EMH & the RW

Bonus Material

Efficient Markets Hypothesis (EMH)

• Q: Can past information be used to build profitable trading rules in financial markets? In particular, can past return realizations tell us anything about expected future returns? Very old questions.

• The *efficient markets hypothesis* (EMH) is a first attempt to address the predictability issue.

• Earliest known version:

"When shares become publicly known in an open market, the value which they acquire there may be regarded as the judgement of the best intelligence concerning them."

- George Gibson, *The Stock Exchanges of London, Paris and New York*, G. P. Putnman & Sons, New York, 1889.

• In 1900, Louis Bachelier, a French PhD student at the time, was the first to propose the "*Random Walk* Model" for security prices.

Efficient Markets Hypothesis (EMH)

- Samuelson (1965)

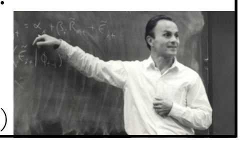
"In an informationally efficient market, price changes must be unforecastable."

- Fama (1970)

"A market in which prices always *fully reflect* available information is '*efficient*'."

If we have new information (a new earnings announcement) prices will adjust immediately (or very fast). Prices (significantly) jump with relevant information. But, they have to jump a proper amount, not too much (over-reaction) or not too little (under-reaction).

Eugene Fama (USA, 1939)



Efficient Markets Hypothesis (EMH)

- Grossman and Stiglitz (1980)

"There must be sufficient profit opportunities --i.e. inefficiencies, frictions-- to compensate investors for the cost of trading and information-gathering. "

Then, under a frictionless world, it is impossible to have efficient prices (& EM). Only when all information gathering & trading costs are zero we can expect prices to fully reflect all available information.

<u>Conundrum</u>: But, if prices reflect fully and instantly all available information, who is going to gather information?

Efficient Markets Hypothesis (EMH)

- Malkiel (1992)

"The market is said to be efficient with respect to some information set… implies that it is impossible to make *economic profits* by trading on the basis of [the information in that set]."

The first sentence of Malkiel's definition expands Fama's definition and suggests a test for efficiency useful in a laboratory.

The second sentence suggests a way to judge efficiency that can be used in empirical work. This is what is usually done in the finance literature.

Example: If Fund managers *outperform* the market consistently, then prices are not efficient with respect to their information set.

Many examples of "*inefficiencies*" with respect to some information sets.

Efficient Markets Hypothesis (EMH)

- The behavioral finance field has found that investors often show predictable and financially ruinous behavior ("irrational"?). Different causes: overreaction, overconfidence, loss aversion, herding, psychological accounting, miscalibration of probabilities, regret, etc.

Examples: Momentum strategies (buying past winners and selling past losers, under-reaction?) and Contrarian strategies (buying past losers and selling past winners, over-reaction?) achieve abnormal returns.

- Lo (2004)

"… much of what behavioralists cite as counterexamples to economic rationality […] are, in fact, consistent with an evolutionary model of individuals adapting to a changing environment."

There is a time dimension. It takes time to adapt to new circumstances.

EMH: Versions

- Efficiency can only be defined with reference to a specific type of information. Fama (1970) defined three classes of information sets:
  - (a) Historical sequence of prices. This set gives **Weak form EMH**.
  - (b) Public records of companies and public forecasts regarding the future performance and possible actions. Sets (a) & (b) create the **Semi-strong form EMH**.
  - (c) Private or inside information. Sets (a), (b) & (c) deliver the **Strong form EMH**.

- Violations:
  - Technical traders devising profitable strategies (weak EMH)
  - Reading a newspaper and devising a profitable trading strategy (semi-strong EMH)
  - Corporate insiders making profitable trades (strong EMH)

EMH: Versions

- Q: Can markets really be strong-form efficient? Very unlikely, plenty of examples of successful trading with private information: Jeffrey Skilling (Enron), Ivan Boesky/Michael Milken (junk bonds), Eugene Plotkin and David Pajcin (from Goldman Sachs, trading on M&A inside information), James McDermott Jr (Keefe, Bruytee & Woods, passed M&A tips to his mistress), Raj Rajaratnam (Galleon Group).

- Perfectly rational factors may account for violations of EMH:
  - Microstructure issues and trading costs.
  - Rewarding investors for bearing certain dynamic risks.
  - Time-varying expected returns due to changing conditions can generate predictability.

EMH: Joint Tests

• We are talking about economic profits, adjusting for risk and costs. Thus, a model for risk adjustment is needed. Results will be conditional on the underlying asset pricing model.

• Fama (1991) remarks that tests of efficiency are *joint tests* of efficiency and some asset pricing model, or benchmark.

Example: Many benchmarks assume constant "normal" returns. This is easier to implement, but may not be correct. Thus, rejections of efficiency could be due to rejections of the benchmark.

• Most tests suggest that if the security return (beyond the mean) cannot be forecasted, then market efficiency is not rejected.

Example: A wrong asset pricing model may reject efficiency. It would be easy to find (demeaned) returns to be forecastable if we use the wrong mean.

EMH: Expectations and Information Set

• The conditional expectation of the stochastic process X_{t+1}, conditioned on information set I_t, can be written as:

$$E[X_{t+1} | I_t] = E_t[X_{t+1}]$$

• Information set, I_t: It describes what we know at time t. The usual assumption is that we do not forget anything. Over time, the information set increases: I_t is contained in I_{t+1}; I_{t+1} is contained in I_{t+2}, etc. That is, we have a sequence $I_0 \subseteq I_1 \subseteq I_2 \ldots \subseteq I_t$. In stochastic processes this sequence is called a "*filtration,*" with notation $\{\mathscr{F}_t\}$.

Technical note: We say a stochastic process $\{X_t\}$ is *adapted* to a filtration $\{\mathscr{F}_t\}$ if X_t is *measurable* $\mathscr{F}_t$ for all t.

Measurable? The event of interest is in $\mathscr{F}_t$.

EMH: Random Prices

- Efficient market: A market where prices are random with respect to an information set ("*filtration*"), I_t.

- Let the price of a security at time t be given by the expectation of some "*fundamental value,*" V*, conditional on I_t:

$$P_t = E[V^* | I_t] = E_t[V^*]$$

- The same equation holds one period ahead so that:

$$P_{t+1} = E[V^* | I_{t+1}] = E_{t+1}[V^*]$$

- The expectation of the price change over the next period is:

$$E_t[P_{t+1} - P_t] = E_t[E_{t+1}[V^*] - E_t[V^*]] = 0$$

since I_t is contained in I_{t+1} ⇒ $E_t[E_{t+1}[V^*]] = E_t[V^*]$ (by Law of IE).

Remark: Under efficiency, financial asset prices are unpredictable.

EMH: Martingale & Fair Games – Definitions

Martingale: A stochastic process P_t is a martingale if:

$$E[P_{t+1} | \Omega_t] = P_t \qquad (\text{or } E_t[P_{t+1}] = P_t)$$

where the information set is Ω_t (what we know at time t, includes P_t).

Submartingale: If $E[P_{t+1} | \Omega_t] \geq P_t$. -$P_t$: Lower bound for $E_t[P_{t+1}]$

Supermartingale: If $E[P_{t+1} | \Omega_t] \leq P_t$. -$P_t$: Upper bound for $E_t[P_{t+1}]$

Fair game model: A stochastic process r_t is a fair game if:

$$E[r_{t+1} | \Omega_t] = 0$$

⇒ if P_t is a martingale or pure random walk, $(P_{t+1} - P_t)$ is a fair game.

Note: Only referring to expected values!

EMH: Martingale & Fair Games – Definitions

• The Martingale process can be setup as a special case of an AR(1) process:

$$p_t = \mu + \phi\, p_{t-1} + \varepsilon_t$$

with $\phi = 1$, $\mu = 0$, & $E_t[\varepsilon_{t+1}] = 0$. A non-stationary process.

Technical detail: Martingale condition is neither a necessary nor a sufficient condition for *rational expectations* models of asset prices (LeRoy (1973), Lucas (1978)).

According to Lucas (1978), in markets where all investors have rational expectations, prices do fully reflect all available information and *marginal-utility weighted* prices follow martingales.

• But, we consider the martingale as an important starting point.

The Random Walk Hypothesis

Definition: Random Walk (RW)

A stochastic process p_t is a RW if:

$$p_t = \mu + p_{t-1} + \varepsilon_t \quad \text{-where } p_t = \ln(P_t)$$

$$\Rightarrow r_t = \mu + \varepsilon_t = \Delta\, p_t$$

Assumptions about ε_t: Uncorrelated with past information, with constant mean (=0) & variance (σ^2). That is,

$$\varepsilon_t \sim D\,(0, \sigma^2),$$

with $E_t[\varepsilon_{t+1}] = 0$, $E_t[\varepsilon^2_{t+1}] = \sigma^2$

If $\mu \neq 0$, the process is called a RW *with a drift*.

• A RW with no drift is a martingale with structure for the error term, ε_t, uncorrelated, zero mean and constant variance.

The Random Walk Hypothesis

• We start testing the EMH by assuming log returns, r_t, follow a RW with a drift. We called this "Random Walk Model":

$$\Rightarrow r_t = \Delta p_t = \mu + \varepsilon_t = \Delta p_t$$

where $\varepsilon_t \sim D(0, \sigma^2)$.

• Different specifications for ε_t produce different testable hypothesis for the EMH-RW Model:

- **RW1**: ε_t is *independent and identically distributed* (*i.i.d.*) ~ $D(0, \sigma^2)$. Not realistic. (Old tests: Cowles and Jones (1937)).

- **RW2**: ε_t is *independent* (allows for heteroskedasticity). Test using filter rules, technical analysis. (Alexander (1961, 1964), Fama (1965)).

- **RW3**: ε_t is *uncorrelated* (allows for dependence in higher moments). Test using autocorrelations, variance ratios, long horizon regressions.

The RW Hypothesis: Autocorrelations & ACF

• Assume r_t is covariance stationary and ergodic. Then

$\gamma_k = \text{cov}(r_t, r_{t-k})$ - Auto-covariance between times t & $t-k$
$\rho_k = \gamma_k / \gamma_0$. - $\text{Var}[r_t] = \gamma_0$

are not time dependent. We estimate both statistics with $\hat{\gamma}_k$ and $\hat{\rho}_k$.

• Under **RW1** Hypothesis (and some assumptions)

$$\sqrt{T}\,\hat{\rho}_k \xrightarrow{a} N(0, 1)$$

$$\Rightarrow \qquad \text{SE}[\hat{\rho}_k] = 1/\sqrt{T}$$

<u>Technical Note</u>: The sample correlation coefficients, $\hat{\rho}_k$, are negatively biased in finite samples. See Fuller (1976).

• To check autocorrelations up to order k, we use the ACF for r_t. Confidence Intervals can be easily approximated by $\pm\, 2/\sqrt{T}$.

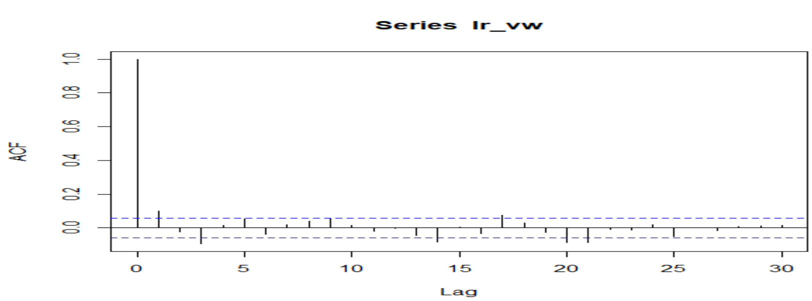
The RW Hypothesis: ACF – Monthly Data

Example: ACF with k = **24 lags** for the **monthly** Equal- and Value-weighted (EW & VW, respectively) CRSP index returns from 1926:Jan – 2022:March (T = 1,155):

```
EMH_da <-
read.csv("http://www.bauer.uh.edu/rsusmel/4397/crsp_ew_vw_m.csv",head=TRUE,sep=",")
lr_vw <- EMH_da$vwretd                # Value weighted CRSP returns (including distributions)
lr_ew <- EMH_da$ewretd                # Equal weighted CRSP returns (including distributions)
T <- length(lr_vw)
SE_rho <- 1/sqrt(T)                   # Asymptotic SE for rho's: |rho| > 2 * SE => significant
> SE_rho
[1] 0.02942449                        # |rho| > 2 * SE => significant
acf_y <- acf(lr_vw)
> acf_y
Autocorrelations of series 'lr_vw', by lag
     0      1      2      3      4      5      6      7      8      9     10     11     12
 1.000 -0.011  0.044 -0.183  0.140 -0.001  0.002 -0.010  0.121 -0.024 -0.003 -0.045 -0.002
    13     14     15     16     17     18     19     20     21     22     23     24
 0.045  0.009 -0.004  0.007  0.010  0.015 -0.010 -0.004 -0.005  0.051 -0.009 -0.015
```

The RW Hypothesis: ACF – Monthly Data

Example (continuation):



Conclusion: There are a few significant autocorrelations (3rd, 4th, and 8th), all smaller than 0.2 in absolute value.

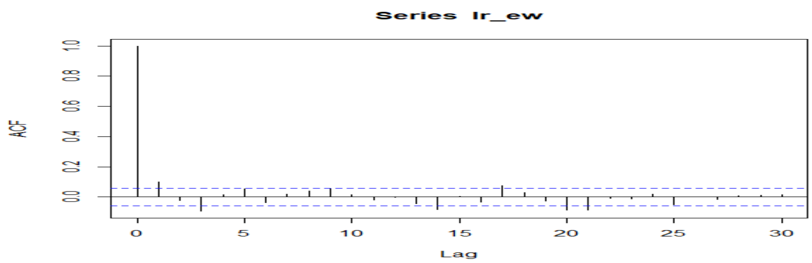
The RW Hypothesis: ACF – Monthly Data

Example (continuation):

```
acf_y <- acf(lr_ew)
> acf_y
Autocorrelations of series 'lr_ew', by lag
```

0	1	2	3	4	5	6	7	8	9	10	11	12
1.000	**0.101**	-0.023	-0.094	0.014	0.056	-0.038	0.018	0.041	0.056	0.014	-0.020	-0.001

13	14	15	16	17	18	19	20	21	22	23	24
-0.044	-0.082	0.004	-0.035	0.074	0.030	-0.028	-0.087	-0.088	-0.007	-0.012	0.019



Conclusion: Again, a few significant autocorrelations, but small in size.

The RW Hypothesis: ACF – Daily Data

Example: ACF with $k = 24$ **lags** for the **daily** Equal- and Value-weighted (EW & VW, respectively) CRSP index returns from 1926:Jan 1 – 2022:March 30 (T = 23,359):

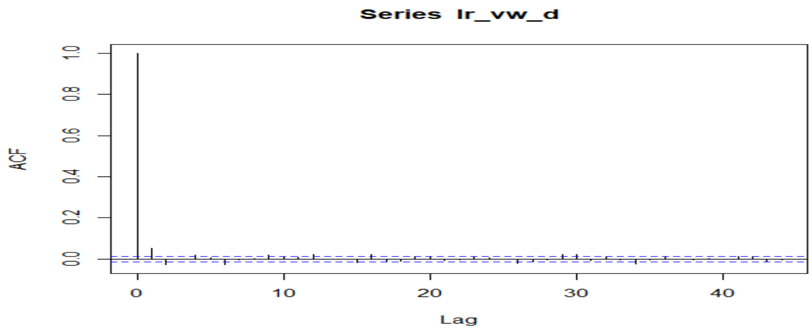
```
EMH_d_da <-
read.csv("http://www.bauer.uh.edu/rsusmel/4397/crsp_ew_vw_d.csv",head=TRUE,sep=",")
lr_vw_d <- EMH_d_da$vwretd          # VW CRSP returns (including distributions)
lr_ew_d <- EMH_d_da$ewretd          # EW CRSP returns (including distributions)
T <- length(lr_ew_d)
SE_rho <- 1/sqrt(T)                 # Asymptotic SE for rho's: |rho| > 2 * SE => significant
> SE_rho
[1] 0.006279628                     # |rho| > 2 * SE => significant
acf_y <- acf(lr_vw_d)
> acf_y
Autocorrelations of series 'lr_vw_d', by lag
```

0	1	2	3	4	5	6	7	8	9	10	11	12
1.000	**0.053**	-0.027	0.001	0.018	0.007	-0.027	-0.003	0.002	0.019	0.014	0.010	0.023

13	14	15	16	17	18	19	20	21	22	23	24
-0.001	-0.001	-0.017	0.024	-0.010	-0.011	0.013	0.014	-0.008	-0.003	0.011	0.006

The RW Hypothesis: ACF – Monthly Data

Example (continuation):



Conclusion: There are many significant autocorrelations, with the exception of the first one, all very small.

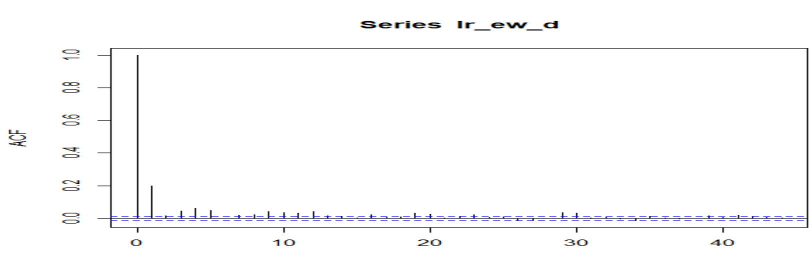
The RW Hypothesis: ACF – Daily Data

Example (continuation):

acf_y <- acf(lr_ew)
> acf_y
Autocorrelations of series 'lr_ew', by lag

0	1	2	3	4	5	6	7	8	9	10	11	12
1.000	**0.198**	0.016	0.046	0.061	0.049	0.000	0.018	0.023	0.042	0.034	0.033	0.040

13	14	15	16	17	18	19	20	21	22	23	24
0.015	0.014	0.004	0.021	0.007	0.008	0.033	0.025	0.003	0.009	0.023	0.007



Conclusion: Lots of significant autocorrelations, but, in general, small.

The RW Hypothesis: Autocorrelation Joint Tests

• We already know two tests to check for zero autocorrelation in a time series: Box-Pierce Q and Ljung-Box tests. We usually rely on the Ljung-Box (1978), LB, test, since it has better small sample properties.

-The Q & LB statistics test a joint hypothesis that the first p autocorrelations are zero: H_0: $\rho_1 = ... = \rho_p = 0$

Under **RW1** and using the asymptotic distribution of $\hat{\rho}_k$:

$$Q = T \sum_{k=1}^{p} \hat{\rho}_k^2 \xrightarrow{d} \chi_p^2.$$

$$LB = T * (T-2) * \sum_{k=1}^{p} \frac{\hat{\rho}_k^2}{T-k} \xrightarrow{d} \chi_p^2.$$

The RW Hypothesis: Autocorrelation Joint Tests

• Q & LB tests are widely use, but they have two main limitations:

(1) The test was developed under the independence (**RW1**) assumption.

If y_t shows dependence, such as heteroscedasticity, the asymptotic variance of $\sqrt{T}\,\hat{\boldsymbol{\rho}}$ is no longer **I**, but a non-diagonal matrix.

There are several proposals to "*robustify*" both Q & LB tests, see Diebold (1986), Robinson (1991), Lobato et al. (2001). The "robustified" Portmanteau statistic uses $\tilde{\rho}_k$ instead of ρ_k:

$$\tilde{\rho}_k = \frac{\hat{\gamma}_k}{\tau_k} = \frac{\sum_{t=k+1}^{T}(y_t - \bar{y})(y_{t-k} - \bar{y})}{\sum_{t=k+1}^{T}(y_t - \bar{y})^2\,(y_{t-k} - \bar{y})^2}$$

Thus, for Q we have:

$$Q^* = T \sum_{k=1}^{p} \tilde{\rho}_k^2 \xrightarrow{d} \chi_p^2.$$

24

The RW Hypothesis: Autocorrelation Joint Tests

(2) The selection of the number of autocorrelations p is arbitrary.

The traditional approach is to try different p values, say 3, 6 & 12. Another popular approach is to let the data "select" p, for example, using AIC or BIC, an approach sometimes referred as "*automatic selection*."

Escanciano and Lobato (2009) propose combining BIC's and AIC's penalties to select p in Q* (BIC for small k and AIC for bigger k).

- It is common to reach different conclusion from Q and Q*.

25

Testing for Autocorrelation: Monthly Evidence

Example: Q and LB tests with $\boldsymbol{p}$ = **3** & **12 lags** for the **monthly** EW & VW CRSP index returns from 1926:Jan – 2022:March (T = 1155):

- Q test for **monthly VW**

```
> Box.test(lr_vw, lag = 4, type="Box-Pierce")

        Box-Pierce test

data:  lr_vw
X-squared = 22.812, df = 4, p-value = 0.000138

> Box.test(lr_vw, lag = 12, type="Box-Pierce")

        Box-Pierce test

data:  lr_vw
X-squared = 34.696, df = 12, p-value = 0.0005234
```

26

Testing for Autocorrelation: Monthly Evidence

Example (continuation):

- LB tests for **monthly VW**

```
> Box.test(lr_vw, lag = 4, type="Ljung-Box")

        Box-Ljung test

data:  lr_vw
X-squared = 22.891, df = 4, p-value = 0.0001332

> Box.test(lr_vw, lag = 12, type="Ljung-Box")

        Box-Ljung test

data:  lr_vw
X-squared = 34.87, df = 12, p-value =  0.0004912
```

27

Testing for Autocorrelation: Monthly Evidence

Example (continuation): Q* tests with automatic lag selection. In R, the package *vrtest* has the Auto.Q function that computes this test. As always, you need to install *vrtest* first.

- Q* test for **monthly VW**

```
> Auto.Q(lr_vw, 12)
$Stat
[1] 3.059582

$Pvalue
[1] 0.08026232
```

Conclusion: Once we take into consideration potential heteroscedasticity in y_t, there is weak evidence for autocorrelation in monthly Value-weighted CRSP index returns.

28

Testing for Autocorrelation: Monthly Evidence

Example (continuation):

- Q test for **monthly EW**

```
> Box.test(lr_ew, lag = 4, type="Box-Pierce")

        Box-Pierce test

data:  lr_ew
X-squared = 61.607, df = 4, p-value = 1.333e-12

> Box.test(lr_ew, lag = 12, type="Box-Pierce")
X-squared = 83.328, df = 12, p-value = 9.531e-13
```

- LB tests for monthly EW

```
> Box.test(lr_ew, lag = 4, type="Ljung-Box")
X-squared = 61.793, df = 4, p-value = .218e-12

> Box.test(lr_ew, lag = 12, type="Ljung-Box")
X-squared = 83.719, df = 12, p-value =  8.02e-13
```

29

Testing for Autocorrelation: Monthly Evidence

Example (continuation): Q* tests with automatic lag selection.

- Q* test for **monthly EW**

```
library(vrtest)

> Auto.Q(lr_ew, 12)
$Stat
[1] 6.487553

$Pvalue
[1] 0.01086324
```

Conclusion: Strong evidence for autocorrelation in monthly EW CRSP returns (the evidence was weaker, once we take into consideration potential heteroscedasticity in y_t, for monthly VW CRSP returns). That is, we reject the RW hypothesis for monthly EW CRSP returns.

30

Testing for Autocorrelation: Daily Evidence

Example: Q and LB tests with ***p*** = **5** & **20 lags** for the **daily** Equal- and Value-weighted (EW & VW, respectively) CRSP index returns from 1926: Jan 1 – 2022 :March 30 (T = 25,359):

```
EMH_d_da <- read.csv("http://www.bauer.uh.edu/rsusmel/4397/crsp_ew_vw_d.csv", head=TRUE,sep=",")

lr_vw_d <- EMH_d_da$vwretd        # Value weighted CRSP returns (with distributions)
lr_ew_d <- EMH_d_da$ewretd        # Equal weighted CRSP returns (with distributions)
T <- length(lr_ew_d)
```

• Q tests for **daily VW**

```
> Box.test(lr_vw_d, lag = 5, type="Box-Pierce")
data:  lr_vw_d
X-squared = 100.64, df = 5, p-value = 2.2e-16

> Box.test(lr_vw_d, lag = 20, type="Box-Pierce")
data:  lr_vw_d
X-squared = 184.68, df = 20, p-value < 2.2e-16
```

31

Testing for Autocorrelation: Daily Evidence

Example (continuation):

• Q* test for daily VW (continuation)

```
> Auto.Q(y, 20)                 # Q* test automatic selection of p
$Stat
[1] 11.73454

$Pvalue
[1] 0.0006135076
```

• Q tests for **daily EW**

```
> Box.test(lr_ew_d, lag = 5, type="Box-Pierce")

data:  lr_ew_d
X-squared = 1213.3, df = 5, p-value = 2.2e-16

> Box.test(lr_ew_d, lag = 20, type="Ljung-Box")

data:  lr_ew_d
X-squared = 1445.4, df = 20, p-value = 2.2e-16
```

32

Testing for Autocorrelation: Daily Evidence

Example (continuation):

- Q* test for daily EW (continuation)

```
> Auto.Q(y, 40)                    # Q* test automatic selection of p
$Stat
[1] 235.7106

$Pvalue
[1] 0
```

Conclusion: Strong evidence for autocorrelation in daily VW & EW CRSP returns. That is, we reject the uncorrelated returns hypothesis as implied by the RW hypothesis for daily VW & EW CRSP returns.

33

The RW Hypothesis: Variance Ratio (VR) Tests

- Intuition: For all 3 RW hypotheses, the variance of RW increments is linear in the time interval. If the interval is twice as long, the variance must be twice as big. That is, the variance of monthly data should be 4 times bigger than the variance of weekly data. (Recall the log approximation rules for *i.i.d.* returns.)

- If r_t is a covariance stationary process (constant first two moment, and covariance independent of time), then for the variance ratio of 2-period versus 1-period returns, VR(2):

$$\text{VR}(2) = \frac{\text{Var}[r_t(2)]}{2*\text{Var}[r_t]} = \frac{\text{Var}[r_t + r_{t+1}]}{2*\text{Var}[r_t]} =$$

$$= \frac{\text{Var}[r_t] + \text{Var}[r_{t+1}] + 2\,\text{Cov}[r_t, r_{t+1}]}{2*\text{Var}[r_t]} = \frac{2\,\sigma^2 + 2\,\gamma_1}{2\sigma^2} = 1 + \rho_1$$

where $r_t(2) = r_t + r_{t+1}$

The RW Hypothesis: VR Tests

- $\text{VR}(2) = \frac{\text{Va}[r_t(2)]}{2*\text{Var}[r_t]} = 1 + \rho_1.$

- Three cases:

  $\rho_1 = 0 \Rightarrow \text{VR}(2) = 1$ (True under **RW1**, random walk)

  $\rho_1 > 0 \Rightarrow \text{VR}(2) > 1$ (mean aversion)

  $\rho_1 < 0 \Rightarrow \text{VR}(2) < 1$ (mean reversion)

- The intuition generalizes to longer horizons:

  $$\text{VR}(q) = \frac{\text{Var}[r_t(q)]}{q * \text{Var}[r_t]} = 1 + 2 * \sum_{k=1}^{q-1}(1 - \frac{k}{q})\, \rho_k.$$

The VR(q) is a particular linear combination of the 1st (q - 1) autocorrelation coefficients (with linearly declining weights).

The RW Hypothesis: VR Tests – Distribution

- Under **RW1**, we have H_0: VR(q) = 1.

  H_1: VR(q) $\neq$ 1.

Technical Note: Under **RW2** and **RW3**, VR(q) = 1 provided

$$1/T \sum_{t=0}^{T} \text{Var}[r_t] \rightarrow \bar{\sigma}^2 > 0$$

we need this assumption, since some "fat-tailed" distributions do not have a well-defined second moment.

- To do any testing we need the sampling distribution of the VRs (estimated variance ratios) under H_0: VR(q) = 1. We use the statistic:

$$\frac{\sqrt{Tq}}{\sqrt{2*(q-1)}}(\widehat{\text{VR}}(q) - 1) \xrightarrow{a} N(\mathbf{0}, 1)$$

This is Cochrane's (1988) VR test. The test rejects H_0 –i.e., the RWH – if the above statistic is greater in absolute value than **1.96**.

The RW Hypothesis: VR Tests – Computations

• For the special case of $q = 2$, we use

$$\sqrt{T}\,(\widehat{\text{VR}}(2) - 1) \xrightarrow{a} \text{N}(\mathbf{0}, 1)$$

• $\text{Var}[r_t(q)]$ is computed using the **MLE formulation**, that is, dividing by T, not by $(T-1)$ (or T minus degrees of freedom).

Example: We have monthly data from Jan 1973. Then, we compute

$$\text{Var}[r_t] = \frac{\sum_{t=1}^{T}(r_t - \bar{\boldsymbol{r}})^2}{T}$$

$$\text{Var}[r_t(2)] = \frac{\sum_{t=1}^{T}(r_t(2) - 2 * \bar{\boldsymbol{r}})^2}{T}.$$

Note: Since the tests are asymptotic tests, in this case, relying on the Normal distribution, dividing by T or by $(T-k)$ does not make any difference.

The RW Hypothesis: VR Tests – Computations

• $\text{Var}[r_t(q)]$ is computed using **non-overlapping returns**.

Example: We compute **non-overlapping bi-monthly returns**, using monthly data from Jan 1973.

(1) monthly returns: r_t is computed as usual. For the first return:

$$r_{t=Jan\ 73} = \ln(P_{t=Jan\ 31,73}) - \ln(P_{t=Jan\ 1,73})$$

(2) bi-monthly returns. The first three $r_t(2)$ are computed as:

$$r_{t=Feb\ 73}(2) = r_{t=Feb\ 73} + r_{t=Jan\ 73}$$
$$r_{t=Apr\ 73}(2) = \text{r}_{t=Apr\ 73} + r_{t=Mar\ 73}$$
$$r_{t=June\ 73}(2) = \text{r}_{t=June\ 73} + r_{t=May\ 73}$$

Note: We have "clean data," with no introduced serial correlation. But, we lose observations. If we have 1,000 monthly returns, using non-overlapping bi-monthly returns we end up with only 500 observations.

The RWH: VR Test Monthly Evidence (VW)

Example: We check the RW Hypothesis, under RW3, for the monthly CRSP EW and VW Index returns. In R, the package *vrtest* has functions to compute the above mentioned VR tests.

- VR tests for **monthly VW**

```
library(vrtest)
kvec <- c(2,3,12)                              #Vector with different q
y <- lr_vw
> vr_1 <- VR.minus.1(y, kvec)                  # Stat should be close to 0 if RW
> vr_1
$VR.auto                                       # VR with Automatic ("optimal) q selection
[1] 0.1954746

$Holding.Periods
[1]  2  3 12

$VR.kvec                                       (VR – 1) stat for each q=kvec[i]
[1] 0.1007011 0.1187365 0.1212423

> sqrt(T*kvec)/sqrt(2*(kvec-1))*vr_1$VR.kvec   # VR test for each q=kvec[i] ~ N(0,1)
[1] 3.422358 3.494666 3.043158
```

The RWH: VR Test Monthly Evidence (EW)

Example (continuation):

- VR tests for **monthly EW**

```
> y <- lr_ew
> vr_1 <- VR.minus.1(y, kvec)                  # Stat should be close to 0 if RW
> vr_1
$VR.auto                                       # VR with Automatic ("optimal) q selection
[1] 0.1954746

$Holding.Periods
[1]  2  3 12

$VR.kvec                                       (VR – 1) stat for each q=kvec[i]
[1] 0.2043236 0.2789327 0.2180176

> sqrt(T*kvec)/sqrt(2*(kvec-1))*vr_1$VR.kvec   # VR test for each q=kvec[i] ~ N(0,1)
[1] 6.943998 8.209583 5.472199
```

Conclusion: Using the VR test (with $q = 2, 3, 12$), we reject the RW Hypothesis $\Rightarrow$ tests are greater in absolute value than **1.96**.

The RWH: VR Tests – Issues

• Several issues has been raised regarding the VR's tests. The main issues are:

(1) Choice of *q*. In the previous examples, we have arbitrarily selected *q*. Similar to the situation with the Q and LB tests, there are suggestions to automatically (or "optimally," according to some loss function) select *q*. Choi (1999) is one example of this approach, (the *vrtest* R package uses this approach in the *Auto.VR* test).

(2) Poor asymptotic approximation. In simulations, it is found that the asymptotic Normal distribution is a poor approximation to the small-sample distribution of the VR statistic. The usual solution is to use a bootstrap (Kim's (2009) bootstrap gives the p-value of the automatic VR test in the *Auto.VR* function).

The RWH: VR Test Monthly Evidence (VW)

Example: We use VR tests with automatic selection and a bootstrap to check the RW Hypothesis for the monthly CRSP EW and VW Index returns. Again, we use AutoBoot.test function in R package *vrtest*.

• Automatic VR tests for **monthly VW**

```
y <- lr_vw
> AutoBoot.test(y, nboot=1000, wild="Normal", prob=c(0.025,0.975))        # Choi (1999)
$test.stat                    (Automatic variance ratio test statistic as in Choi (1999))
[1] 2.509324

$VRsum                        (1+ weighted sum of autocorrelation up to the optimal order)
[1] 1.195475

$pval
[1] 0.064

$CI.stat
     2.5%     97.5%
-2.836631  2.612363

$CI.VRsum
     2.5%     97.5%
0.8323731 1.1927214
```

The RWH: VR Test Monthly Evidence (EW)

Example (continuation):.

- Automatic VR tests for **monthly EW**

```
y <- lr_ew
> AutoBoot.test(y, nboot=1000, wild="Normal", prob=c(0.025,0.975))        # Choi (1999)
$test.stat                    (Automatic variance ratio test statistic as in Choi (1999))
[1] 4.173898
$VRsum                        (1+ weighted sum of autocorrelation up to the optimal order)
[1] 1.382554
$pval
[1] 0.021
$CI.stat
     2.5%     97.5%
-3.262026  3.359002
$CI.VRsum
     2.5%     97.5%
0.7687769 1.2610106
```

Conclusion: Using the Automatic VR test and a bootstrap, we have strong evidence against the RW Hypothesis for EW, but weak for VW.

The RWH: VR Tests – LM's Modifications

- Lo & MacKinlay (LM, 1988, 1989) propose modifications to the test:

- Allow for **overlapping returns**, and, thus, using more observations. But, overlapping returns will be autocorrelated, even if underlying process is not. We need to adjust for this feature.

- Use unbiased estimators of variances –i.e., divide by $(T - df)$.

$$M_1(q) = \frac{\sqrt{3*T*q}}{\sqrt{2*(2q-1)*(q-1)}}(\overline{VR}(q) - 1) \xrightarrow{a} N(\mathbf{0}, 1),$$

where $\overline{VR}(q)$ is the VR statistic computed using overlapping returns.

- Allow for possible **heteroscedasticity** of returns (more realistic)

$$M_2(q) = \frac{(\overline{VR}(q) - 1)}{\sqrt{\phi(q)}} \xrightarrow{a} N(\mathbf{0}, 1),$$

where

$$\phi(q) = \sum_{j=1}^{q}\left[\frac{2(q-j)}{q}\right]^2 * \left\{\frac{\sum_{t=j+1}^{T}(r_t - \bar{r})^2(r_{t-j} - \bar{r})^2}{[\sum_{t=1}^{T}(r_t - \bar{r})^2]^2}\right\}.$$

The RWH: LM Tests Monthly Evidence

Example: We check the RW Hypothesis, under RW3, for the monthly CRSP EW and VW Index returns using the LM's tests: M1 and M2. Again, we use the R package *vrtest*.

- Automatic VR tests for **monthly VW**

```
library(vrtest)
kvec <- c(2,3,12)                    #Vector with different q
y <- lr_vw
> Lo.Mac(y, kvec)                    # LM's tests M1 & M2 ~ asymptotic N(0,1)
$Stats
        M1        M2
k=2  3.422358 1.7485059
k=3  2.706957 1.4241521
k=12 1.099060 0.6373211
```

Conclusion: We reject H_0 (RW Model) using M1 for $q = 2, 3$; but, once we allow for heteroscedasticity (M2 tests), we cannot reject H_0.

The RWH: LM Tests Monthly Evidence

Example (continuation):

- Automatic VR tests for **monthly EW**

```
y <- lr_ew
> Lo.Mac(y, kvec)                    # LM's tests M1 & M2 ~ asymptotic N(0,1)
$Stats
        M1        M2
k=2  6.943998 2.5480302
k=3  6.359116 2.5009114
k=12 1.976326 0.9975538
```

Conclusion: Strong rejection of RW using M1, especially for $q = 2, 3$; but, using M2 test with $q = 12$, we cannot reject the RW Hypothesis.

Consistent with previous result, stronger evidence for EW returns than for VW returns.

The RWH: VR & LM Tests – Issues ($\xrightarrow{a}$ N(0,1))

• Several issues has been raised regarding the LM's tests:

(1) Poor asymptotic approximation. The asymptotic standard normal distribution provides a poor approximation to the small-sample distribution of the VR statistic. LM's tests tend to be biased and right-skewed, in finite samples.

• Proposed solutions:
- **Alternative asymptotic distributions**, as in Richardson and Stock (1989) or Chen and Deo (2006).

- **Bootstrapping**, as in Kim (2006) or Malliaropulos and Priestley (1999).

The RWH: VR & LM Tests – Issues (Joint Tests)

(2) Joint tests. The LM's tests are individual tests, where H_0 is tested for a specific value of q. But, under H_0, VR(q) = 1, for all q. LM's tests ignore the joint nature of testing for the RW Hypothesis.

• Proposed solutions:
- RS statistic, a Wald Test, as proposed by Richardson and Smith (1993):

$$RS(q) = T\,(\mathbf{VR} - \iota)'\,\Phi^{-1}\,T\,(\mathbf{VR} - \iota) \xrightarrow{d} \chi^2_q.$$

where **VR** is the (q×1) vector of q sample variance ratios, ι is the (q×1) unit vector, and Φ is the covariance matrix of **VR**.

- QP statistic, a Wald Test based on a "power transformed" VR statistic, as proposed by Chen and Deo (2006). QP asymptotically follows a χ^2_q distribution. This test is a one-sided test (H_1: VR(q) < 1 for all q.)

The RWH: VR & LM Tests – Issues (Joint Tests)

- Proposed solutions (continuation):

- CD statistic, a join test, as proposed by Chow and Denning (1993):

$$\text{CD} = \sqrt{T}\max_{1\leq i\leq m} |M_2(q_i)|$$

which follows a complex distribution, the studentized maximum modulus [SMM] distribution with *m* and *T* degrees of freedom (*m* is the number of k values). This SMM distribution is tabulated in Hahn and Hendrickson (1971) and Stoline and Ury (1979).

In general, we use the simulated critical values obtained by simulations as done by Chow and Denning themselves or a bootstrap as in Kim (2006).

The RWH: VR & LM Tests – Monthly Evidence

Example: We check the monthly LM test results using a bootstrap instead of the asymptotic distribution. We use the *Boot.test* function in the R package *vrtest*, which provides two bootstrapped p-values: one for the LM statistic and the other one for the CD statistic.

- VR tests for **monthly VW**

```
> y <- lr_vw
> Lo.Mac(y, kvec)                # LM's tests M1 & M2
$Stats
          M1        M2
k=2  3.422358 1.7485059
k=3  2.706957 1.4241521
k=12 1.099060 0.6373211

> Boot.test(y, kvec, nboot=1000, wild="Normal", prob=c(0.025,0.975))       #Kim's Bootstrap
$Holding.Period
[1]  2  3 12

$LM.pval                         (Bootstrap p-values for the Lo-MacKinlay M2 tests)
[1] 0.067 0.157 0.503
```

The RWH: VR & LM Tests – Monthly Evidence

Example (continuation):

```
> Lo.Mac(y, kvec)                # LM's tests M1 & M2
$Stats
         M1        M2
k=2  3.422358 1.7485059
k=3  2.706957 1.4241521
k=12 1.099060 0.6373211

> Boot.test(y, kvec, nboot=1000, wild="Normal", prob=c(0.025,0.975))        #Kim's Bootstrap
$Holding.Period
[1]  2  3 12

$LM.pval                         (Bootstrap p-values for the Lo-MacKinlay M2 tests)
[1] 0.067 0.157 0.503

$CD.pval                         (Bootstrap p-value for the Chow-Denning test)
[1] 0.153

$CI                              (C.I. for Lo-Mackinlay M2 tests from Bootstrap distribution)
        2.5%    97.5%
k=2  -1.825961 1.827630
k=3  -1.847447 1.855263
k=12 -1.712367 2.152280
```

The RWH: VR & LM Tests – Monthly Evidence

Example (continuation):

```
> Wald(y, kvec)                              # RS Wald test
$Holding.Period
[1]  2  3 12

$Wald.stat
[1] 12.42735

$Critical.Values_10_5_1_percent
[1]  6.251389  7.814728 11.344867

> Chen.Deo(y, kvec)                          # QP Wald test
$Holding.Period
[1]  2  3 12

$VRsum
[1] 0.07335402

$QPn
[1,] 3.154226

$ChiSQ.Quantiles_1_2_5_10_20_percent
[1] 11.344867  9.837409  7.814728  6.251389  4.641628
```

The RWH: VR & LM Tests – Monthly Evidence

Example (continuation):

- VR tests for **monthly EW**

```
> y <- lr_ew
> Lo.Mac(y, kvec)                # LM's tests M1 & M2
$Stats
        M1        M2
k=2  6.943998 2.5480302
k=3  6.359116 2.5009114
k=12 1.976326 0.9975538

> Boot.test(y, kvec, nboot=1000, wild="Normal", prob=c(0.025,0.975))        #Kim's Bootstrap
$Holding.Period
[1] 5 20 60

$LM.pval                         (Bootstrap p-values for the Lo-MacKinlay M2 tests)
[1] 0.001 0.004 0.279

$CD.pval                         (Bootstrap p-value for the Chow-Denning test)
[1] 0.017
```

The RW Hypothesis: VR Test Monthly Evidence

Example (continuation):

```
$LM.pval                         (Bootstrap p-values for the Lo-MacKinlay M2 tests)
[1] 0.001 0.004 0.279

$CD.pval                         (Bootstrap p-value for the Chow-Denning test)
[1] 0.017

$CI                              (C.I. for Lo-Mackinlay M2 tests from Bootstrap distribution)
        2.5%     97.5%
k=2  -1.754012 1.708415
k=3  -1.710910 1.816157
k=12 1.563058 2.092434

> Wald(y, kvec)                  # RS Wald test
$Holding.Period
[1]  2  3 12

$Wald.stat
[1] 52.68679

$Critical.Values_10_5_1_percent
[1]  6.251389  7.814728 11.344867
```

The RW Hypothesis: VR Test Monthly Evidence

Example (continuation):

```
> Chen.Deo(y, kvec)                    # QP Wald test
$Holding.Period
[1]  2 3 12

$VRsum
[1] 0.1442001

$QPn
[1,] 6.524497

$ChiSQ.Quantiles_1_2_5_10_20_percent
[1] 11.344867  9.837409  7.814728  6.251389  4.641628
```

Conclusion: Consistent with previous result, solid evidence (only the Wald test rejects H_0) for the RW for VW returns, but weak evidence for EW returns.

The RW Hypothesis: VR Test Daily Evidence

Example: We check the RW Hypothesis, under RW3, for the daily CRSP EW and VW Index returns.

- VR tests for **daily VW**

```
kvec <- c(5, 20, 60)                 #Vector with different q
y <- lr_vw
vr_1 <- VR.minus.1(y, kvec)          # Stat should be close to 0 if RW
> vr_1
$VR.auto                  (value of VR-1 with automatic selection of holding vectors)
[1] 0.08049192

$Holding.Periods
[1]  5  20 60

$VR.kvec                  (the values of VR-1 for the chosen holding periods)
[1] 0.06015875  0.11155693  0.16958754

> sqrt(T*kvec)/sqrt(2*(kvec-1))*vr_1$VR.kvec  # VR test for each q=kvec[i] (~ N(0,1) dist)
[1] 1.616329 2.750494 4.109789
```

The RW Hypothesis: VR Test Daily Evidence

Example (continuation):

```
> AutoBoot.test(y, nboot=300, wild="Normal", prob=c(0.025,0.975))          # Choi (1999)
$test.stat
[1] 4.354851

$VRsum
[1] 1.080492

$pval
[1] 0.02333333

$CI.stat
     2.5%     97.5%
-3.423189  4.067023

$CI.VRsum
     2.5%     97.5%
0.9483973 1.0656480
```

The RW Hypothesis: VR Test Daily Evidence

Example (continuation):

```
> Lo.Mac(y, kvec)                 # LM's tests M1 & M2
$Stats
          M1        M2
k=5  4.372645 1.757401
k=20 3.574490 1.573525
k=60 3.057608 1.536068

> Boot.test(y, kvec, nboot=1000, wild="Normal", prob=c(0.025,0.975))        #Kim's Bootstrap
$Holding.Period
[1]  2  3 12

$LM.pval                          (Bootstrap p-values for the Lo-MacKinlay M2 tests)
[1] 0.06333333 0.08000000 0.07333333

$CD.pval                          (Bootstrap p-value for the Chow-Denning test)
[1] 0.11333

$CI                               (C.I. for Lo-Mackinlay M2 tests from Bootstrap distribution)
          2.5%     97.5%
k=5  -1.602225 2.333427
k=20 -1.594718 1.935643
k=60 -1.748524 1.782090
```

The RW Hypothesis: VR Test Daily Evidence

Example (continuation):

```
> Wald(y, kvec)                                # RS Wald test
$Holding.Period
[1]  5 20 60

$Wald.stat
[1] 21.19834

$Critical.Values_10_5_1_percent
[1]  6.251389  7.814728 11.344867

> Chen.Deo(y, kvec)                            # QP Wald test

$VRsum
[1] 0.05863072

$QPn
        [,1]
[1,] 3.639522

$ChiSQ.Quantiles_1_2_5_10_20_percent
[1] 11.344867  9.837409  7.814728  6.251389  4.641628
```

The RW Hypothesis: Overall Evidence

- Tests results are based on CRSP value-weighted (VW) and equal weighted (EW) indices from **1925** & individual securities from **1962**.
- Daily, weekly and monthly returns from VW and EW indices show significant (positive) autocorrelation.
- VR(q) > 1 statistics reject RW3 for EW index but not VW index. Market capitalization or size may be playing a role. Rejection of RW stronger for smaller firms. Their returns more serially correlated.
- For individual securities, VR(q) < 1, suggesting small and negative correlations (and not significant).
- VR tests in other countries and financial markets. Tests also tend to reject the RWH, with stronger rejections for smaller markets and less liquid markets.

The RW Hypothesis: Implications

- The rejection of the RWH does not necessarily imply a violation of the EMH.

- Main implication: Theoretical pricing models should be able to explain the pattern of serial correlation.

- Side Question: How can portfolios show VR(q) > 1 when individual securities show VR(q) < 1?