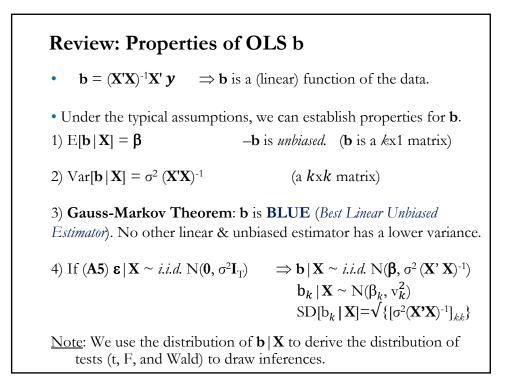


Review: CLM & OLS

Classical linear regression model (CLM) - Assumptions:
(A1) DGP: y = X β + ε is correctly specified (& linear!).
(A2) E[ε|X] = 0
(A3) Var[ε|X] = σ² I_T
(A4) X has full column rank -rank(X) = k, where T ≥ k.
Objective function: S(x; β) = ∑_{i=1}^T ε_i² = ε'ε = (y - Xβ)' (y - Xβ) ⇒ b = (X'X)⁻¹ X' y (kx1) vector
b is an estimate of the marginal effect (first derivative) on (A1).



Review: Properties of OLS b

5) If (A5) is not assumed, we still can obtain a (limiting) distribution for **b**. Under additional assumptions –mainly, the matrix **X'X** does not explode as T becomes large–, as $T \rightarrow \infty$,

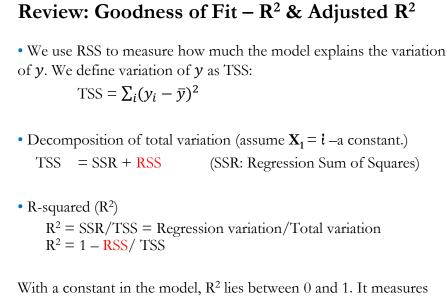
> (i) $\mathbf{b} \xrightarrow{p} \mathbf{\beta}$ (**b** is consistent) (ii) $\mathbf{b} \xrightarrow{a} N(\mathbf{\beta}, \sigma^2 (\mathbf{X}, \mathbf{X})^{-1})$ (**b** is asymptotically normal)

• Properties (1)-(4) are called *finite* (or *small*) sample properties.

• Properties (5.i) and (5.ii) are called *asymptotic* properties, they only hold when T is large (actually, as $T \rightarrow \infty$). We use (5.ii) to draw inferences.

<u>Note</u>: If not sure about the applicability of the *asymptotic* distribution, use bootstrap to draw inferences.

Review: Fitted Values, Residuals & s^2 • OLS estimates β with **b**. Now, we define *fitted values* as: $\hat{y} = X \mathbf{b}$ Now we define the estimated error, e (also called *residuals*): $e = y - \hat{y}$ It can be shown that e is uncorrelated with $X: X'e = 0 \implies e \perp X$ • Using e, we define a measure of unexplained variation: Residual Sum of Squares (RSS) = $e'e = \sum_i e_i^2$ • We use RSS to calculate s^2 , the unbiased estimator of σ^2 : $s^2 = \text{RSS} / / (T - k) = \sum_i e_i^2 / (T - k)] = e'e/(T - k)$ • Then, the estimator of $\text{Var}[\mathbf{b} | \mathbf{X}] = s^2 (\mathbf{X'X})^{-1}$



how much of total variation of y is explained by the regression (SSR).

Review: Goodness of Fit – R^2 & Adjusted R^2

• Main problem with R^2 : R^2 never falls when regressors (say z) are added to the regression. This occurs because RSS decreases with more information.

<u>Solution</u>: Incorporate a penalty for number of parameters in \mathbb{R}^2 . This is what *Adjusted*- \mathbb{R}^2 does:

$$\overline{R}^2 = 1 - \frac{s^2}{\text{TSS}/(T-1)} \qquad (s^2 = \text{RSS}/(T-k))$$

There is a trade-off in s^2 : higher k decreases numerator, RSS, but, it also decreases denominator, (T - k).

 \Rightarrow maximizing $\overline{R}^2 \ll \min[\operatorname{RSS}/(T-k)] = s^2$

We can use \overline{R}^2 to compare models. There are other popular goodness of fit measures with penalties for number of parameters: AIC & BIC.

Review: Testing Only One Parameter

• We are interested in testing a hypothesis about one parameter in our linear model: $y = X \beta + \varepsilon$

1. Set H_0 and H_1 (about only one parameter): H_0 : $\beta_k = \beta_k^0$ H_1 : $\beta_k \neq \beta_k^0$

2. Appropriate T(X): *t-statistic*:

$$t_k = \frac{b_k - \beta_k^0}{s_{b,k}} \sim t_{T-k}$$

- 3. Compute t_k , \hat{t} , using b_k , β_k^0 , s, and $(X'X)^{-1}$. Get *p*-value(\hat{t}).
- 4. <u>Rule</u>: Set an α level. If *p*-value($\hat{\mathbf{t}}$) < α \Rightarrow Reject \mathbf{H}_0 : $\beta_k = \beta_k^0$ Alternatively, if $|\hat{\mathbf{t}}| > t_{T-k,1-\alpha/2}$ \Rightarrow Reject \mathbf{H}_0 : $\beta_k = \beta_k^0$

Review: Testing Only One Parameter

• Special case: $H_0: \beta_k = 0$ $H_1: \beta_k \neq 0.$

Then,

$$t_k = \frac{b_k}{s_{b,k}} \sim t_{T-k}$$

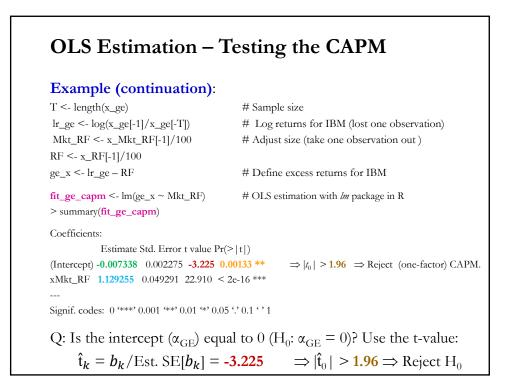
This special case of t_k is called the **t-value** or **t-ratio** (also "t-stats").

• Usually,
$$\alpha = 5\%$$
, when $T - k > 30$, then $t_{T-k,1-\alpha/2} = 1.96$
Rule for $\alpha = 5\%$: if $|\hat{t}_k| > 1.96 \approx 2$, test is "significant" ($\Rightarrow \beta_k \neq 0$).

Note: t-distribution is symmetric. Then,

 $|t_{T-k,\alpha/2}| = t_{T-k,1-\alpha/2}$

OLS Estimation – Testing the CAPM Example: We test the CAPM for GE. Recall that the CAPM states: $\mathbf{E}[r_{i=GE,t}-r_f] = \beta_{i=GE} \mathbf{E}[(r_{m,t}-r_f)].$ According to the CAPM, equilibrium excess returns are only determined by excess market returns -i.e., the CAPM is a one factor model. There is no constant or extra factors besides the market. A linear data generating process (DGP) consistent with the CAPM is: $(r_{GE,t} - r_f) = \alpha_{GE} + \beta_{GE} (r_{m,t} - r_f) + \varepsilon_{GE,t},$ t = 1, ..., TThus, we test the CAPM by testing H_0 (CAPM holds): $\alpha_{GE} = 0$ H₁ (CAPM rejected): $\alpha_{GE} \neq 0$. SFX_da <read.csv("http://www.bauer.uh.edu/rsusmel/4397/Stocks_FX_1973.csv",head=TRUE,sep=",") # Extract IBM price data x_ge <- SFX_da\$GE x_Mkt_RF <- SFX_da\$Mkt_RF # Extract Market excess returns (in %) x_RF <- SFX_da\$RF # Extract risk free rate (in %)



OLS Estimation – Testing the CAPMExample (continuation): $\Rightarrow |\hat{t}_{\alpha}| > 1.96$ \Rightarrow Reject H₀ (CAPM) at 5% levelConclusion: The CAPM is rejected for IBM at the 5% level.Note: You can also reject H₀ by looking at the *p-value* of intercept.Interpretation: Given that the intercept is significant (& negative). GE*underperformed* relative to what the CAPM expected:- GE excess returns: mean(ge_x) = -0.0009589826- GE excess returns (CAPM) = 1.129255 * mean(Mkt_RF)= 1.129255 * 0.0056489 = 0.006378998- Ex-post difference: -0.000959 - 0.006379 = -0.007338 ($\approx \alpha_{GE}$)

OLS Estimation – The 3-Factor F-F Model

• The CAPM is routinely rejected. A popular alternative is the empirically derived 3-Factor Fama-French Model (1993) with:

a) *Size* factor (SMB) measured as returns of small (size portfolio) minus returns of big (size portfolio)

b) *Value* factor or book-to-market factor (HML), measured as returns of high (B/M portfolio) minus returns of low (B/M portfolio).

• Then, a linear DGP generating this model is:

$$(r_{i,t}-r_f) = \alpha + \beta_1 (r_{m,t}-r_f) + \beta_2 SMB_t + \beta_3 HML_t + \varepsilon_t.$$

• Under this model, the main drivers of expected returns are sensitivity to the market, sensitivity to size, and sensitivity to value stocks, as measured by the book-to-market ratio.

OLS Estimation – The 3-Factor F-F Model

• The 3-factor FF model produces expected excess returns: $E[r_{i,t} - r_f] = \beta_1 E[r_{m,t} - r_f] + \beta_2 E[SMB_t] + \beta_3 E[HML_t]$

A significant constant would be evidence against this model: something is missing in the model.

In 2014, Fama & French added two more factors: RMW & CMA.
RMW measures the return of the portfolio of most profitable firms ("robust") minus the portfolio least profitable ("weak").
CMA measures the return of a portfolio of firms that invest conservatively minus a portfolio of firms that invest aggressively.

• Again, the 5-factor FF model produces expected excess returns: $E[r_{i,t} - r_f] = \beta_1 E[r_{m,t} - r_f] + \beta_2 E[SMB_t] + \beta_3 E[HML_t] + \beta_4 E[RMW_t] + \beta_5 E[CMA_t]$

Review: Is GE's Beta equal to 1?

Example: For the 3-Factor Fama-French Model for GE returns we want to test if the 3 F-F factors are significant. The model:

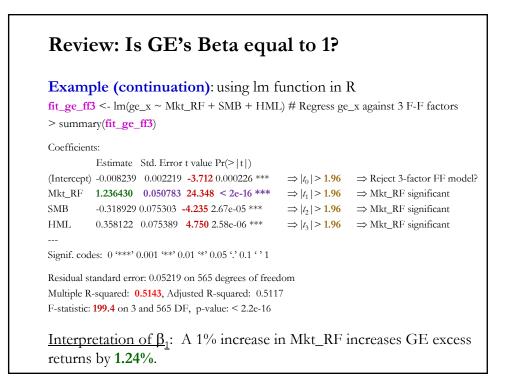
 $(r_{GE,t} - r_f) = \alpha + \beta_1 (r_{m,t} - r_f) + \beta_2 SMB_t + \beta_3 HML_t + \varepsilon_t.$

Before testing H_0 : $\beta_1 = 1$, we check the adequacy of the model:

- Check R² and interpret it
- Goodness of Fit test and interpret it
- Signs of coefficients and interpret them.

Then, we test

$$H_0: \beta_1 = 1$$
$$H_1: \beta_1 \neq 1.$$



Review: Is GE's Beta equal to 1?

Example (continuation): using lm function in R

Interpretation of R^2 : The 3 F-F factors explain 51% of the variability of GE returns.

Interpretation of F-test (Goodness of Fit Test):

F-statistic: 199.4 on 3 and 565 DF, p-value: < 2.2e-16

 \Rightarrow Very low *p-value*. That is, strong rejection of H_0 : (No joint significance of 3 F-F factors).

The t-stats point out that the 3 F-F factors are significant drivers of GE excess returns.

<u>Interpretation of constant</u> (α_{GE}): The significant constant signals that something is missing from the model. It constant, α_{GE} , is also negative: GE underperformed relative to the 3-factor F-F model.

Review: Is GE's Beta equal to 1? Example (continuation): • Q: Is GE's market beta (β_1) equal to 1? That is, $H_0: \beta_1 = 1$ vs. $H_1: \beta_1 \neq 1$ $\Rightarrow \hat{t}_1 = \frac{b_1 - \beta_1^o}{s_{b,k}} = \frac{1.28643 - 1}{0.050783} = 4.655733$ Decision Rule: $|\hat{t}_1 = 4.6557| > 1.96 \Rightarrow \text{Reject } H_0: \beta_1 = 1 \text{ at } 5\% \text{ level.}$ Conclusion: GE systematic market risk is greater than the market. Note: \hat{t}_1 can be calculated using summary(βt_ge)\$coef, which gets the whole Im matrix. > $t_b_1 < (\text{summary(fit_ge})$coef[2,1] - 1//\text{summary(fit_ge})$coef[2,2]}$ > t_b_1

Review: Is GE's Beta equal to 1? Example (continuation): • $(1 - \alpha/2)$ % CI for GE's market beta (β_k) : $[b_k + t_{T-k,\alpha/2} * \text{Est SE}(b_k), b_k + t_{T-k,1-\alpha/2} * \text{Est SE}(b_k)]$ For $\alpha = 5$ %: $\Rightarrow [1.28643 - 1.96 * 0.050783, 1.28643 + 1.96 * 0.050783] = \beta_1 \in [1.186895, 1.385965]$ with 95% confidence Clearly, $\beta_1 = 1$ is outside the range \Rightarrow GE is riskier than the market.

Review: General Linear Hypothesis – \mathbf{H}_{0} : $\mathbf{R}\boldsymbol{\beta} = \mathbf{q}$ • Suppose we are interested in testing *J* joint hypotheses. **Example:** We want to test that in the 3 FF factor model that the SMB and HML factors have the same coefficients, $\beta_{SMB} = \beta_{HML} = \beta^{0}$. We can write linear restrictions as \mathbf{H}_{0} : $\mathbf{R}\boldsymbol{\beta} - \mathbf{q} = \mathbf{0}$, where \mathbf{R} is a *J*x*k* matrix and \mathbf{q} a *J*x1 vector. In the above example (*J*=2), we write: $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} \beta_{1} \\ \beta_{MKt} \\ \beta_{SMB} \\ \beta_{HML} \end{bmatrix} = \begin{bmatrix} \beta^{0} \\ \beta^{0} \end{bmatrix}$ **Review: General Linear Hypothesis** – \mathbf{H}_0 : $\mathbf{R}\beta = \mathbf{q}$ • Q: Is $\mathbf{R}\mathbf{b} - \mathbf{q}$ close to 0? Two different approaches to this questions. **Approach (1).** Wald test. We base the answer on the discrepancy vector: $\mathbf{m} = \mathbf{R}\mathbf{b} - \mathbf{q}$. Then, we construct a Wald statistic: $\mathcal{W} = \mathbf{m}' (\operatorname{Var}[\mathbf{m} | \mathbf{X}])^{-1} \mathbf{m}$ to test if \mathbf{m} is different from 0. $\mathcal{W}^* = (\mathbf{R}\mathbf{b} - \mathbf{q})' {\mathbf{R}[\mathbf{s}^2(\mathbf{X}'\mathbf{X})^{-1}]\mathbf{R}\}^{-1}(\mathbf{R}\mathbf{b} - \mathbf{q})$ - If (A5) is assumed: $\mathbf{F} = \mathcal{W}^*/\mathcal{J} \sim F_{\mathcal{J},\mathcal{T}-\mathcal{K}}$. - If (A5) is not assumed, results are only asymptotic: $\mathcal{J}^* F \xrightarrow{d} \chi_{\mathcal{J}}^2$

Review: Wald Test Statistic for H₀: $\mathbf{R\beta} - \mathbf{q} = \mathbf{0}$ **Example:** In the 3 FF factor model for GE (*T*=571), we test: $H_0: \beta_{Mkt} = 1, \beta_{SMB} = -0.1 \text{ and } \beta_{HML} = 0.3.$ $H_1: \beta_{Mkt} \neq 1 \text{ and/or } \beta_{SMB} \neq -0.1 \text{ and/or } \beta_{HML} \neq 0.3. \Rightarrow J = 3$ library(car) linearHypothesis(fit_ge_ff3, c("Mkt_RF = 1","SMB = -0.1", "HML= 0.3"), test="F") # exact test Hypothesis: Mkt_RF = 1 SMB = -0.1 HML = 0.3 Model 1: restricted model Model 2: ge_x ~ Mkt_RF + SMB + HML Res.Df RSS Df Sum of Sq F Pr(>F) 1 568 1.6067 2 565 1.5389 3 0.067761 8.2927 2.094e-05 ***

Review: General Linear Hypothesis – H_0 : $R\beta = q$

• Q: Is **Rb** – **q** close to **0**?

Approach (2). F test.

We base the answer on a model loss of fit when restrictions are imposed: RSS must increase (or R^2 must go down).

Steps:

1. Estimate Restricted Model, get RSS_R

2. Estimate Unrestricted Model, get RSS_U

$$F = \frac{\frac{RSS_R - RSS_U}{(k_U - k_R)}}{\frac{RSS_U}{(T - k_U)}} \sim F_{J,T-k}. \quad (\text{where } J = k_U - k_R)$$

• The F-test constructed using a variable that can divide the data into 2 categories to compute $RSS_R \& RSS_U$ is usually referred as *Chow test*.

Review: F Test - Are SMB and HML Priced?

Example: We want to test if the additional FF factors (SMB, HML) are significant for GE (T=570).

Unrestricted Model:

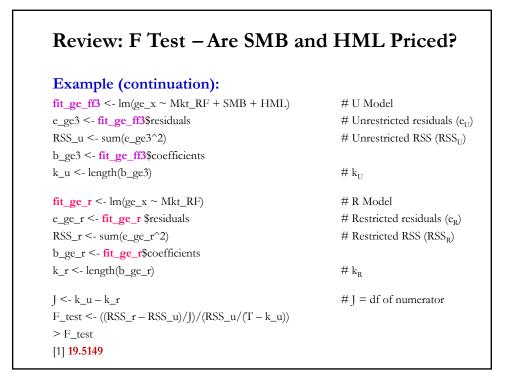
(U)
$$(r_{GE,t} - r_f) = \alpha + \beta_1 (r_{m,t} - r_f) + \beta_2 SMB_t + \beta_3 HML_t + \varepsilon_t$$

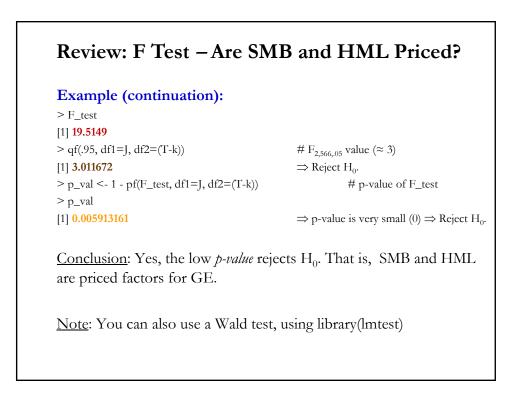
Hypothesis: $H_0: \beta_2 = \beta_3 = 0$ $H_1: \beta_2 \neq 0$ and/or $\beta_3 \neq 0$

Then, the Restricted Model:

(R)
$$(r_{GE,t} - r_f) = \alpha + \beta_1 (r_{m,t} - r_f) + \varepsilon_t$$

Test:
$$F = \frac{(RSS_R - RSS_U)/J}{RSS_U/(T - k_u)} \sim F_{J,T-k}, \qquad J = (k_U - k_R) = 4 - 2 = 2$$





Review: F Test – Are SMB and HML Priced?

Example (continuation):

>library(Intest)
> waldtest(fit_ge_ff3, fit_ge_r)
Wald test
Model 1: ge_x ~ Mkt_RF + SMB + HML
Model 2: ge_x ~ Mkt_RF
Res.Df Df F Pr(>F)
1 566
2 568 -2 19.5149 0.005913161 ***
--Signif. codes: 0 **** 0.001 *** 0.01 ** 0.05 *. 0.1 * 1