)/(1.91) = .8195$ (which is the same estimate we got in Example 9. Are you surprised?). ¶

We will call $ß_i$ *market risk*. A high $ß_i$ means asset i has a lot of market risk. In this case, asset i has a large component of systematic risk. Investors will require compensation (higher $E[r_i]$) for holding assets with large amounts of market risk.

Example RC.11: Go back to Example RC.9. An international investor who invests USD 1 in the U.S. will require higher expected returns than a similar investment in Singapore. ¶

III. Capital Asset Pricing Model (CAPM)

The CAPM provides an answer to the following questions:

i. Exactly what is the risk-return relationship between ß and E[r]?
ii. What is the optimal investment strategy for an individual?

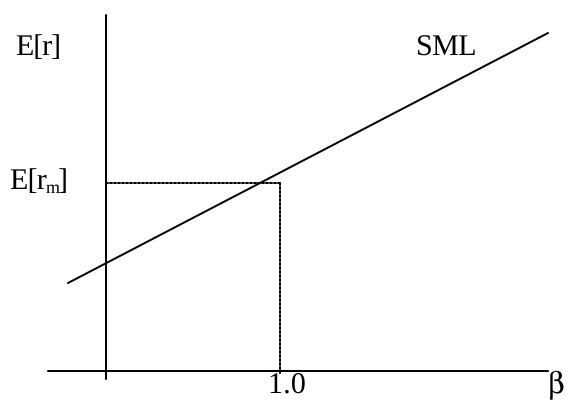
The CAPM shows there is a linear relationship between ß and E[r], called the <u>Security Market Line</u> (SML):

$$E[r_i] = r_f + ß_i\,(E[r_m] - r_f),$$

where
$E[r_i]$ = the expected rate of return on security i.
$E[r_m]$ = the expected rate of return on the market portfolio.
r_f = the riskless rate (usually, the U.S. T-Bill rate).

The risk premium on an asset ($E[r_i]-r_f$) is proportional to its ß. The risk premium on the market portfolio ($E[r_m]-r_f$) is the factor of proportionality. If Country Index A has twice the market risk (i.e., twice the ß) of Country Index B, it should earn twice the risk premium.

Graphically:



In equilibrium all assets and portfolios will plot on the SML.

It is important to realize that the CAPM is consistent with what we did earlier:

$E[r_i] = \alpha_i + ß_i\,E[r_m]$.

<u>Note</u>: Set $\alpha_i = (1-ß_i)\,r_f$ and you get the CAPM.

Now, we can provide another measure of abnormal returns:

If asset i has an $E[r_i]$ above that given by the SML, asset i gains abnormal returns.

3.A <u>The Efficient Frontier</u>

Recall that for a portfolio of two assets (S and J):

$E[r_p] = \omega_s\,E[r_s] + \omega_j\,E[r_j]$.

$\sigma_p^2 = \omega_s^2\, \sigma_s^2 + \omega_j^2\, \sigma_j^2 + 2\, \omega_1\, \omega_2\, \rho_{sj}\, \sigma_s\, \sigma_j$.

Let's examine this relationship in a different way than before. We want to find the expected return and the SD for the portfolio for all possible combinations of weights.

Example RC.12: Go back to Table RC.5 and analyze the weekly returns of the country indices of Singapore (S) and Japan (J). From Table RC.6, ρ_{sj} = .26. Then,
$E[r_s] = 0.20$ $\quad$ $E[r_j] = 0.46$
$\sigma_s = 3.48$ $\quad$ $\sigma_j = 2.73$.

Look at the risk-return tradeoff of portfolios of these two stocks:

ω_s	0.00	0.33	0.50	0.67	1.00
ω_j	1.00	0.67	0.50	0.33	0.00
$E[r_p]$	0.46	0.37	0.33	0.29	0.20
σ_p	2.73	2.40	2.48	2.71	3.48

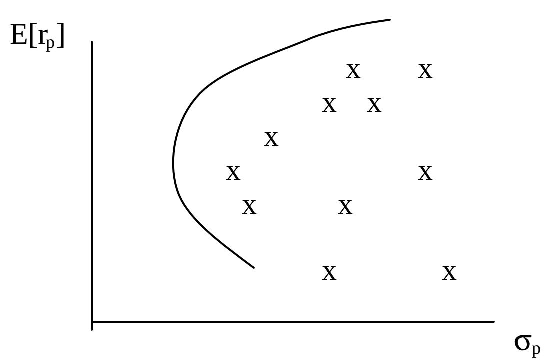
An investor who likes returns and dislikes volatility will find that Japan dominates Singapore. That is, Japan has a higher $E[r_p]$ and lower σ_p than Singapore. Japan, however, does not dominate a 50-50 portfolio: Japan has higher $E[r_p]$ and σ_p.

Moreover, if we plot the highest $E[r_p]$ for each level of σ_p, we find that (ω_s=1.00,ω_j=0) does not belong in this plot. ¶

In Example RC.12, we can improve the risk-return tradeoff with the addition to the portfolio of a third asset (given that they are not perfectly correlated).

The plot of the highest $E[r_p]$ for each level of σ_p is called the *efficient frontier*. A portfolio in the frontier is called mean-variance efficient.

Graphically,



Individual assets and portfolios with diversifiable risk lie inside the frontier (in Example RC.12, Singapore lies inside the frontier).

Example RC.13: Figure RC.2 shows two curves: the Singapore-Japanese frontier (full line) and the Pacific frontier (dotted line). Adding assets pushes the efficient frontier NW (by eliminating diversifiable risk). ¶

3.B Adding the Riskless Security

The CAPM assert that if an investor is going to hold risky assets, he/she should hold them in the form of the market portfolio.

Suppose you hold a proportion ω of your wealth in the riskless asset and (1-ω) of your wealth in the market portfolio:

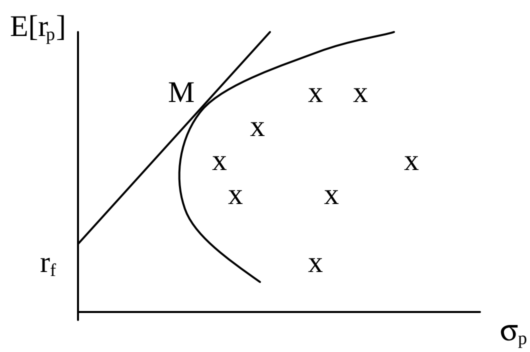
$E[r_p] = \omega\, r_f + (1 - \omega)\, E[r_m]$.
$\sigma_p = (1 - \omega)\, \sigma_m$. (because $\sigma_f^2=0$ and $\sigma_{fm}=0$)

Solving for ω in the second equation and replacing it into the first equation yields,

$E[r_p] = r_f + (\sigma_p/\sigma_f)\,(E[r_m] - r_f)$.

This relationship is called the Capital Market Line (CML).

Our picture of the efficient frontier makes the best strategy clear. Draw a line from r_f (the risk free rate and zero standard deviation) tangent to the efficient frontier. Let's label this risky portfolio M.



Absent inside information, everyone will choose from the tangent menu.

Everyone will hold some combination of r_f and portfolio M: this gives us the CML. Since everyone holds M as their portfolio of risky assets, M must be the *market portfolio* (in equilibrium). Because investors have different tolerances for risk, they do not put the same percentage of their wealth in the market portfolio. More risk-averse investors allocate a higher percentage of their wealth to the riskless asset.

? The Market Portfolio is Efficient

Since M is in the efficient frontier, M is a mean-variance efficient portfolio. The efficiency of M is crucial in the CAPM. All the porfolios in the efficient frontier can be generated by two efficient portfolios (r_f and M). In practice, however, there is no guarantee that the market portfolio is efficient. ?

3.C <u>Major conclusions</u>

The CAPM has very strong implications:

i. ß is the relevant measure of the risk of a security.

ii. The risk premium ($E[r_i]-r_f$) is proportional to ß.

iii. Every investor will hold a portfolio constructed from two funds: the riskless asset and the market portfolio. (This principle is called the two-fund separation theorem.)

The above implications are testable. The CAPM, however, is not directly testable because expectations are not observed. In practice, tests assume that the "ex-post" (i.e., observed) return is drawn from the same distribution as the "ex-ante" (unobserved) return. This assumption is called the *rational expectations* assumption.

In addition to the rational expectations assumption, the identification of the market portfolio M is crucial to empirical testing of the CAPM. The market portfolio should include all risky assets in the economy. And the economy does not only refer to the U.S. economy. It should include all risky assets in the world. In general, test of the CAPM concentrate in one country only and use proxies to M, like the S&P 500, which exclude bonds, real estate, small stocks, etc.

Domestic studies of the CAPM conclude that CAPM does not exactly hold, but the insights hold. Among the problems: there seems to be more than one factor influencing asset returns (for example, size and book-to-market ratio).

A one factor or International CAPM is hard to defend on theoretical grounds in the presence of exchange risk, different national policies, and market imperfection. Furthermore, all the problems associated with the practical use of the domestic CAPM are even more valid for the international version. The construction of a World Market portfolio is a very difficult task. In an ideal world, this world market portfolio includes all the world's human capital plus the whole range of financial instruments: stocks, bonds, short-term deposits in foreign currency, gold, etc.). Restrictions on investment, differential taxes, transaction costs make the design of a complete world market equilibrium model a hopeless operational task.

3.D <u>A multi-country CAPM</u>

The theoretical framework of the CAPM can be readily extended to the international situation. Subject to restrictive assumptions, this model makes the claim that the risk that is priced in the market is measured by the international beta of an asset, that is, the beta relative to the world market portfolio hedged against exchange risk.

? **Domestic Factors or World Factors** Academic research has provided strong support for domestic factors. Assets within a country are more affected by domestic factors than world factors. ?

This empirical fact implies that a multi-country or top-down approach is needed. According to the simple, descriptive, multi-country model each asset is influenced by its domestic market factor, which in turn is influenced by the single world market factor. In other words, an asset is indirectly sensitive to the world market factor through its national market factor.

The sensitivity a country displays to the world factor is the result of many influences, including its degree of international trade and investment, domestic monetary and economic policy, regulation and control of capital flows. Thus the world beta of an asset ($ß_{iw}$) is the product of its domestic beta ($ß_i$) and the sensitivity of the domestic country factor to the world market factor ($ß_{cw}$). That is,

$ß_{iw} = ß_i\, ß_{cw}$.

Example RC.14: Suppose we only look at stock markets, like in Example RC.9. In that example, we calculated the beta for Singapore, $ß_{cw}$=.8195. Suppose a particular stock, Singapore Computers, has a beta with the country market portfolio (the Singapore Index) of $ß_i$= .9651. Therefore, the world beta of Singapore Computers is:

$ß_{iw}$ = (.9651)x(.8195) = .7909. ¶

Although this theoretical framework is useful for structuring the investment process, most money managers do not regard the world capital market as fully efficient. The first step in the multi-country approach requires analysts trying to forecast national market returns and currency movements. These forecast lead to international asset allocation with an under (or over) weighing of some markets relative to the world index. In the second step, individual companies are valued within the context of their national markets.

Superior performance can come from two sources: country selection and individual selection within a national market.

Beta coefficients have to be estimated, which is no simple task. The simplest method used by large investment institution is to employ a simple regression model over past data (like in Example RC.9). Some analysts adjust the regression betas to factor in certain statistical properties and other information.

Summary:

i. A financial analyst should compute both the expected return and risk of assets relative to his/her domestic market. This allows a list of assets with abnormally high-expected returns given their risk level to be drawn.

ii. From this list, the manager can select assets to achieve the desired international asset allocation among markets, which is based on general forecasts of national financial markets (which is based on general forecasts of markets, interest rates, currencies, etc.).

Example RC.15: If a manager (based on forecasts) is bullish on a particular national market, he/she should select assets with high betas for that market. On the other hand, if the manager is bullish in the currency but quite uncertain about the national market itself, he/she should select only low-beta assets in that market. ¶

IV. Arbitrage Pricing Theory (APT)

The multi-country CAPM is an attempt to overcome the simple one factor model used in the International CAPM. The multi-factor approach, however, has its limitations. Although a domestic asset is mostly influenced by the domestic factor, it seems unlikely that all domestic influences can be summarized in a single domestic factor identical for every firm. For example, asset prices react somewhat differently to oil or currency movements.

The arbitrage pricing theory (APT) does not have the limitations imposed by the CAPM or multi-country CAPM. APT retains the distinction between diversifiable and non-diversifiable risk, but imposes fewer restrictive assumptions in its derivations of asset returns than does the CAPM. A great advantage of the APT is that it can be applied to a subset of investments, so that we do not need to consider every single world asset as we did in the CAPM. It starts with a descriptive model, where the return on an asset is determined by a number of *common factors* plus a term specific to the asset, and leads to a theory of what returns should be in an efficient market.

4.A Descriptive Model

The derivation of APT requires two major assumptions. First, agents believe that there is a set of identifiable factors, which generate the variability of all asset returns. Second, there are no arbitrage opportunities available. From the first assumption, we write the following multi-factor model, where r_i is the rate of return on the asset i and α is a constant term,

$$r_i = \alpha + \gamma_1 \delta_1 + \gamma_2 \delta_2 + ... + \gamma_k \delta_k + \varepsilon_i,$$

where $\delta_1, \delta_2, ... \delta_k$ are the k factors common to all assets, and ε_i is a random term specific to asset i (and independent of all δs and other εs). The ε term is the source of all idiosyncratic or diversifiable risk, and γ_j, loading factor for factor j, represents the sensitivity of this asset to factor j (j=1,...,k). The APT model also assumes that the factors and error terms have a mean of zero. The γs vary among assets. Some assets may be highly sensitive to certain factors and less sensitive to others.

4.B Theory

Arbitrage ensures that only the γs should be priced, so that $E(r_i)$ is a linear function of these betas:

$$E[r_i] = r_f + \gamma_1 \lambda_1 + \gamma_2 \lambda_2 + ... + \gamma_k \lambda_k,$$

where λ_j is the expected risk premium associated to factor j. A clear intuition is behind this equilibrium relation. If the indicated factors truly generate the movements of all asset returns and if current asset prices allow no arbitrage, then, expected returns are approximately linearly related to the covariance between the asset returns and the factors.

APT is a more general model than the traditional CAPM. The CAPM's intuition that covariance risk --risk that cannot be diversified away— underlies the pricing of assets is also present in the APT model. The APT structure, however, allows for more than one source of risk.

According to the CAPM, we need to identify all securities in the world in order to construct the world market portfolio and derive the theoretical pricing relation. This is not necessary for the APT because it focuses on the relative pricing of the securities under study, the common factors being exogenous.

Now, you should be wondering what are the factors that determine the return on an asset. Unfortunately, there is not a strong theory to guide us to select the factors. Instead, these factors must be estimated from the data; that is, unlike the CAPM, they are not specified by the theory. In the field of international finance, many APT studies and practitioners focus on three sets of factors:

i. The first set of factors is international in nature and may be linked to a real variable such as economic growth, a monetary variable such as inflation or changes in interest rate, or energy costs such as oil prices.

ii. The second set of factors is purely domestic, reflecting the deviation of a nation from the world economy path. It could range from a single domestic factor to a more detailed set of domestic factors such as deviation from world real output growth, inflation, etc.

iii. The third set of factors is common to companies within the same industry, regardless of the country they are in. These are known as industry-wide factors.

4.C Practical use of the model

We will use a simple four-factor model for stock prices. In this model each company stock price is affected by only four factors, and three of them vary among companies according to their nationality and industrial sector. The four factors are:

1. World economic activity. This factor would explain why all stock prices in the world tend to be positively correlated.

2. Domestic real growth. The domestic real growth rate relative to the average world real growth rate would be a country-specific factor.

3. Currency movements. A firm would have a positive or negative beta with respect to this country-specific factor depending on both its import-export structure and the currency denomination of its financing.

4. Industry-wide conditions. This last factor would be industry-specific across national boundaries.

The next step is for financial analysts to provide estimates for the model's sensitivity coefficients, γs. Once the γs are estimated, it is possible to calculate expected returns on securities. Where markets are not efficient, an analyst would expect to find securities with an expected return that is larger than the equilibrium value given by the theory. These are the securities a portfolio manager should add to their portfolio.

Example RC.16: Suppose a manager analyzes the Chilean mining company, Minas de Chile (MdC). She has the following information: world economic activity is expected to grow by 5% and the sensitivity of MdC to the world factor (γ_1) is .30; expected domestic real growth relative to world real growth rate is 4% and γ_2=.60; expected currency change is 7% and γ_3=.70; and the world mining industry is expected to have a return of 10% and γ_4=.25. The risk free rate (r_f) is 3%. Suppose portfolios associated to these factors have similar expected rates of returns. Then,

$E[r_i] = 3 + .30x(5-3) + .60x(4-3) + .70x(7-3) + .25x(10-3) = 8.75$. ¶

Example RC.17: Assume a manager believes in the following scenario: a worldwide slowdown in real growth and inflation and a better relative economic performance of Mexico. He/she will then invest in companies that are less sensitive to changes in real growth, most favorably influenced by the reduction in inflation, and directly affected by better performance of the Mexican economy. ¶

Interesting readings

Section I of this chapter is based on "Market Efficiency and Equity Pricing: International Evidence and Implications for Global Investing," by Gabriel Hawawini, in **World Equity Markets**.

Tallman, E.W. (1989). "Financial Asset Pricing Theory: A Review of Recent Developments," Economic Review, Federal Reserve Bank of Atlanta.

Section RC.D and Section IV are based on **International Investments**, by Bruno Solnik.

A good general reference for the whole chapter can be found in **A Random Walk down Wall Street**, by Burton G. Malkiel (see chapters 1, 5, 6, 7 and 9). This book is a classic on the subject and it is very easy to read).

APPENDIX RC

I. Basic Statistical Concepts

1.A Probability Distributions for Continuous Random Variables

1.A.1 Cumulative Distribution function (cdf)

The cdf $F_X(x)$ of a continuous random variable X measures the probability that X does not exceed the value x, as a function of x; that is,

$F_X(x) = P(X \leq x)$.

1.A.2 Probability Density Function (pdf)

Let X be a continuous random variable, and let x be any number lying in the range of values this random variable can take. The pdf, $f_X(x)$, of the random variable is a function with the following properties:

(i) $f_X(x) \geq 0$ for all values of x

(ii) Suppose this density function is graphed. Let δ and ß be two positive values of the random variable X, with δ < ß. Then the probability that X lies between δ and ß is the area under the density function between these points.

$P(\delta < X < ß) = ?_{\delta}^{ß} f_X(x)\, dx$

(iii) $?_{-\infty}^{\infty} f_X(x)\, dx = 1$.

Note: When X is a discrete random variable, the above definitions have to be rewritten. Let $p_X(x)$ be the probability of X=x. Then, in the above definitions,

(i) becomes $p_X(x) \geq 0$, for all values of x, and

(ii) becomes $P(\delta < X < ß) = \Sigma_{X<ß}\, p_X(x)\, x - \Sigma_{X<\delta}\, p_X(x)\, x$.

(iii) becomes $\Sigma_A\, p_X(x)\, x = 1$, where $x \varepsilon A$.

1.A.3 Expectations for continuous random variables

Suppose that a random experiment leads to an outcome that can be represented by a continuous random variable. If T replications of this experiment are performed, then the *expected value* of the random variable is the average of the values taken, as the number of replications tends to infinite. We will denote the expected value of a random variable X as E[X].

Formally,

$E[X] = ?\ x\ f_X(x)\ dx.$

The *mean* of X, μ_X, is defined as the expected value of X. That is, $\mu_X = E[X]$.

Note: When X is a discrete variable, then $E[X] = \Sigma_A\ p_X(x)\ x$.

Example A.RC.1: At time t=0, a stock index sells for p_0 = USD 105 per share. The probability distribution over the three possible states of the economy (recession, stable, expansion) next year might be:

	State 1	State 2	State 3
Price (p_1)	USD 100	USD 120	USD 140
Probability	.20	.60	.20

Therefore,

$E[P_1] = \mu_P$ = .2 x USD 100 + .6 x USD 120 + .2 x USD 140
= USD 20 + USD 72 + USD 28 = USD 120. ¶

The *variance* of X, σ^2_X, is defined as the expectation of the squared deviation, $(X-\mu)^2$, of the random variable from its mean. That is, $\sigma^2_X = E[(X-\mu_X)^2]$.

Formally,

$Var[X] = ?\ (x-\mu)^2\ f(x)\ dx.$

A different expression for σ^2_X is given by $\sigma^2_X = E[(X^2)] - \mu_X^2$.

Note: It should be clear, by now, that when X is a discrete variable, then $Var[X] = \Sigma_A\ (x-\mu)^2\ p_X(x)$, where $x\varepsilon A$.

The variance should be non-negative. The square root of the variance of X is called the *standard deviation* of X, σ_X.

Example A.RC.2: We want to calculate the standard deviation for the stock index P presented in Example A.RC.1.

$Var[P_1] = \sigma^2_p = .2x(100-120)^2 + .6x(120-120)^2 + .2x(140-120)^2 =$

$= .2x400 + .6x0 + .2x400 = 160.$

The standard deviation (SD) is the square root of the variance:

$\sigma_p = (160)^{1/2}$ = USD 12.65. ¶

Let X be a continuous random variable with mean μ_X and variance σ^2_X, and let δ and ß be any fixed numbers. Define the random variable Z as Z = δ + ß X. Then,

$\mu_Z = \delta + ß\, E[X] = \delta + ß\, \mu_X,$
$\sigma^2_Z = Var(\delta + ß\, X) = ß^2\, \sigma^2_X.$

Let X and Y be a pair of continuous random variables, with means μ_X and μ_Y respectively. The expected value of $(X-\mu_X)(Y-\mu_Y)$ is called the *covariance* between X and Y. That is,

$Cov(X,Y) = \sigma_{XY} = E[(X-\mu_X)(Y-\mu_Y)].$

We can rewrite the above expression as

$\sigma_{XY} = E(XY) - \mu_X\, \mu_Y.$

The covariance measures the comovements of two random variables. If the $\sigma_{XY}>0$ when X increases, Y tends to increase and vice versa. If the $\sigma_{XY}<0$ when X increases, Y tends to decrease and vice versa. If the random variables X and Y are independent, then $\sigma_{XY}=0$. The converse of this last statement, however, is not necessarily true.

The comovement between two random variables can be measured by the correlation coefficient. The correlation between the random variables X and Y is

$Corr(X,Y) = \rho_{XY} = \sigma_{XY}/(\sigma_Y\sigma_Y).$

The correlation coefficient is a number between -1 and 1.

It is easy to calculate means and variances for linear combinations of random variable. For example, define Z = δX + ßY. Then,

$\mu_Z = \delta\, \mu_X + ß\, \mu_Y,$
$\sigma_Z = \delta^2\, \sigma^2_X + ß^2\, \sigma^2_Y + 2\, \delta ß\, \sigma_{XY}.$

1.B The Normal Distribution

Suppose the random variable X has a pdf given by:

$f_X(x) = [1/(2\pi\sigma^2)^{1/2}]\, \exp\{-(x-\mu)^2/(2\sigma^2)\}$ $\qquad -\infty < x < \infty,$

where μ and σ^2 are any number such that $-\infty<\mu<\infty$, and $0<\sigma^2<\infty$. Then X is said to follow a *normal distribution*. We will use the following notation: $X \sim N(\mu,\sigma^2)$.

If Z follows a *standard normal distribution* then $Z \sim N(0,1)$. Using the formula of the pdf normal distribution, tables for the cdf of the standard normal distribution have been tabulated.

1.B.1 Useful Results of the Normal Distribution

Let $X \sim N(\mu,\sigma^2)$. Then,

(i) $E[X] = \mu$.

(ii) $Var(X) = E[(X-\mu)^2] = \sigma^2$

(iii) The shape of the pdf is a symmetric bell-shaped curve centered on the mean.

(iv) Let $Z = (X-\mu)/\sigma$. Then, $Z \sim N(0,1)$.

(v) If δ and ß are any numbers with $\delta <$ ß, then

$$P[\delta < X < ß] = P[(\delta-\mu)/\sigma < Z < (ß-\mu)/\sigma] = F_Z[(ß-\mu)/\sigma] - F_Z[(\delta-\mu)/\sigma],$$

where F_Z represents the cdf of Z.

Example A.RC.3: Let X be the annual stock returns (in percentage points) in the U.S. Assume that $X \sim N(11.44, 16.22^2)$. (The mean and variance of X have been obtained from annualizing the U.S. weekly mean return and variance from Table RC.5.) Suppose you are the manager of a portfolio that tracks the U.S. Index. You want to find the probability that your portfolio's return (X) is lower than -30% next year (i.e., a market crash).

$P[X < -30] = P[Z < (-30-11.44)/16.22] = P[Z < -2.55] = F_Z[-2.55] = 1 - F_Z[2.55] = 1 - .9878 = .0122.$

That is, the probability that next year stock return is lower than 30% is 1.22%. ¶

Example A.RC.4: Go back to Example A.RC.3. Now, suppose you want to determine a minimum return, say τ%, with probability .95. That is, you want to find the probability that your portfolio's return exceeds a level τ with probability .05. That is,

$.95 = P[X > \tau] = P[Z > (\tau-11.44)/16.22] = 1 - F_Z[(\tau-11.44)/16.22].$

From the Normal Table, we obtain that if $F_Z(z) = .05$, $z = -1.645$.

Then, $(\tau-11.44)/16.22 = -1.645 \quad \Rightarrow \tau = 11.44 - 1.645\ (16.22) = -15.323.$

That is, there is a 95% probability that next year's portfolio return will be bigger than -15.323%. ¶

Using the above properties, it is very easy to construct confidence intervals for the random variable X, which is normally distributed with mean μ and variance σ^2. The key to construct confidence intervals is to select an appropriate z value, such that

$$X \,\varepsilon [\mu \pm z\,\sigma] \quad \text{with a probability } (1\text{-}\alpha).$$

For a 99% confidence interval (i.e, α=.01) z is equal to 2.58.

For a 98% confidence interval (i.e, α=.01) z is equal to 2.33.

For a 95% confidence interval (i.e, α=.05) z is equal to 1.96 ($\approx$2).

For a 90% confidence interval (i.e, α=.10) z is equal to 1.645.

1.C Sample Estimates

In practice, the distribution of rates of return is estimated with past observations, under the assumption that all observations are identically and independently distributed. If we have a total of T observations $\{X_1, X_2, ..., X_T\}$ the mean and the variance can be estimated by their sample counterparts:

$$\overline{x} = (1/T)\Sigma_i{}^T X_i$$

The equation for the sample variance can be restated as

$$\sigma^2 = (1/T)\Sigma_i{}^T(X_i - \overline{x})^2$$

For the majority of financial series sampled at a frequency lower than weekly, the second term is very small (negligible) relative to the first.

Example A.RC.5: The weekly returns on the Japanese stock market from 1980-1996 (T=847) have a mean of .257% and a standard deviation of 3.095%. The squared average return is $(847/846)(.00257)^2$=.0000066. The variance term is $(.03095)^2$=.0009579, which is much greater than the squared mean return. In these situations, ignoring the mean in the estimation of the variance does not significantly affect the estimate. ¶

1.C.1 Sample Distributions and Estimation Error

The difference between a particular value for the sample estimator and the true value of the corresponding population parameter is called *estimation error*, or *sampling error*. This difference is not an error in the sense that anyone made a mistake; it is simply due to the fact that the sample estimators are computed from a subset of the population rather than from the entire population. In order to know the magnitude of the estimation error, we need to study the sampling distributions of the sample estimators.

1.C.1.i Sample Distribution of the Mean

Suppose each random variable of T observations is drawn from a population with mean μ and variance σ^2. The sample is composed of $X_1, X_2, \ldots, X_T$. Each X_i is considered a random variable with the same mean μ and variance σ^2. We want to make inferences about the population mean μ. We will study the sampling distribution of the random variable .

$$E[\bar{x}] = E[(1/T)\Sigma_i^T X_i] = (1/T)E[\Sigma_i^T X_i] = (1/T)(T\mu) = \mu$$

That is, the mean of the sample population is the population mean. This conclusion states that if samples of T observations are repeatedly and independently drawn from a population, then as the number of samples becomes large, the average of the sample means gets closer to the true population mean.

Suppose we sample seventeen years of Japanese stock returns and we obtain a weekly return of .257%. How good an approximation might this be to the expected (mean) weekly return for the whole population (i.e., future and past history) of Japanese stock returns? The answer to this question depends on the spread, or variance, in the sampling distribution of .

$$\mathrm{Var}(\bar{x}) = \mathrm{Var}[(1/T)\Sigma_i^T X_i] = (1/T^2)\mathrm{Var}[\Sigma_i^T X_i] = (1/T^2)(T\sigma^2) = \sigma^2/T.$$

That is, the variance of the sampling distribution of the sample mean decreases as the sample size T increases.

The squared root of the variance of the sample mean is called *standard error*:

$$SE(\bar{x}) = \sigma/T^{1/2}.$$

Example A.RC.6: Consider the data for Japanese weekly stock returns from Example A.RC.5. We want to calculate the standard error for . Using the sample estimates we obtain:

$$SE(\bar{x}) = s/T^{1/2} = (.03095)/(847)^{1/2} = .00106 \text{ (or .11\%)}.$$

A 95% confidence interval around the point estimate of μ = 0.257% is given by (.037%,.467%). At the 5% level, it is significantly different that zero. But, we should note that the confidence interval is quite wide, which indicates substantial estimation error. That is, even after seventeen years, the expected return, μ, is measured with low precision. ¶

If the sample is taken $X_1, X_2, \ldots, X_T$ from a normal population, the distribution of the sample mean is:

$$\bar{x} \sim N(\mu, \sigma^2/T).$$

1.C.1.ii Sample Distribution of the Sample Variance

When we defined s^2, we used in the denominator (T-1), instead of T. This was done to insure that the sample variance on average is equal to the true population variance. That is,

$E[s^2] = \sigma^2$.

In order to continue to characterize the sampling distribution, we have to assume an underlying distribution for the population X. For many applications in finance and economics a normal distribution is not unreasonable (at least, as a crude approximation). Before establishing the distribution of the sample variance, we will introduce the chi-squared distribution.

1.C.1.ii.1 The Chi-squared Distribution

Let $Z = X^2$. If the random variable X follows a standard normal distribution, then the random variable Z follows a chi-squared distribution with 1 degree of freedom. That is, $Z \sim \chi^2_1$.

The chi-squared distribution has the *additive property*. That is, if $Z = X_1 + X_2$, where $X_1 \sim \chi^2_{n1}$ and $X_2 \sim \chi^2_{n2}$, then $Z \sim \chi^2_{n1+n2}$.

In general, if $X_i \sim N(\mu_i, \sigma^2_i)$, then $Z = \Sigma_i^T [(X_i - \mu_i)/\sigma_i] \sim \chi^2_T$.

The chi-squared family of distributions is often used in finance and economics. The distributions are defined only for positive values, which is consistent with the possible values for the variance. Fractiles of the χ^2_v -chi-squared with v degrees of freedom- are tabulated.

The mean of a chi-squared distribution is $E[\chi^2_v] = v$, while the variance is given by $Var[\chi^2_v] = 2v$.

Confidence intervals for a χ^2_v are constructed using standard procedures:

$P(\chi^2_{v,1-\alpha/2} < \chi^2_v < \chi^2_{v,\alpha/2}) = 1-\alpha$.

1.C.1.ii.2 Sample distribution of the sample variance: Continuation

Now, we are ready to derive the distribution for the following random variable:

$(T-1)\, s^2/\sigma^2 \sim \chi^2_{T-1}$.

Using the results from the chi-squared distribution, it can easily be checked that $E[s^2] = \sigma^2$:

$E[(T-1)s^2/\sigma^2] = (T-1) \Rightarrow E[s^2] = \sigma^2$.

Similarly, the variance of ^{2} is derived is by:

$Var[(T\text{-}1)s^2/\sigma^2] = 2\ (T\text{-}1) \quad \Rightarrow Var[s^2]=2(T\text{-}1)\ \sigma^4/(T\text{-}1)^2 = 2\sigma^4/(T\text{-}1).$

We want to construct a confidence interval for the population variance. To derive this confidence interval, let rewrite the standard confidence interval for a chi-squared variable:

$P(\chi^2_{v,1-\alpha/2} < \chi^2_v < \chi^2_{v,\alpha/2}) = P(\chi^2_{v,1-\alpha/2} < (T\text{-}1)s^2/\sigma^2 < \chi^2_{v,\alpha/2}) = 1 - \alpha$

After some easy algebra, we derive:

$P[(T\text{-}1)s^2/\chi^2_{v,\alpha/2} < \sigma^2 < (T\text{-}1)s^2/\chi^2_{v,1-\alpha/2}] = 1\text{-}\alpha.$

Example A.RC.7: Reconsider the data for Japanese weekly stock returns from Example A.RC.5. We want to calculate a 95% confidence interval for σ^2 --i.e., α=.05. First, we need to find $\chi^2_{846,.025}$ and $\chi^2_{846,.975}$:
$\chi^2_{846,.975}$=767.3 and $\chi^2_{846,.025}$=914.8.

$P[(846)(.03095)^2/(914.8) < \sigma^2 < (846)(.03095)^2/(767.3)] = .95$

$P[\ .000886 < \sigma^2 < .001056\ ] = .95$

A 95% confidence interval around the point estimate of σ, s=3.095%, is given by (2.98%,3.25%). The confidence interval is quite compact around the sample point estimate. Compared to the sample mean estimate, σ is measured with accuracy. ¶

When T is large (for example, T is above 30), the chi-squared distribution converges to a normal distribution. That is, for large T,

$s^2 \sim N(\sigma^2, \sigma^4/(T\text{-}1)].$

In this case, the standard error of the sample standard deviation is given by:

$SE(s) = \sigma/(2\ T)^{1/2}.$

Example A.RC.8: For financial and economic data, ordinarily T is substantially larger than 30. Thus, we can use the normal approximation to calculate confidence intervals for the population standard deviation. Using our sample estimates from Example A.RC.5, we estimate the standard error for the sample variance of Japanese weekly returns:

$SE(s) = s/(2\ T)^{1/2} = .03095/(2\ 847)^{1/2} = .000752$ (or .075%).

That is, a 95% confidence interval for σ is given by (2.945%,3.245%). Compared with the results from Example A.RC.7, we can see that the approximation works very well. ¶

From the above examples, we conclude that volatility is estimated with a much higher precision than the expected return.

II. Calculating Rates of Return

Suppose that we observe for a given asset a series of prices over time, $P=\{P_1, P_2,....,P_T\}$. We want to calculate the rate of return on this given asset. We also observe a series of interim payments that this asset has paid to its holder over time, $D=\{D_1, D_2,....,D_T\}$. We want to calculate the rate of return of this asset a time interval.

The rate of return of any asset is defined as the capital gain plus any interim payment:

$$r^A_t = (P_t + D_t - P_{t-1})/P_{t-1}. \qquad \text{(A.RC.1)}$$

This definition implicitly assumes that the interim payment D is reinvested only at the end of the month. Formula (A.RC.1) calculates the *arithmetic rate of return*.

We can also calculate returns using logarithms:

$$r^G_t = \ln[(P_t + D_t)/P_{t-1}] = \ln(P_t + D_t) - \ln(P_{t-1}). \qquad \text{(A.RC.2)}$$

Formula (A.RC.2) calculates the *geometric rate of return*.

To compare both formulas, assume that $D_t=0$, for all t. This is not an unreasonable assumption for many assets. For example, exchange rates do not pay any dividends, and many stocks do not pay any dividends or make very small dividend payments relative to P_t.

Geometric returns are very handy when we want to calculate multi-period returns. For example, we want to calculate a weekly return based on daily observations: $P_{M,1}$, $P_{T,1}$, $P_{W,1}$, $P_{Th,1}$, $P_{F,1}$, $P_{M,2}$.

$$\begin{aligned} r^G_{M,2\text{-}M,1} &= \ln(P_{M,2}/P_{M,1}) = \ln(P_{M,2}) - \ln(P_{M,1}) \\ &= \ln(P_{M,2})-\ln(P_{F,1}) + \ln(P_{F,1})-\ln(P_{Th,1}) + \ln(P_{Th,1})-\ln(P_{W,1}) + \ln(P_{W,1})-\ln(P_{T,1}) + \ln(P_{T,1})- \\ &\quad - \ln(P_{M,1}) = r^G_{M,2\text{-}F,1} + r^G_{F,1\text{-}Th,1} + r^G_{Th,1\text{-}W,1} + r^G_{W,1\text{-}T,1} + r^G_{T,1\text{-}M,1}. \end{aligned}$$

That is, the geometric weekly rate is the same as sum of the geometric daily returns over the week. Decomposing arithmetic returns is not this simple.

Geometric returns are very convenient to translate rates of return from one currency into another. For example, suppose you want to translate the returns of your Japanese investment into your local currency, say, the USD. At time 0 you buy Japanese government (JBG) for P_0=JPY 100,000 at exchange rate S_0=.01 USD/JPY. At time 1 you sell the JGB and receive P_1=JPY 110,000. The exchange rate is S_1=.009 USD/JPY. You want to calculate the rate of return (in USD) of your investment.

$$r^A = (P_1S_1-P_0S_0)/(P_0S_0) = (\text{USD } 990 - \text{USD } 1{,}000)/\text{USD } 1000 = .01 \ (1\%).$$

$r^G = \ln[(P_1S_1)/(P_0S_0)] = \ln(P_1/P_0) + \ln(S_1/S_0)] = \ln(110{,}000/100{,}000) + \ln(.009/.01)$
$= 0.09531 + (-0.1054) = -.01009\ \ (\approx 1\%)$.

When we use the geometric rate of return, the USD return on the JGB investment is equal to the sum of two rates of returns: the rate of return on the JGB and the rate of return on the JPY/USD exchange rate.

Geometric returns are also convenient to calculate rates of returns for exchange rates. Recall that an exchange rate is a ratio of two currencies. It can easily be shown that the rate of return of the currency in the numerator relative to the currency in the denominator is exactly equal to the negative rate of return of the currency in the denominator relative to the currency in the numerator. That is,

$$r^G_{FC/DC} = -r^G_{DC/FC}$$

When we use arithmetic rates of return the above result does not hold.

Example A.RC.9: The JPY has depreciated against the USD from .01 USD/JPY to .009 USD/JPY. The geometric rate of return of a U.S. investor who bought JPY is:

$$r^G_{JPY/USD} = \ln(S_{1,USD/JPY}/S_{0,USD/JPY}) = \ln(.009/.01) = -.1054\ \ (-10.54\%).$$

The geometric rate of return of a Japanese investor who bought USD is:

$$\begin{aligned} r^G_{USD/JPY} &= \ln[(1/S_{1,USD/JPY})/(1/S_{0,USD/JPY})] \\ &= \ln(S_{0,USD/JPY}/S_{1,USD/JPY})] \\ &= \ln(.01/.009) = .1054\ \ (10.54\%). \end{aligned}$$

The arithmetic rate of return of a U.S. investor who bought JPY is:

$$r^A_{JPY/USD} = (S_{1,USD/JPY}-S_{0,USD/JPY})/S_{0,USD/JPY} = -.001/.01 = -.10\ \ (-10).$$

The arithmetic rate of return of a Japanese investor who bought USD is:

$$\begin{aligned} r^A_{USD/JPY} &= [(1/S_{1,USD/JPY})-(1/S_{0,USD/JPY})]/(1/S_{0,USD/JPY}) \\ &= [111.1111 - 100]/100 = .1111\ \ (11.11\%). \end{aligned}$$ ¶

Since $r^G_{FC/DC} = -r^G_{DC/FC}$, the variance of the FC/DC exchange rate is equal to the variance of the DC/FC exchange rate.

Example A.RC.10: If the volatility (standard deviation) of the USD/JPY is 12%, then the volatility of the JPY/USD is also 12%. ¶

You probably have noticed in the above examples that the geometric rate of return and the arithmetic rate of return are very close. This result is especially true when the arithmetic rate of return is very small. Note that

$$r^G_t = \ln(P_t/P_{t-1}) = \ln(1+r^A_t).$$

Now, use a Taylor expansion to rewrite $\ln(1+r^A_t)$ around r^A_0. Ignoring third order terms and higher (they will not matter for small r^A_t), the above expression becomes:

$$r^G_t = \ln(1+r^A_0) + [1/(1+r^A_0)]\ (r^A_t-r^A_0) + [1/(1+r^A_0)]\ (r^A_t-r^A_0) + [-1/(1+r^A_0)]^2\ (r^A_t-r^A_0)^2.$$

For $r^A_0=0$, r^G_t equals to:

$$r^G_t = r^A_t + (r^A_t)^2.$$

For small r^A_t, the term $(r^A_t)^2$ (and r^A_t at higher powers) is negligible, therefore, $r^G_t \approx r^A_t$.

2.A Time Aggregation

We have already seen that aggregating geometric rates of returns was quite simple. For example, if we have daily returns, the monthly rate of return is (assuming 20 days in a month):

$$r^G_M = r^G_1 + r^G_2 + r^G_3 + ... + r^G_{20}.$$

Then, the expected rate of return is equal to:

$$E[r^G_M] = E[r^G_1] + E[r^G_2] + E[r^G_3] + ... + E[r^G_{20}].$$

The calculation of an aggregated variance is more difficult, because covariances are involved. For example, if we are interested in the three-day return, the three-day variance is given by:

$$Var(r^G_M) = Var(r^G_{,1}) + Var(r^G_2) + Var(r^G_3) + 2\ Cov(r^G_1,r^G_2) + 2\ Cov(r^G_1,r^G_3) + 2\ Cov(r^G_2,r^G_3).$$

If we assume that the rates of return are uncorrelated (i.e., markets are efficient), the calculation of an aggregated variance is very simple. Under this assumption, $Cov(X_i,r_j)=0$, for $i\neq j$. Then, for the three-day variance we get:

$$Var(r^G_M) = Var[r^G_{,1}] + Var[r^G_2] + Var[r^G_3].$$

If we assumed that the returns are iid with mean μ and variance σ^2, then the monthly (20-day) return and variance are:

$$E[r^G_M] = 20\ \mu,$$
$$Var(r^G_M) = 20\ \sigma^2.$$

In general, let μ and σ^2 be measured in a given base frequency. Then

$\mu_f = \mu\ T$,
$\sigma^2_f = \sigma^2\ T$,

where f is the frequency selected (days, months, years) and T represents the frequency selected relative to the base frequency. The standard deviation is calculated by taking the square root of σ^2_f:

$\sigma_f = \sigma\ T^{1/2}$,

Example A.RC.11: Consider the weekly Japanese stock market data of Example A.RC.5: μ_w=.00257, and σ_w=(.03095). That is, the base frequency is weekly. You want to calculate the daily and annual rate of return and standard deviation for the Japanese stock market.

(1) Daily (i.e., f=d=daily and T=1/5)
μ_d = (.00257) (1/5) = .000514 (0.0514%)
σ_d = (.03905) $(1/5)^{1/2}$ = .01746 (0.0175%)

(2) Annual (i.e., f=a=annual and T =52)
μ_a = (.00257) (52) = .1336 (13.36%)
σ_a = (.03905) $(52)^{1/2}$ = .2816 (28.16%) ¶

Recall that the calculation for the standard deviation is done assuming that rates of return are uncorrelated. If rates of return show a low correlation, then the above calculation should be taken as an approximation.

2.B Filtering Volatility

In the previous section we have shown that we can obtain an estimated annualized volatility using different time-frames. For example, we can use 10 days, 30 days, 100 days. An important question remains to be answered: which time-frames is the correct one? There is, unfortunately for traders, no straight answer, and this disagreements help create a market. Some traders with a long memory used many years of information, while others use only the most recent 20 days.

Filtering is a method for taking into account that events in the past might need to be treated differently. That is, they might need a different weight. A popular filtering method is the *exponential decay*. Under exponential decay, the weight of an event in the calculation of volatility is proportional to its distance away from the present

Let λ be the decay factor. This decay factor indicates the importance of past events in the present calculation of volatility. The decay factor, λ, is a number between zero and one. If λ is close to one, then events in the past have a considerable weight in today's volatility. The formula for volatility is given by:

Install Equation Editor and double-click here to view equation.

when T is very large (higher than 1,000), $\Omega=1/(1-\lambda)$. Note that we have ignored μ in the volatility calculation, as suggested in Section I of this Appendix.

Example A.RC.12: Consider the data for Japanese weekly stock returns from Example A.RC.5. We want to calculate the volatility using the exponential decay method. Using a $\lambda=.97$, we obtain the weekly volatility:

d = .018922,

which implies an annualized volatility of 13.64%.

A sharp contrast with the .03095 weekly volatility from Example A.RC.5, which implies an annual volatility of 22.31%. ¶

Reference:

Statistics for Business and Economics, by Paul Newbold, published by Prentice Hall.

Exercises:

1.- Look at Table RC.1 (the U.K. column) and perform a t-test for the first four autocorrelations and a Q(K) test for the joint significance of the first 8 autocorrelations.

2.- You are a U.S. portfolio manager who cares about GBP returns. Your research department has forecasted returns and betas for the coming year.

Country	Expected return	Beta
U.K.		
r_f	8	
Stock Market	10	1.0
Company A	13	1.2
Company B	9	0.9
Company C	11	1.5
Italy		
r_f	11	
Stock Market	13	1.0
Company D	12	0.8
Company E	16	2.0
Company F	14	1.1
ITL/GBP	-6	
Japan		
r_f	6	
Stock Market	14	1.0
Company G	16	1.3
Company H	18	1.1
Company I	12	0.9
JPY/GBP	+5	

You believe in the multi-country approach, how would you structure your portfolio?

3.- Is asset prices forecasting consistent with market efficiency? What would be the expected asset price movement in an efficient market?

4.- "An asset with high volatility should never be added to a portfolio with low volatility." What do you think of this statement?

5.- Over the past seven years the best performing stock markets have been found in emerging countries. Many of these emerging markets have substantially reduced their inflation rate, used free market policies and open the economy. Should emerging markets continue to outperform if the world capital market is efficient?

6.- The estimated volatility of a domestic asset is 11% (annualized standard deviation of returns). A foreign asset has a volatility of 27% and a correlation of 0.43. What is the volatility of a portfolio invested for 75% in the domestic asset and 25% in the foreign asset?

7.- Suppose you analyze the French gourmet coffee company, Belabu. You have the following information: world economic activity is expected to grow by 5.5% and the sensitivity of Belabu to the world factor (γ_1) is .25; expected domestic real growth relative to world real growth rate is -1% and γ_2=.30; expected currency change is 8% and γ_3=.80; and the retail coffee sector is expected to have a return of 15% and γ_4=.35. The risk free rate (r_f) is 5%. Suppose portfolios associated to these factors have similar expected rates of returns. Belabu's expected return is 4%. Suppose you are an investment analyst and believe on the International APT. Based on the IPT Model, is Belabu a buy, hold or sell?

8.- You are given the following sample estimates for weekly returns for the following countries (estimated using data from January 1980 to May 1996, T=847):

Country	Mean	Standard Deviation
Australia	0.125%	3.223%
Hong Kong	0.252%	4.454%
France	0.180%	2.899%
Germany	0.186%	2.686%
U.K.	0.191%	2.745%

i. Calculate a 95% confidence interval for the weekly population mean and the population variance for each series. Which population parameter seems to be estimated with greater precision?

ii. Calculate a 95% confidence interval for the annual population mean and the population variance