Eurocurrency Contracts

Futures Contracts, FRAs, & Options

Eurocurrency Futures

- **Eurocurrency time deposit**
  Euro-xxx: The currency of denomination of the xxx instrument is not the official currency of the country where the xxx instrument is traded.

  **Example:** Euro-deposit (xxx = a deposit)
  A Mexican firm deposits USD in a Mexican bank. This deposit qualifies as a Eurodollar deposit.

  The interest rate paid on Eurocurrency deposits is called LIBOR.

  Eurodeposits tend to be *short-term*: 1 or 7 days; or 1, 3, or 6 months.
Typical Eurodeposit instruments:

- **Time deposit**: Non-negotiable, registered instrument.
- **Certificate of deposit**: Negotiable and often bearer.

**Note I**: Eurocurrency deposits are direct obligations of commercial banks accepting the deposits and are not guaranteed by any government. They are *low-risk* investments, but Eurodollar deposits are *not risk-free*.

**Note II**: Eurocurrency deposits play a major role in the international capital market. They serve as a benchmark interest rate for corporate funding.

• Eurocurrency time deposits are the *underlying asset* in Eurodollar currency futures.

**Eurocurrency futures contract**

A Eurocurrency futures contract calls for the delivery of a 3-mo Eurocurrency time USD 1M deposit at a given interest rate (LIBOR).

Similar to any other futures a trader can go *long* (a promise to make a future 3-mo deposit) or *short* (a promise to take a future 3-mo loan).

With Eurocurrency futures, a trader can go:
- *Long*: Assuring a yield for a future USD 1M 3-mo deposit
- *Short*: Assuring a borrowing rate for a future USD 1M 3-mo loan.

The *Eurodollar futures* contract should reflect the market expectation for the future value of LIBOR for a 3-mo deposit.
• **Q: How does a Eurocurrency futures work?**

Think of a futures contract on a time deposit (TD), where the expiration day, \( T_1 \), of the futures precedes the maturity date \( T_2 \) of the TD. Typically, \( T_2-T_1: 3\)-months.

Such a futures contract locks you in a 3-mo. interest rate at time \( T_1 \).

<table>
<thead>
<tr>
<th>Today</th>
<th>( T_1 )</th>
<th>( T_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Cash deposit</td>
<td>Cash payout</td>
</tr>
</tbody>
</table>

**Example:** In June you agree to buy in mid-Sep a TD that expires in mid-Dec.

Value of the TD (you receive in mid-Dec) = USD 100.

Price you pay in mid-Sep = USD 99.

\[ \Rightarrow \text{3-mo return on mid-Dec } \frac{100-99}{99} = 1.01\% \text{ (or 4.04\% annually.)} \]

• Eurocurrency futures work in the same way as the TD futures:

“A Eurocurrency futures represents a futures contract on a eurocurrency TD having a principal value of USD 1M with a 3-mo maturity.”

- Traded at exchanges around the world. Each market has its own reset rate: LIBOR, PIBOR, FIBOR, TIBOR, etc.
- Eurodollar futures price is based on 3-mo. LIBOR.
- Eurodollar deposits have a face value of USD 1,000,000.
- Delivery dates: March, June, September, & December.
- Delivery is only "in cash," –i.e., no physical delivery.
- The (forward) interest rate on a 3-mo. CD is quoted at an annual rate. The eurocurrency futures price is quoted as:

\[ 100 – \text{the interest rate of a 3-mo. euro-USD deposit for forward delivery} \]

**Example:** The interest rate on the forward 3-mo. deposit is 6.43%

\[ \Rightarrow \text{The Eurocurrency futures price is 93.57}. \]
Note: If interest rates go up, the Eurocurrency futures price goes down, so the short side of the futures contract makes money.

• Minimum Tick: USD 25.
  Since the face value of the Eurodollar contract is USD 1M
  ⇒ one basis point has a value of USD 100 for a 360-day deposit.
  For a 3-month deposit, the value of 1 bp is **USD 25** (= USD100/4).

**Example:**

Eurodollar futures Nov 20:  93.57
Eurodollar futures Nov 21:  93.54
  ⇒ Short side gains USD 75 = 3 x **USD 25**. ¶

• **Calculation of forward 3-mo LIBOR**

  A: Eurodollar futures reflect market expectations of forward 3-month rates. An implied forward rate \( f \) indicates approximately where short-term rates may be expected to be sometime in the future.

**Example:** 3-month LIBOR spot rate = **5.44%** (91 day period)
6-month LIBOR spot rate = **5.76%** (182 day period)
3-month forward rate = \( f \)

\[
(1 + 0.0576 \times 182/360) = (1 + 0.0544 \times 91/360) \times (1 + f \times 91/360) \\
\Rightarrow f = [(1 + 0.0576 \times 182/360)/(1 + 0.0544 \times 91/360) - 1] \times (360/91) \\
f = 0.059975 (6.00\%)
\]
Example: From the WSJ (Oct. 24, 1994) Eurodollar contracts quotes:

<table>
<thead>
<tr>
<th>Terminology</th>
</tr>
</thead>
</table>
| **Amount**: A Eurodollar futures involves a face amount of USD 1M.  
⇒ To hedge USD 10M, we need 10 futures contracts.  |
| **Duration**: Duration measures the time at which cash flows take place.  
For money market instruments, all cash flows generally take place at the maturity of the instrument.  |

A 6-mo. deposit has approximately twice the duration of a 3-mo. deposit.  
⇒ Value of 1 bp for 6-mo. is approximately USD 50.  

Hedge a USD 1 million six-month deposit beginning in March with:  
(1) 2 March Eurodollar futures *(stack hedge)*.  
(2) 1 March Eurodollar futures and 1 June Eurodollar futures *(strip hedge)*.
Slope: Eurodollar contracts are used to hedge other interest rate instruments. The rates on these underlying instruments may not be expected to change one-for-one with Eurodollar interest rates.

If we define $f$ as the interest rate in an Eurodollar futures contract, then

$$slope = \frac{\Delta \text{ underlying interest rate}}{\Delta f}.$$  (think of delta)

If T-bill rates have a slope of .9, then we would only need 9 Eurodollar futures contracts to hedge USD 10M of 3-mo T-bill.

Notation:

$F_A$: Face amount of the underlying asset to be hedged

$D_A$: Duration of the underlying asset to be hedged.

$n$: Number of Eurodollar futures needed to hedge underlying position:

$$n = \left(\frac{F_A}{1,000,000}\right) \times \left(\frac{D_A}{90}\right) \times slope.$$ 

Example: To hedge USD 10M of 270-day commercial paper with a slope of .935 would require approximately 28 contracts:

$$n = \left(\frac{F_A}{1M}\right) \times \left(\frac{D_A}{90}\right) \times slope = \left(\frac{10M}{1M}\right) \times \left(\frac{270}{90}\right) \times .935 = 28.05$$
Q: Who uses Eurocurrency futures?
A: Speculators and Hedgers.

Hedging
Short-term interest rates futures can be used to hedge interest rate risk:
- You can lock future investment yields (Long Hedge).
- You can lock future borrowing costs (Short Hedge)

Example:
(1) Long Hedge (a promise to make a future 3-mo deposit).
Bank A is offered a 3-mo USD 1M deposit in 2-mo. Buying eurocurrency futures allow Bank A to lock a profit on the future deposit.

(2) Short Hedge (a promise to take a future 3-mo. loan).
A company wants to borrow for 6-mo from Bank A in 1-mo. Selling eurocurrency futures allows Bank A to lock a profit on the future loan.

Eurodollar Strip Yield Curve and the CME (IMM) Swap
Typical quote of 4 successive Eurodollar futures:

<table>
<thead>
<tr>
<th></th>
<th>Price</th>
<th>Yield</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mar 95</td>
<td>93.57</td>
<td>6.43</td>
</tr>
<tr>
<td>Jun 95</td>
<td>93.12</td>
<td>6.88</td>
</tr>
<tr>
<td>Sep 95</td>
<td>92.77</td>
<td>7.23</td>
</tr>
<tr>
<td>Dec 95</td>
<td>92.46</td>
<td>7.56</td>
</tr>
</tbody>
</table>

Days and Period Covered:
- 92 = March 95 – June 95
- 92 = June 95 – September 95
- 91 = September 95 – December 95
- 91 = December 95 – March 96

Successive eurodollar futures give rise to a strip yield curve:
- March future involves a 3-mo. rate: begins in March and ends in June.
- June future involves a 3-mo. rate: begins in June and ends in Sep.
- Etc...
  ⇒ This strip yield curve is called Eurostrip.

Note: Compounding the interest rates (yield) for 4 successive eurodollar contracts defines a one-year rate implied from four 3-mo. rates.
A **CME swap** involves a trade whereby one party receives one-year fixed interest and makes floating payments of the three-months LIBOR.

CME swap payments dates: Same as Eurodollar futures expiration dates.

**Example**: On August 15, a trader does a Sep-Sep swap. Floating-rate payer makes payments on the third Wed. in Dec, & on the third Wed. of the following Mar, June, and Sep. Fixed-rate payer makes a single payment on the third Wed. in Sep.

Note: Arbitrage ensures that the one-year fixed rate of interest in the CME swap is similar to the one-year rate constructed from the Eurostrip.

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**Pricing Short-Dated Swaps**

Swap coupons are routinely priced off the Eurostrip.

**Key to pricing swaps**: The swap coupon is set to equate the present values of the fixed-rate side and the floating-rate side of the swap.

- Eurodollar futures contracts provide a way to do that.

- The estimation of the fair mid-rate is complicated a bit by:
  1. the convention is to quote swap coupons for generic swaps on a s.a. bond basis, and
  2. the floating side, if pegged to LIBOR, is usually quoted money market basis.
Pricing Short-Dated Swaps

Notation: If the swap has a tenor of \( m \) months and is priced off 3-mo Eurodollar futures, then pricing will require \( n \) sequential futures series, where \( n = m/3 \).

Example: If the swap is a 6-mo swap \( (m=6) \Rightarrow \) we need 2 Eurodollar futures contracts. ¶

• Procedure to price a swap coupon involves three steps:
  
i. Calculate the implied effective annual LIBOR for the full duration (full-tenor) of the swap from the Eurodollar strip.
  
ii. Convert the full-tenor LIBOR (quoted on money market basis), to its fixed-rate equivalent \( \text{FRE}_{0,3n} \) (quoted on annual bond basis).
  
iii. Restate the fixed-rate equivalent on the same payment frequency as the floating side of the swap. The result is the swap coupon \( \text{SC} \).

Pricing Short-Dated Swaps: Details

• Three steps:
  
i. Calculate the implied effective annual LIBOR for the full duration (full-tenor) of the swap from the Eurodollar strip:

\[
\begin{align*}
  r_{0,3n} &= \prod_{t=1}^{n} \left[ 1 + r_{3(t-1),3t} \frac{A(t)}{360} \right]^{\tau} - 1, \\
  \tau &= 360/ \sum A(t) 
\end{align*}
\]

ii. Convert the full-tenor LIBOR, which is quoted on money market basis, to its fixed-rate equivalent \( \text{FRE}_{0,3n} \), which is stated as an annual effective annual rate (annual bond basis):

\[
\text{FRE}_{0,3n} = r_{0,3n} \times (365/360).
\]

iii. Restate the fixed-rate equivalent on the same payment frequency as the floating side of the swap. The result is the swap coupon \( \text{SC} \). This adjustment is given by

\[
\text{SC} = [(1 + \text{FRE}_{0,3n})^{1/k} - 1] \times k, \quad k = \text{frequency of payments}.
\]
Example:

Situation: It's October 24, 1994. H Bank wants to price a one-year fixed-for-floating interest rate swap against 3-mo LIBOR starting on Dec 94. Fixed rate will be paid quarterly (quoted quarterly bond basis).

<p>| Eurodollar Futures, Settlement Prices (October 24, 1994) |
|---------------------------------|----------------|----------------|----------------|</p>
<table>
<thead>
<tr>
<th>Implied</th>
<th>Price 3-mo. LIBOR</th>
<th>Notation Days</th>
<th>Number of Days (A(t))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dec 94</td>
<td>94.00</td>
<td>6.00</td>
<td>0 x 3 90</td>
</tr>
<tr>
<td>Mar 95</td>
<td>93.57</td>
<td>6.43</td>
<td>3 x 6 92</td>
</tr>
<tr>
<td>Jun 95</td>
<td>93.12</td>
<td>6.88</td>
<td>6 x 9 92</td>
</tr>
<tr>
<td>Sep 95</td>
<td>92.77</td>
<td>7.23</td>
<td>9 x 12 91</td>
</tr>
<tr>
<td>Dec 95</td>
<td>92.46</td>
<td>7.56</td>
<td>12 x 15 91</td>
</tr>
</tbody>
</table>

Housemann Bank wants to find the fixed rate that has the same present value as four successive 3-mo. LIBOR payments.

(1) Calculate implied LIBOR rate using (i).

Swap is for twelve months, $n = 4$.

\[
f_{0,12} = \left[ (1 + .06x(90/360)) \times (1 + .0643x(92/360)) \times (1 + .0688x(92/360)) \right]^{360/365} - 1 = .06760814. \] (money mkt basis)

(2) Convert this money market rate to its effective equivalent stated on an annual bond basis.

\[
F_{RE_{0,12}} = .06760814 \times (365/360) = .068547144. \] (bond basis)

(3) Coupon payments are quarterly, $k = 4$. Restate this effective annual rate on an equivalent quarterly bond basis.

\[
S_C = \left[ \frac{1 + .068547144}{4} - 1 \right] \times 4 = .0668524 \] (quarterly bond basis)

⇒ The swap coupon mid-rate is 6.68524%.
**Example:** Now, Housemann Bank wants to price a one-year swap with *semiannual* \((k = 2)\) fixed-rate payments against 6-month LIBOR.

The swap coupon mid-rate is calculated to be:

\[
SC = [(1 + 0.068547144)^{1/2} - 1] \times 2 = 0.06741108 \text{ (s.a. bond basis).}
\]

A dealer can quote swaps having tenors out to the limit of the liquidity of Eurodollar futures on any payment frequency desired.

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**Gap Risk Management**

*Gap risk:* Assets and liabilities have different maturities. Eurocurrency futures are used to hedge gap risk.

**Example:** Gap Risk Management

**Situation:** It's March 20.

- A bank can lend a 6-mo Euro-EUR deposit at 4.25%, with a value date on March 24 and maturity date on September 24 (183 days).
- A Swiss bank observes a rate of 4% on 3-mo euro-EUR deposits, with a value date of March 24. The deposit matures on June 24 (92 days).
- June Euro-EUR futures are trading at 96.13 (or, yield = 3.87%).

March 24 92 days June 24 91 days Sep 24

- 3-mo deposit = 4%
- 6-mo loan = 4.25%
- 183 days
Gap Risk Management

Example (continuation):

• Gap risk: The bank receives a 3-mo deposit and lends for 6-mo.
  ⇒ Risk: The interbank deposit interest rate on June 24 is uncertain.
  ⇒ Gap risk: It can be managed using Jun Euro-EUR futures.

• Bank considers lending a 6-mo deposit at 4.25%, funded by two 3-mo deposits: the 1st at 4%; the 2nd one at the June Euro-EUR rate.

Q: Is it profitable for the bank?
Yes, if bank can get a 3-mo deposit starting in June at a lower rate than \( f \).

Gap Risk Management

Calculations: We calculate \( f \) & compare it with the June Euro-EUR rate.
Implied forward rate, \( f \) (break even):

\[
[1 + .0425 \times (183/360)] = [1 + .04 \times (92/360)] \times [1 + f \times (91/360)]
\]

⇒ \( f = 4.457\% \).

• As long as the bank can ensure that it will pay a rate less than 4.457% for the 2nd 3-mo. period, the bank will make a profit.

• June Euro-EUR are at 3.87% < \( f = 4.457\% \).
  ⇒ Shorting one June Euro-EUR at 96.13, makes the bank a profit.
Forward Rate Agreements (FRA)

**FRA Contract**

An FRA involves two parties: A buyer and a seller. The parties agree on fixing the interest rate at $f$, agreed rate, on a nominal sum of money, $N$, during a future period of time, the FRA period.

- Seller pays the buyer (increased interest cost) if $i$ (market rate) > $f$
- Buyer pays the seller (increased interest cost) if $i$ < $f$.

- The contract is settled in cash at the beginning of the FRA period. That is, an FRA is a cash-settled interbank forward contract on $i$.

Terminology: An agreement on a 3-mo. interest rate for a 3-mo. period beginning 6-mo from now and terminating 9-mo from now (“6x9”).

⇒ this agreement is called "six against nine," or 6x9.

<table>
<thead>
<tr>
<th>FRA starts</th>
<th>Cash Settlement</th>
</tr>
</thead>
<tbody>
<tr>
<td>Today (t=0)</td>
<td>6 months</td>
</tr>
<tr>
<td></td>
<td>3 months</td>
</tr>
<tr>
<td></td>
<td>9 months</td>
</tr>
</tbody>
</table>

FRA Contract Notation:

- $f$ = Agreed rate at $t=0$ (“Today”), expressed as a decimal.
- $S$ = Settlement rate (market rate, $i$), observed at beginning of FRA period
- $N$ = Nominal contract amount,
- $ym$ = Days in the FRA period, and
- $yb$ = Year basis (360 or 365).

- Then, if $i > f$, seller pays the buyer: $N * (i - f) * (ym/yb)$.
- if $i < f$, buyer pays the seller.

- Think of the buyer as short interest rate exposure (gets paid when $i ↑$)
Note: Cash settlement is made at the beginning of the FRA period, then, the denominator discounts the payment back to that point.

Example: A bank buys a 3X6 FRA for USD 2M with $f = 7.5\%$. (Bank pays if $i < f$; gets paid if $i > f$.) There are 91 days in the FRA period. Suppose, in 3 months, at the beginning of the FRA period, $i = 9\%$.  

Summary:

$N = \text{USD 2M}$,  
$ym = 91$,  
$yb = 360$,  
$f = 7.5\%$,  
$i = 9\%$.  
($i > f \Rightarrow \text{Bank gets paid}$.)

• Bank receives cash at the beginning of the FRA-period from the seller:  
$$\text{USD 2M} \times (0.09 - 0.075) \times (91/360) = \text{USD 7,414.65}$$

Example (continuation):

Check: The bank borrowing cost is $f = 7.5\%$:  
$$\text{USD 2M} \times .075 \times (91/360) = \text{USD 37,916.67}.$$  

Bank’s CFs at the end of the 6-mo (FRA) period:

• Net borrowing cost on USD 2M:  
$$\text{USD 2M} \times .09 \times (91/360) = \text{USD 45,500.00}$$

minus (FRA adjustment)  
$$\text{USD 7,414.65} \times [1 + .09 \times (91/360)] = \text{USD -7,583.33}$$

Net borrowing cost = $\text{USD 37,916.67}$
**FRA and Arbitrage**

- An FRA is an interbank-traded equivalent of the implied forward rate.
- Consider how a bank would construct *FRA bid & ask* rates by reference to interbank bid & ask rates on Eurodeposits ("Cash").

**Example:** On Sep 24, a Eurobank wants USD 100M of 6-mo deposit. It is offered USD 100M of 9-mo deposit at the bank's bid rate (10.5625%).

Current rates:

<table>
<thead>
<tr>
<th>Cash</th>
<th>FRA</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>bid</strong> asked</td>
<td><strong>bid</strong> asked</td>
</tr>
<tr>
<td>6 months</td>
<td>10.4375</td>
</tr>
<tr>
<td>9 months</td>
<td>10.5625</td>
</tr>
</tbody>
</table>

- **Q:** Should the bank take the 9-mo deposit?
  The 9-mo deposit becomes a 6-mo deposit by selling a 6X9 FRA. That is, the bank sells off (lends) the last 3-mo in the FRA market.

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**Example (continuation):**

Days from September 26 to June 26 (9-mo deposit) = 273 days.
Days from March 26 to June 26 (6X9 FRA) = 92 days.

- The interest paid at the end of nine months to the depositor is:
  USD 100 M * (10.5625%) * (273/360) = **USD 8,009,895.83**.

- Interest earned on lending for 6-mo in the interbank market, then another 3-mo at the FRA rate is:
  USD 100M * [(1+.104375*(181/360)) * (1+.1048*(92/360)) – 1]
  = **USD 8,066,511.50**.

There is a net profit of **USD 56,615.67** at the end of nine months:

⇒ Bank takes the 9-mo deposit at the bid’s rate of **10.5625%**.
Example (continuation):
Q: Is Arbitrage possible?
A: Check if **USD 8,066,511.50** > 9-mo borrowing cost in Cash market.

The bank would have to buy a deposit (borrow) for 9 months in the interbank (Cash) market at **10.6875%**:

\[
\text{USD } 100 \text{ M } \times (0.106875) \times \left( \frac{273}{360} \right) = \text{USD 8,104,687.50}. 
\]

\[\Rightarrow\text{ No arbitrage: Interest paid on the deposit (**USD 8,104,687.50**) } > \text{Interest earned on lending for 6-mo in the Cash market } \& \text{ another 3-mo at the FRA rate (**USD 8,066,511.50**).} \]

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**Eurodollar Futures Options and Other Derivatives**

**Example**: CME eurodollar put.

A CME eurodollar put (call): Buyer pays a premium to acquire the right to go short (long) one CME eurodollar futures contract at the opening price given by the put's (call's) strike price.

- Options are American.
- Expiration (T): Last trade date for the futures contract.
- Strike prices are in intervals of .25 in terms of the CME index.

**Example**: A dealer *buys a put* on June Eurodollar futures with a strike of 93.75. If exercised, it gives the right to go *short* one eurodollar futures contract at an opening price of 93.75.
Example: On Tuesday, November 1, 1994, the *WSJ* published the following quotes for eurodollar and LIBOR futures options.

### EURODOLLAR (CME)

<table>
<thead>
<tr>
<th>Strike</th>
<th>$ million; pts. of 100%</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Calls-Settle</td>
</tr>
<tr>
<td></td>
<td>Dec</td>
</tr>
<tr>
<td>9350</td>
<td>0.56</td>
</tr>
<tr>
<td>9375</td>
<td><strong>0.33</strong></td>
</tr>
<tr>
<td>9400</td>
<td>0.14</td>
</tr>
<tr>
<td>9425</td>
<td>0.03</td>
</tr>
<tr>
<td>9450</td>
<td>0.00</td>
</tr>
<tr>
<td>9475</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Est. vol. 56,820;
Fri vol. 80,063 calls; 72,272 puts
Op. Int. Fri 939,426 calls; 1,016,455 puts

- Premium quotes: in percentage points (1 bp = USD 25).

### Examples: June 95 put and call

1. Consider the June 95 put, with a strike price of $X_{Z,p} = 93.75$. A price of **0.69** would represent **USD 25 * 0.69 = USD 1,725**.

2. Consider the June 95 call, with a strike price of $X_{Z,c} = 93.50$. A price of **0.18** would represent **USD 25 * 0.18 = USD 450**.

**Example:** Buying insurance.

**Underlying Position:** Short a June 1995 eurodollar futures at $Z = 93.99$.

UP’s problem: Potential unlimited loss.

**Solution:** Buy insurance: Long a *June 1995 call* ($X_{Z,c} = 93.50$ & C = **0.18**). The spot interest rate is 6%.

Hedging Position’s (Long June 1995 call) cost:
- Call premium paid: **USD 25 * 0.18 = USD 450**.
- Add 6% carrying cost: **USD 450 * [1 + .06 * (30/360)] = USD 452.25**
Simulate net payoffs for different $Z$ in 30 days: 93.00; 93.50; & 94.50.

**Scenario #1**: In 30 days, $Z = 93.00$ (< $X_{Z,c}$, call no exercised)
- UP (futures) payoff: 93.99 – 93.00 = 0.99 or **USD 2,475** (= 99 * **USD 25**)
- HP (not exercise): 0.
  ⇒ OP Net payoff: **USD 2,475** – **USD 452.25** = **USD 2,022.75**.

**Scenario #2**: In 30 days, $Z = 93.50$ (= $X_{Z,c}$, call no exercised/indifferent)
- UP payoff: 93.99 – **93.50** = 0.49 or **USD 1,225** (= 49 * **USD 25**)
- HP (no exercise): 0
  ⇒ OP Net payoff: **USD 1,225** – **USD 452.25** = **USD 772.75**.

**Scenario #3**: In 30 days, $Z = 94.50$ (> $X_{Z,c}$, call exercised)
- UP payoff: 93.99 – 94.50 = -0.51 or **USD -1,275** (= -51 * **USD 25**)
- HP (exercise): 94.50 – **93.50** = 1.00 or **USD 2,500** (= 100 * **USD 25**)
  ⇒ OP Net payoff: **USD 1,225** – **USD 452.25** = **USD 772.75**.

**Payoff Matrix** (in 30 days) for possible $Z$ prices: 93, 93.50, 94.50, 95.

<table>
<thead>
<tr>
<th>Futures Price</th>
<th>Future Payoff</th>
<th>Call Payoff</th>
<th>Option Cost</th>
<th>Carrying Cost</th>
<th>Total Cost</th>
<th>Total Cost (USD)</th>
</tr>
</thead>
<tbody>
<tr>
<td>93.00</td>
<td>.99</td>
<td>0.00</td>
<td>.18</td>
<td>.0009</td>
<td>.8091</td>
<td><strong>2022.75</strong></td>
</tr>
<tr>
<td><strong>93.50</strong></td>
<td>.49</td>
<td>0.00</td>
<td><strong>.18</strong></td>
<td>.0009</td>
<td><strong>.3091</strong></td>
<td><strong>772.75</strong></td>
</tr>
<tr>
<td>94.50</td>
<td>-.51</td>
<td>1.00</td>
<td><strong>.18</strong></td>
<td>.0009</td>
<td><strong>.3091</strong></td>
<td><strong>772.75</strong></td>
</tr>
<tr>
<td>95.00</td>
<td>-1.01</td>
<td>1.50</td>
<td><strong>.18</strong></td>
<td>.0009</td>
<td><strong>.3091</strong></td>
<td><strong>772.75</strong></td>
</tr>
</tbody>
</table>

Note: Minimum payoff (floor): **USD 772.75** (= **30.91** * **USD 25**)

- By buying the call, the trader has limited his/her possible exposure on the future to **-.3091** basis points (or a minimum profit of **USD 772.75**).

- This sum can be approximated: $Z – X_{Z} – C = 93.99 – **93.50** – **.18** = **.31**.

Note: Usually a put establishes a floor. Here, the intuition is reversed.
Note: Q: A call establishes a floor?
A: Recall that \( Z = 100 - f \) \( \Rightarrow \) The cap is really a floor on future interest costs, given by \( f \). Not on \( Z! \)

When \( Z \leq 100 - X \) \( \Rightarrow f \geq X \).
Thus a call on \( f \), which pays off when \( f > X \), is equivalent to a put on \( Z \), which pays off when \( Z < 100 - X \).

Example: Let \( X_{\text{interest rate, call}} = 6.50 \).
• A call on the forward rate \( f \) has a positive exercise value when \( f > 6.50 \).
• This is equivalent to an eurodollar futures price \( Z < 100 - 6.50 = 93.50 \).
  \( \Rightarrow \) The value of an interest rate call with \( X_{t,\text{call}} = 6.50 \) is equal to the value of an eurodollar futures put with \( X_{Z,\text{put}} = 93.50 \). ¶
• Summary: The value of a call on \( f \) with strike price \( X_{i,c} \) is equal to the value of a put on \( Z = 100 - f \) with strike price \( X_{Z,\text{put}} = 100 - X_{i,c} \).

Valuation of futures options
Q: How should eurodollar futures options be priced?
A: Use the Black-Scholes formula.

• Underlying asset (uncertain variable): the forward interest rate \( f \).
  Key: The forward interest rate, \( f \), embodied in the futures price.

• The value of a European call on the forward interest rate \( f \) is given by:
  \[ c_t = B_T(T) \ast \left[ f \ast N(d1) - X \ast N(d2) \right], \]
  \[ d1 = \ln(\frac{f}{X}) + \frac{0.5 \ast v^2 \ast T}{v \ast \sqrt{T}} \]
  and \[ d2 = \ln(\frac{f}{X}) - \frac{0.5 \ast v^2 \ast T}{v \ast \sqrt{T}} \]

  \( B_T(T) \): Price of futures contract with expiration date \( T \),
  \( N(.) \): Cumulative normal distribution,
  \( v^2 \): Variance of \( B_T \).
• The European put price is obtained from the put-call parity:

\[ p = c + B \times (X - f). \]

• The European put and call will have equal values when the forward interest (or FRA) rate is equal to the strike price.

**Example:** Table XV.B (European options on interest rates).

• Assume \( v = .15 \).
• \( T = 90/365 = .2466 \).
• Discount rate: 8% (\( B = 1/(1+.08/4) = .98039 \)).
• Option premium is paid today, and the cash value of the option payoff is paid at option expiration.

<table>
<thead>
<tr>
<th>( f )</th>
<th>( X )</th>
<th>Call</th>
<th>Put</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.5</td>
<td>7.0</td>
<td>.541</td>
<td>.008</td>
</tr>
<tr>
<td></td>
<td>.218</td>
<td>.551</td>
<td>.060</td>
</tr>
<tr>
<td></td>
<td>.051</td>
<td>.001</td>
<td>.011</td>
</tr>
<tr>
<td>8.0</td>
<td>7.5</td>
<td>1.471</td>
<td>.071</td>
</tr>
<tr>
<td></td>
<td>.561</td>
<td>.233</td>
<td></td>
</tr>
</tbody>
</table>

Table XV.B

**Value of European Options on Forward Interest Rates**
Example (continuation):
i. Calculations for the call and put option with $X = 7$ and $f = 7.5$

A. Call
Substituting into d1 and d2:
\[
d1 = \frac{\ln(f/X) + .5 \cdot v^2 \cdot T}{v \cdot \sqrt{T}} \quad \text{and} \quad d2 = \frac{\ln(f/X) - .5 \cdot v^2 \cdot T}{v \cdot \sqrt{T}}.
\]
\[
d1 = \left[ \frac{\ln(7.5/7) + .5 \cdot (.15)^2 \cdot .2466}{.15 \cdot .2466} \right] = .9635
\]
\[
d2 = \left[ \frac{\ln(7.5/7) - .5 \cdot (.15)^2 \cdot .2466}{.15 \cdot .2466} \right] = .8890
\]

* Cumulative normal distribution at $z = .9635$: .3324.

Recall: since $d1$ is positive, we have to add .50%.
\[
\rightarrow \quad N(d1 = .9635) = .3324 + .50 = .8324
\]
\[
N(d2 = .8890) = .8130
\]

Example (continuation):
i. Calculations for the call and put option with $X = 7$ and $f = 7.5$

\[
N(d1 = .9635) = .8324
\]
\[
N(d2 = .8890) = .8130
\]

\[
c = B_t(T) \cdot [f \cdot N(d1) - X \cdot N(d2)] = .98039 \cdot [.75 \cdot .8324 - 7 \cdot .8130] = .5408
\]

B. Put
Substituting into put-call parity:
\[
p = c + B \cdot (X - f) = .5408 + .98039 \cdot (7 - 7.5) = .050805.
\]
Example (continuation):

**Example:** Interpretation of option values in Table XV.B.

Let’s pick: $X = 7.0$ & $f = 7.5 \Rightarrow e = .541$.

- Since $X$ and $f$ are in percent, the prices ($c$ & $p$) is also stated in percent.

To translate this price to a dollar amount: we have to know the option size and the duration in days of the forward interest period.

- Suppose the option is based on 3-mo LIBOR.
- Nominal amount of USD 10 million.
- There are 92 days in the 3-mo period.

Then the dollar cost of the option is:

$$ .541 \times (1/100) \times (92/360) \times USD \, 10,000,000 = USD \, 13,825.56. $$

Example (continuation):

- The values in Table XV.B also assume that the option premium is paid today, and that the cash in the option payoff is received at *expiration*, which is the beginning of the forward interest or FRA period.

For example, suppose the cash in the option payoff will not be received until the end of the forward interest period (92 days).

Then, the table value (for $X = 7.0$, $f = 7.5$) must be *discounted* by the forward interest rate $f = 7.5$ for 92 days:

$$ .541 / [1 + .075 \times (92/360)] = .5308258. $$

This corresponds to an option premium of

$$ .5308258 \times (1/100) \times (92/360) \times USD \, 10M = USD \, 13,565.55. $$
• At the CME, Eurodollar options are *American*. To price CME Eurodollar options we use the American option pricing equations.

**Example**: The Eurodollar future is \( Z = 93.00 \). We want to get the value of a future call with strike price of \( X_Z = 92.50 \).

First, we calculate:

\[
\begin{align*}
  f &= 100 - 93.00 = 7.00 \\
  X &= 100 - 92.50 = 7.50.
\end{align*}
\]

Table XV.C is the same as Table XV.B, but the eurodollar futures prices and strikes have been substituted for their interest rate equivalent, and the options are *American* instead of European.

<table>
<thead>
<tr>
<th>( X_Z \rightarrow )</th>
<th>91.50</th>
<th>92.00</th>
<th><strong>92.50</strong></th>
<th>91.50</th>
<th>92.00</th>
<th><strong>92.50</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>( Z \downarrow )</td>
<td>92.00</td>
<td>.071</td>
<td>.233</td>
<td>.555</td>
<td>.564</td>
<td>.233</td>
</tr>
<tr>
<td>92.50</td>
<td>.011</td>
<td>.061</td>
<td>.219</td>
<td>1.004</td>
<td>.555</td>
<td>.219</td>
</tr>
<tr>
<td><strong>93.00</strong></td>
<td>.001</td>
<td>.008</td>
<td>.051</td>
<td>1.500</td>
<td>1.002</td>
<td>.545</td>
</tr>
</tbody>
</table>

**Caps, Floors, and Collars**

"*Cap*" on interest rates: \( i \) do not rise above some ceiling level.

"*Floor*" on interest rates: \( i \) do not fall too low.

*Collar*: A long cap and a floor.

• **Motivation**: Financial cost insurance.

**Example**: Collar on 6-mo LIBOR

6-mo LIBOR: 8.50%.

Two parties negotiate a collar: Cap 6-mo LIBOR at 9%, Floor 6-mo LIBOR at 7.5%.

**Note**: If the cap level is low enough (say 8.25) and the floor level is high enough (say 8.25), one is left with a fixed-rate contract.
**Incomplete Example**: Cap.

A LIBOR borrower buys an interest rate cap of 9% on 6-mo. LIBOR. The cap lasts T days.

Buyer of the cap: Pays an up-front price for the cap, a *premium*, at t=0.

When 6-mo. LIBOR rises above 9% in any loan period, the cap buyer (exercises option!) will be compensated for the increased interest cost.

**Note**: At t=0, the market interest rate on the first 6-mo. interval (say, from Jan 30 to July 30) is already known. It is typically *excluded* from the cap.

![Diagram](image)

**Example**: A Cap.

- On December 17, 2018, a LIBOR borrower buys a 2-yr interest rate cap of 9%, with 6-mo. LIBOR payments on January 30 and July 30.
- On Jan 30, 2019 the option to cap interest cost at 9% begins. It covers the new 6-mo. interval starting on July 30, 2019 and extending to Jan 30.
- \(i\) (= 6-mo LIBOR) for this period will be fixed on July 30, 2019, but interest will be paid on the following January 30, 2020.

![Diagram](image)

- 6-mo LIBOR is fixed at 9.5% on July 30, 2019.
- On Jan 30, 2020 (184 days later) the cap writer will pay the cap buyer: \[
\text{USD} \ 10,000,000 \times \frac{(9.5 - 9)}{100} \times \frac{184}{360} = \text{USD} \ 25,555.56.
\]
**Example:** A Cap.

⇒ The cap is a series of European call options on the interest rate, where the call strike price is the cap rate.

First option (#1) begins at the beginning of the cap period (Jan 30, 2019) & expires on first interest reset date. Underlying variable: 6-mo implied forward (or FRA) rate, $f$, from July 30, 2019 to January 30, 2020.

<table>
<thead>
<tr>
<th>#1 Option begins</th>
<th>#1 Option matures (i is set)</th>
<th>$f$?</th>
<th>#1 Option Settlement</th>
</tr>
</thead>
<tbody>
<tr>
<td>January 30, 2019</td>
<td>July 30, 2019</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

#1 Option expires on July 30, 2019 because the rate is set or determined on that date. But the cash value of the option will not be received until the following January 30, 2020.

---

**Example (continuation):**

#2 Option begins on July 30, 2019 and expires on following January 30, 2020 (second reset date). The underlying variable is $f$ from January 30, 2020 to July 30, 2020 (182 days, leap year).

<table>
<thead>
<tr>
<th>#3 Option begins</th>
<th>#3 Option matures (i is set)</th>
<th>$f$?</th>
<th>#3 Option Settlement</th>
</tr>
</thead>
<tbody>
<tr>
<td>January 30, 2020</td>
<td>182 days</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

#3 Option begins on January 30, 2020 and expires on July 30, 2020 (third reset date). The underlying variable is $f$.

<table>
<thead>
<tr>
<th>#3 Option begins</th>
<th>#3 Option matures (i is set)</th>
<th>$f$?</th>
<th>#3 Option Settlement</th>
</tr>
</thead>
<tbody>
<tr>
<td>January 30, 2020</td>
<td>184 days</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

• Similarly, a floor is a series of European put options on the interest rate, where the put strike price is the interest floor.

• A collar is a combination of calls and puts.
Valuation of a Cap
A cap is a series of European options. The value of the cap is equal to the sum of the value of all the options imbedded in the cap.

Example: Consider a 2-year interest rate cap of 9% on 6-mo LIBOR.
- Cap amount is **USD 10 million**.
- The cap trades on January 28 for effect on January 30.
- Reset dates: July 28 and January 28, and take effect two days later.
- There are 181 days from January 30 to July 30 (182 on leap year).
  • At the time the cap is purchased, offered rates on time deposits are:

<table>
<thead>
<tr>
<th>Period</th>
<th>Offered Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>6 month</td>
<td>8.00</td>
</tr>
<tr>
<td>12 mo.</td>
<td>8.50</td>
</tr>
<tr>
<td>18 mo.</td>
<td>8.65</td>
</tr>
<tr>
<td>24 mo.</td>
<td>8.75</td>
</tr>
</tbody>
</table>

Example (continuation):
• There are 3 options in the cap. Let’s analyze the first one: Option #1.
• The first six months' rate of interest is already determined at 8%.
• Option #1 is thus written on the second six-month period.
• Underlying variable: The "6 against 12" FRA rate.

STEP 1
• Calculating the implied forward rate from the formula:

\[
[1 + .085 \times (365/360)] = [1 + .080 \times (181/360)] \times [1 + f \times (184/360)]
\]

\[\Rightarrow f = .08644.\]
• The option expires in six-months, but does not settle until the end of the second six-month period, which is one year from today.

STEP 2
• The discount rate on the option is 8.50%. The discount factor is

\[ [1 + .085 \times (365/360)] = 1.08618.\]
### Example (continuation):

Note: Other forward (FRA) rates and discount factors may be calculated in a similar way.

<table>
<thead>
<tr>
<th>Option #</th>
<th>Implied Forward Rate</th>
<th>Discount Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>8.644</td>
<td>1.08618</td>
</tr>
</tbody>
</table>

**STEP 3**

- Impute volatilities to each time period. Based on recent activity in the market for caps, these are assumed to be 15 percent \( (v = .15) \).

**STEP 4**

- Calculate Call Value \( (c) \) and amount paid for #1 Option.
- Apply Black-Scholes: \( c = .2029 \).
- Since there are 365 – 181 = 184 days in the interest period, this corresponds to a USD amount of
- Amount paid = \( (.2029/100) \times (184/360) \times USD\ 10M = USD\ 10,371.78 \).

### Example (continuation):

**STEP 4 (continuation)**

- Calculate Call Value \( (c) \) and total amount paid for the cap.

We now have the information needed to price each option and, thus, the cap:

<table>
<thead>
<tr>
<th>Option #</th>
<th>T(365) ( \frac{f}{X} )</th>
<th>( v ) (adjusted)</th>
<th>( \text{USD} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>181 8.644 9</td>
<td>.150 1/1.08618</td>
<td>.2029 10,371.78</td>
</tr>
<tr>
<td>2</td>
<td>184 8.242 9</td>
<td>.150 1/1.13119</td>
<td>.1966 9,886.40</td>
</tr>
<tr>
<td>3</td>
<td>181 7.998 9</td>
<td>.150 1/1.17743</td>
<td>.2071 10,583.48</td>
</tr>
</tbody>
</table>

Value of cap (USD) = 10,371.78 + 9,886.40 + 10,583.48 = 30,841.66.

---
Cap Packaging

- Caps and floors are usually written by companies with existing floating rate borrowings, such as banks.
- Banks often hedge their option writing by borrowing funds at a variable rate with an interest cap.

Example: Bertoni Bank faces the following alternative operations:

a. Lend money to company A at LIBOR + ⅞%.
b. Borrow money from investors at LIBOR + ⅘% with a cap at 10%.
c. Sell a cap option at 10% to company B for ½% per year.

An alternative for Bertoni Bank is to lend to company A at (LIBOR + ⅞%) and borrow from investors at (LIBOR + ⅛%) without any cap. In effect, the margin is equal to ¾%.

Example (continuation):
Let's analyze the operation. Bertoni Bank's net income is given by:

\[(\text{LIBOR} + ⅞\%) - \min(\text{LIBOR} + ⅘\%, 10) + ½ - \max(0, \text{LIBOR} - 10)\].

If LIBOR remains below 10%, Bertoni Bank's net income per year is:

\[(\text{LIBOR} + ⅞\%) - (\text{LIBOR} + ⅘\%) + ½ = 1\%

If LIBOR increases beyond 10%, Bertoni Bank's net income per year is:

\[(\text{LIBOR} + ⅞\%) - (10) + ½ - (\text{LIBOR} - 10) = ⅘ + ½ = 1.375\%\]
Recall:
• In a previous example, we made an interest adjustment to the price of the zero-coupon or discount bond price $B_t$.
• The adjustment reflected the fact that each one of the series of call options involved in the interest rate cap expired at the beginning of the interest period.
• But the option payoff was only received at the end of the period.
• If the number of days in the period is $d_{tm}$, then, in the option formula (XV.4) we replace $B_t(T)$ with $B_t(T+d_{tm})$. At expiration,
  $$B_t(T+d_{tm}) = \frac{1}{1 + f \cdot \left(\frac{d_{tm}}{360}\right)}$$
  (XV.6)
  where $f$ is the interest rate fixed at time $t+T$.

Thus, if $f > X$, the call payoff is
  $$\left(\frac{1}{1 + f \cdot \left(\frac{d_{tm}}{360}\right)}\right) \cdot (f - X).$$
  (XV.7)

Recall: If $f > X$, the call payoff is: $\left(\frac{1}{1 + f \cdot \left(\frac{d_{tm}}{360}\right)}\right) \cdot (f - X)$.

• Compare the above payoff with the value of an FRA: They are the same, provided the option strike price $X$ is the rate agreed (A) in the FRA.
• Similarly, if $f < X$, the call payoff will be zero, but the absolute value of (XV.7) will be the payoff to the corresponding put.

• Thus for LIBOR options involved in a cap, floor, or collar, we may replace equation (XV.4), (XV.5)
  $$c_t = B_t(T+d_{tm}) \cdot [f \cdot N(d1) - X \cdot N(d2)]$$
  (XV.8)
  $$p = c + B_t(T+d_{tm}) \cdot (X - f).$$
  (XV.9)

(The values of $d1$ and $d2$ remain unchanged.)
Then if I go long a call and short a put, $c_t - p_t$, each with strike price $X$ corresponding to the agreed rate in an FRA, the payoff at option expiration will be:

$$c_t - p_t = (f - X) / [1 + f^* (dtm/360)],$$

(The payoff to the buyer of an FRA!)

To summarize:

Long a LIBOR call + Short a LIBOR put = FRA bought.

Similarly,

Short a LIBOR call + Long a LIBOR put = FRA sold.

Note: the equivalence is in terms of value. But the cash flow on an FRA is received at the beginning of the FRA period, whereas the cash flow for the options is received at the end of the FRA period.

Example: Go back to previous Example.

• You want to buy an FRA with $A = 7$, when $f = 7.5$.
• From Table XV.B, we obtain $c$ and $p$ with $X = 7.0$ and $f = 7.5$.
• Thus, the value of the FRA is .49 ($= .541 - .051$).

<table>
<thead>
<tr>
<th>$f$</th>
<th>7.0</th>
<th>7.5</th>
<th>8.0</th>
<th>8.5</th>
<th>7.5</th>
<th>8.0</th>
<th>8.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>X:</td>
<td>.541</td>
<td>.988</td>
<td>1.471</td>
<td>.051</td>
<td>.008</td>
<td>.001</td>
<td></td>
</tr>
<tr>
<td>7.5</td>
<td>.218</td>
<td>.551</td>
<td>.992</td>
<td>.218</td>
<td>.060</td>
<td>.011</td>
<td></td>
</tr>
<tr>
<td>8.0</td>
<td>.060</td>
<td>.233</td>
<td>.561</td>
<td>.551</td>
<td>.233</td>
<td>.071</td>
<td></td>
</tr>
</tbody>
</table>