Lecture 9-d
Time Series: Exponential Smoothing & Evaluation of Forecasts

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Review: ARIMA Models – Box-Jenkins

- How do we select *p, q,* and *d* for an ARIMA model?

- Box-Jenkins Approach

1) Make sure data is stationary –check a time plot. If not, differentiate.

2) Using ACF & PACF, guess small values for *p* & *q*.
If order choice not clear, use AIC, AIC Corrected (AICc), BIC, or HQC (Hannan and Quinn (1979)).

3) Estimate order *p, q*. (ML, MM, OLS for AR; Innovation algorithm for MA, Hannan-Rissanen algorithm for ARMA)

4) Run diagnostic tests on residuals (Check ACF, LB tests).
⇒ Are they white noise? If not, add lags (*p* or *q*, or both).

- Value parsimony. When in doubt, keep it simple (KISS).

- Looks simple, but there are a lot of nuances to the process.

Review: ARIMA Models – Box-Jenkins

• With non-stationary series, we talked about trends:
  - Deterministic vs Stochastic

When faced with non-stationarity, we remove pattern, either by detrending (deterministic trends) or differencing (stochastic trends).

• Similar situation arises when we have seasonal patterns:
  - Deterministic vs Stochastic

Again, the approach is to remove the seasonal pattern. Once removed we follow Box-Jenkins to select an ARIMA model.

• Then, we forecast.
MA(q) forecasts become mean forecasts after q periods
AR(p) are very correlated forecasts, using past p forecasts.
ARMA(p,q) are a combination of both; after q periods, AR dominates.

Forecasting From Simple Models: ES

• Industrial companies, with a lot of inputs and outputs, want quick and inexpensive forecasts. Easy to fully automate. In general, we use past Y_t to forecast future Y_t's, usually referred as the *level's forecasts*.

• Exponential Smoothing Models (ES) fulfill these requirements.

• In general, these models are limited and not optimal, especially compared with Box-Jenkins methods.

• Goal of these models: Suppress the short-run fluctuation by smoothing the series. For this purpose, a weighted average of all previous values works well.

• There are many ES models. We will go over the Simple Exponential Smoothing (SES) & Holt-Winter's Exponential Smoothing (HW ES).

Simple Exponential Smoothing: SES

- We "*smooth*" the series Y_t to produce a quick forecast, S_{t+1} also called *level's forecast*. Smooth? The graph of S_t is less jagged than the graph of original series Y_t.

- We use the observed time series at time t: $Y_1, Y_2, \ldots, Y_t$.

- The equation for the level is: $S_t = \alpha Y_{t-1} + (1-\alpha) S_{t-1}$

where

- α: The smoothing parameter, $0 \leq \alpha \leq 1$.
- Y_t: Value of the observation at time t.
- S_t: Value of the smoothed observation at time t –i.e., the forecast.

- The equation can also be written as an *updating equation*:

$$S_t = S_{t-1} + \alpha (Y_{t-1} - S_{t-1}) = S_{t-1} + \alpha * (\text{past forecast error})$$

SES: Forecast and Updating

- From the updating equation for S_t:

$$S_t = S_{t-1} + \alpha (Y_{t-1} - S_{t-1})$$

we compute the forecast:

$$S_{t+1} = \alpha Y_t + (1-\alpha) S_t = S_t + \alpha (Y_t - S_t)$$

That is, a simple updating forecast: last period forecast + adjustment.

For the next period, we have:

$$S_{t+2} = \alpha Y_{t+1} + (1-\alpha) S_{t+1} = \alpha S_{t+1} + (1-\alpha) S_{t+1} = S_{t+1}$$

Then, the ℓ-step ahead forecast is:

$S_{t+\ell} = S_{t+1}$ ⇒ A naive forecast!

Note: SES forecasts are not very interesting after $\ell > 1$.

SES: Forecast and Updating

Example: An industrial firm uses SES to forecast sales:

$$S_{t+1} = S_t + \alpha * (Y_t - S_t)$$

The firm estimates α = **0.25**. The firm observes Y_t = **5** and, last period's forecast, S_t = **3**.

Then, the forecast for time *t*+1 is:

$$S_{t+1} = \mathbf{3} + \mathbf{0.25} * (\mathbf{5} - \mathbf{3}) = \mathbf{3.50}$$

The forecast for time *t*+1, and any period after time *t*+1, is:

$$S_{t+\ell} = S_{t+1} = \mathbf{3.50} \qquad \text{for } \ell > 1.$$

Later, the firm observes: Y_{t+1} = **4.77**, Y_{t+2} = **3.15**, & Y_{t+3} = **1.85**. Then, the MSE:

$$\text{MSE} = \frac{1}{3} * [(\mathbf{4.77} - \mathbf{3.50})^2 + (\mathbf{3.15} - \mathbf{3.50})^2 + (\mathbf{1.85} - \mathbf{3.50})^2] = \mathbf{1.486}.$$

SES: Forecast and Updating

Example (continuation):

Note: If α = **0.75**, then

$$S_{t+1} = \mathbf{3} + \mathbf{0.75} * (\mathbf{5} - \mathbf{3}) = \mathbf{4.50}$$

A bigger α gives more weight to the more recent observation –i.e., Y_t.

Again, the forecast for time *t*+1 and any period after time *t*+1 is:

$$S_{t+\ell} = S_{t+1} = \mathbf{4.50} \qquad \text{for } \ell > 1.$$

SES: Exponential?

- Q: Why Exponential?

For the observed time series $\{Y_1, Y_2, \ldots, Y_T, Y_{T+1}\}$, using backward substitution, $S_{t+1} = \hat{Y}_t(1)$ can be expressed as a weighted sum of previous observations:

$$S_{t+1} = \alpha Y_t + (1-\alpha) S_t = \alpha Y_t + (1-\alpha)[\alpha Y_{t-1} + (1-\alpha) S_{t-1}]$$
$$= \alpha Y_t + \alpha(1-\alpha) Y_{t-1} + (1-\alpha)^2 S_{t-1}$$
$$\Rightarrow \hat{Y}_t(1) = S_{t+1} = c_0 Y_t + c_1 Y_{t-1} + c_2 Y_{t-2} + \cdots$$

where c_i's are the weights, with

$$c_i = \alpha(1-\alpha)^i; \; i = 0, 1, \quad \ldots; \; 0 \leq \alpha \leq 1.$$

- We have decreasing weights, by a constant ratio for every unit increase in lag.

- Then, $\hat{Y}_t(1) = \alpha(1-\alpha)^0 Y_t + \alpha(1-\alpha)^1 Y_{t-1} + \alpha(1-\alpha)^2 Y_{t-2} + \cdots$

$$\hat{Y}_t(1) = \alpha Y_t + (1-\alpha)\hat{Y}_{t-1}(1) \quad \Rightarrow S_{t+1} = \alpha Y_t + S_t$$

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SES: Exponential Weights

- $c_i = \alpha\,(1-\alpha)^i; \qquad i = 0, 1, \ \ldots; \ 0 \leq \alpha \leq 1.$

$c_i = \alpha(1-\alpha)^i$	$\alpha = 0.25$	$\alpha = 0.75$
c_0	0.25	0.75
c_1	0.25 * 0.75 = 0.1875	0.75 * 0.25 = 0.1875
c_2	.25 * 0.75^2 = 0.140625	0.75 * 0.25^2 = 0.046875
c_3	.25 * 0.75^3 = 0.1054688	0.75 * 0.25^3 = 0.01171875
c_4	.25 * 0.75^4 = 0.07910156	0.75 * 0.25^4 = 0.002929688
⋮		
c_{12}	.25 * 0.75^{12} = 0.007919088	0.75 * 0.25^{12} = 4.470348e-08

Decaying weights. Faster decay with greater α, associated with faster learning: we give more weight to more recent observations.

- We do not know α; we need to estimate it.

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SES: Selecting α

- For $S_{t+1} = \alpha Y_t + (1-\alpha)S_t$, choose α between 0 and 1.

- If α = 1, it becomes a naive model; if α ≈ 1, more weights are put on recent values. The model fully utilizes forecast errors.

- If α is close to 0, distant values are given weights comparable to recent values. Set α ≈ 0 when there are big random variations in Y_t.

- In empirical work, 0.05 ≤ α ≤ 0.3 are used (α ≈ 1 is used rarely).

- α is often selected as to minimize the MSE.

- Numerical Minimization Process:

- Take different α values ranging between 0 and 1.

- Calculate 1-step-ahead forecast errors for each α.

- Calculate MSE for each case.

Choose α which has the min MSE: $e_t = Y_t - S_t \Rightarrow \min \sum_{t=1}^{n} e_t^2 \Rightarrow \alpha$ 11

SES: Selecting α – MSE

$$S_{t+1} = \alpha Y_t + (1-\alpha)S_t$$

Time	Y_t	S_{t+1} (α=0.10)	$(Y_t - S_t)^2$
1	5	-	-
2	7	(0.1)5 +(0.9)5 = 5	4
3	6	(0.1)7 + (0.9)5 = 5.2	0.64
4	3	(0.1)6 + (0.9)5.2 = 5.28	5.1984
5	4	(0.1)3 + (0.9)5.28 = 5.052	1.107
		TOTAL	10.945

$$MSE = \frac{SSE}{n-1} = 2.74$$

- Calculate this for α = 0.2, 0.3,…, 0.9, 1 and compare the MSEs. Choose α with minimum MSE.

Note: $Y_{t=1}$ = 5 is set as the initial value for the recursive equation. 12

SES: Initial Values

- We have a recursive equation:

$$S_{t+1} = \alpha Y_t + (1 - \alpha) S_t$$

we need initial values, S_1 (or Y_0).

- Approaches:

– Set S_1 equal to Y_1. Then, $S_2 = Y_1$.

– Take the average of, say first 4 or 5 observations. Use this average as an initial value.

– Estimate S_1 (similar to the estimation of α.)

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SES: Forecasting Examples

Example 1: We want to forecast log changes in **U.S. monthly dividends** (T=1796) using SES. First, we estimate the model using the R function *HoltWinters*(), which has as a special case SES: set beta=FALSE, gamma=FALSE. We use estimation period T=1750.

```
mod1 <- HoltWinters(lr_d[1:1750], beta=FALSE, gamma=FALSE)
> mod1
Holt-Winters exponential smoothing without trend and without seasonal component.

Call:
HoltWinters(x = lr_d[1:1750], beta = FALSE, gamma = FALSE)

Smoothing parameters:
 alpha: 0.289268                          ⇒ Estimated α
 beta : FALSE
 gamma: FALSE

Coefficients:
        [,1]
a 0.004666795                             ⇒ Forecast
```

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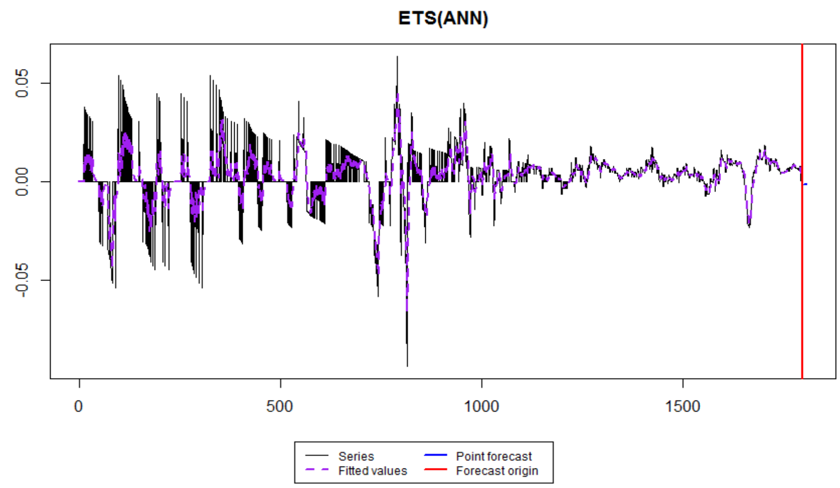
SES: Forecasting Examples

Example 1 (continuation):



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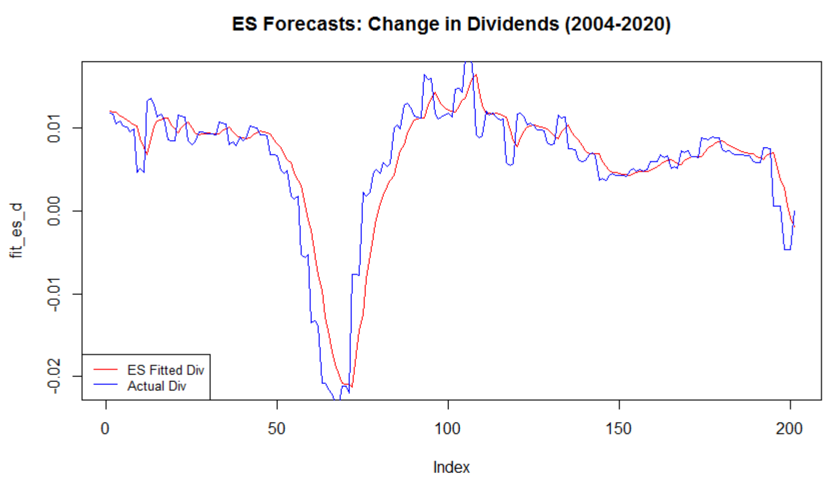
SES: Forecasting Examples

Example 1 (continuation):



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SES: Forecasting Examples

Example 1 (continuation): Now, we do one-step ahead forecasts

```
T_last <- nrow(mod1$fitted)                    # number of in-sample forecasts
h <- 25                                        # forecast horizon
ses_f <- matrix(0,h,1)                         # Vector to collect forecasts
alpha <- 0.29
y <- lr_d
T <- length(lr_d)
sm <- matrix(0,T,1)
T1 <- T – h + 1                                # Start of forecasts
a <- T1                                        # index for while loop
sm[a-1] <- mod1$fitted[T_last]                 # last in-sample forecast
while (a <= T) {
        sm[a] = alpha * y[a-1] +  (1-alpha) * sm[a-1]
a <- a + 1
}

ses_f <- sm[T1:T]
ses_f
f_error_ses <- sm[T1:T] - y[T1:T]              # forecast errors
MSE_ses <- sum(f_error_ses^2)/h                # MSE
plot(ses_f, type="l", main ="SES Forecasts: Changes in Dividends")
```

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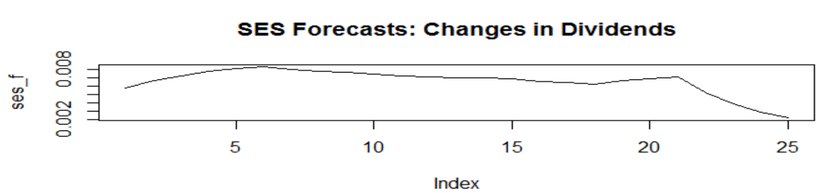
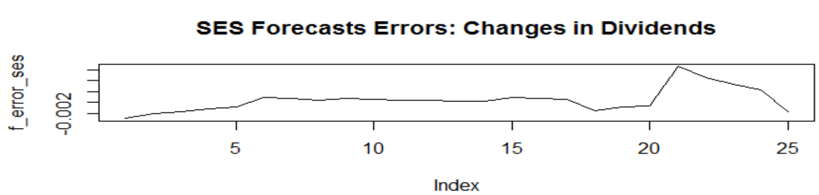
SES: Forecasting Examples

Example 1 (continuation):

```
> ses_f
f_error_ses <- sm[T1:T] - y[T1:T]
> plot(ses_f, type="l", main ="SES Forecasts: Changes in Dividends")
```





SES: Forecasting U.S. Dividends

Example 1 (continuation): *h-step-ahead* forecasts

```
> forecast(mod1, h=25, level=.95)
     Point Forecast        Lo 95       Hi 95
1751    0.004666795 -0.01739204 0.02672563
1752    0.004666795 -0.01829640 0.02762999
1753    0.004666795 -0.01916647 0.02850006
1754    0.004666795 -0.02000587 0.02933947
1755    0.004666795 -0.02081765 0.03015124
1756    0.004666795 -0.02160435 0.03093794
1757    0.004666795 -0.02236816 0.03170175
1758    0.004666795 -0.02311098 0.03244457
1759    0.004666795 -0.02383445 0.03316804
1760    0.004666795 -0.02454001 0.03387360
1761    0.004666795 -0.02522891 0.03456250
1762    0.004666795 -0.02590230 0.03523589
1763    0.004666795 -0.02656117 0.03589476
1764    0.004666795 -0.02720642 0.03654001
...
```

Note: Constant forecasts, but C.I. gets wider (as expected) with h.

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SES: Forecasting Examples

Example 2: We want to forecast **log monthly U.S. vehicles** (1976-2020, T=537) using SES.

```
mod_car <- HoltWinters(l_car[1:512], beta=FALSE, gamma=FALSE)
> mod_car
Holt-Winters exponential smoothing without trend and without seasonal component.

Call:
HoltWinters(x = l_car[1:512], beta = FALSE, gamma = FALSE)

Smoothing parameters:
 alpha: 0.4888382                          ⇒ Estimated α
 beta : FALSE
 gamma: FALSE

Coefficients:
      [,1]
a 7.315328
```

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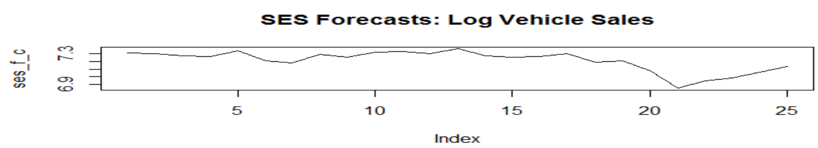
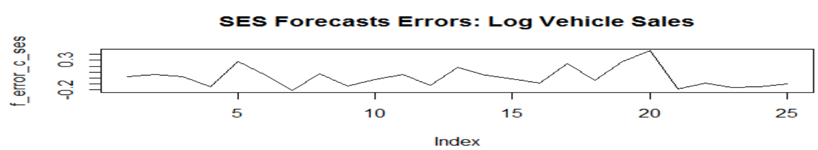
SES: Forecasting Examples

Example 2 (continuation): Now, we do one-step ahead forecasting

```
ses_f_c <- sm_c[T1:T]
f_error_c_ses <- sm_c[T1:T] - y[T1:T]
> plot(ses_f_c, type="l", main ="SES Forecasts: Log Vehicle Sales")
```




```
> plot(f_error_c_ses, type="l", main ="SES Forecasts Errors: Log Vehicle Sales")
```




```
MSE_ses <- sum(f_error_c_ses^2)/h
> MSE_ses
[1] 0.027889
```

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SES: Remarks

- Some computer programs automatically select the optimal α using a line search method or non-linear optimization techniques.

- We have a recursive equation, we need initial values for S_1.

- This model ignores trends or seasonalities. Not very realistic, especially for manufacturing facilities, retail sector, and warehouses.

- Deterministic components, D_t, can be easily incorporated.

- The model that incorporates both a trend and seasonal features is called *Holt-Winter's ES*.

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Holt-Winters (HW) Exponential Smoothing

• We introduce trend (T_t) & seasonality (I_t) factors. Since we also produce smooth forecasts for T_t & I_t, this method is also called *triple exponential smoothing*.

• The *h*-step ahead forecast is a combination of the smooth forecasts of S_t (Level), T_t (Trend) & I_{t+h-s} (Seasonal).

• Both, T_t & I_t, can be included as *additively* or *multiplicatively* factors. In this class, we consider an additive trend and the seasonal factor as additive or multiplicative. We produce *h*-step ahead forecasts:

• For the additive model: $\hat{Y}_t(h) = S_t + h\, T_t + I_{t+h-s}$

• For the multiplicative model: $\hat{Y}_t(h) = (S_t + h\, T_t) * I_{t+h-s}$

<u>Note</u>: Seasonal factor is multiplied in the *h*-step ahead forecast

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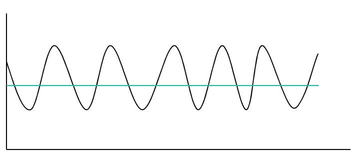
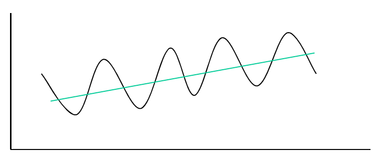
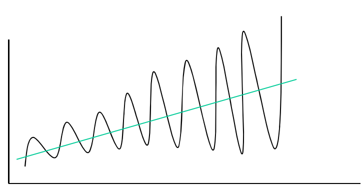
Holt-Winters (HW) ES: Trend & Seasonality



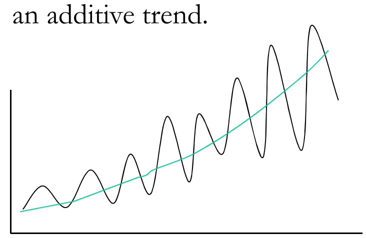
1. No trend and additive seasonal variability.



2. Additive seasonal variability with an additive trend.



3. Multiplicative seasonal variability with an additive trend.



4. Multiplicative seasonal variability with a multiplicative trend.

<u>Note</u>: We will use Model 2 (Additive) and Model 3 (Multiplicative).

Holt-Winters (HW) ES: Additive

• Additive model (additive trend and additive seasonality) forecast:

$$\hat{Y}_t(h) = S_t + h\,T_t + I_{t+h-s}$$

where s is the number of periods in seasonal cycles (=4 for quarters).

• Components:

- The level, S_t, is a weighted average of seasonal adjusted Y_t and the non-seasonal forecast (S_{t-1}+ T_{t-1}):

$$S_t = \alpha(Y_t - I_{t-s}) + (1-\alpha)(S_{t-1} + T_{t-1})$$

- The trend, T_t, is a weighted average of T_{t-1} and the change in S_t.

$$T_t = \beta(S_t - S_{t-1}) + (1-\beta)T_{t-1}$$

- The seasonality is also a weighted average of seasonal index of s last year, I_{t-s}, and the current seasonal index ($Y_{t-1} - S_{t-1} - T_{t-1}$):

$$I_t = \gamma(Y_t - S_{t-1} - T_{t-1}) + (1-\gamma)I_{t-s}$$

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Holt-Winters (HW) ES: Additive

• Then, the model for the *h*-step ahead forecast

$$\hat{Y}_t(h) = S_t + h\,T_t + I_{t+h-s}$$

has three equations:

$$S_t = \alpha(Y_t - I_{t-s}) + (1-\alpha)(S_{t-1} + T_{t-1})$$

$$T_t = \beta\,(S_t - S_{t-1}) + (1-\beta)\,T_{t-1}$$

$$I_t = \gamma(Y_t - S_{t-1} - T_{t-1}) + (1-\gamma)I_{t-s}$$

• We have only three smoothing parameters:

α = level coefficient

β = trend coefficient

γ = seasonality coefficient

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Holt-Winters (HW) ES: Multiplicative

• In the multiplicative seasonal case (with an additive trend), we have the *h*-step ahead forecast:

$$\hat{Y}_t(h) = (S_t + h\, T_t) * I_{t+h-s}$$

• Details for *multiplicative* seasonality –i.e., Y_t/I_t– and *additive* trend

- The forecast, S_t, now shows the average Y_t adjusted ($\frac{Y_t}{I_{t-s}}$).
- The trend, T_t, is a weighted average of T_{t-1} and the change in S_t.
- The seasonality is also a weighted average of I_{t-S} and the Y_t/S_t

• Then, the model has three equations:

$$S_t = \alpha \frac{Y_t}{I_{t-s}} + (1-\alpha)(S_{t-1} + T_{t-1})$$
$$T_t = \beta\,(S_t - S_{t-1}) + (1-\beta)\,T_{t-1}$$
$$I_t = \gamma \frac{Y_t}{S_t} + (1-\gamma)\,I_{t-s}$$

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Holt-Winters (HW) ES: Multiplicative

• We think of (Y_t/S_t) as capturing *seasonal effects*.

s = # of periods in the seasonal cycles

(s = 4, for quarterly data; s = 12, for monthly)

• Again, we have only three parameters:

α = smoothing parameter

β = trend coefficient

γ = seasonality coefficient

• Q: How do we determine these 3 parameters?

- Ad-hoc method: α, β and γ can be chosen as values between

$0.02 < \alpha, \gamma, \beta < 0.2$

- Optimal method: Minimization of the MSE, as in SES.

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Holt-Winters (HW) ES: Multiplicative

Example: An industrial firm uses HW ES to forecast sales next two quarters (h = 1, 2, & 3; with s = 4):

$$\hat{Y}_t(h) = \hat{Y}_{t+h} = (S_t + h\,T_t) * I_{t+h-s}$$

with S_t, T_t, & I_t factors given by:

$$S_t = \alpha \frac{Y_t}{I_{t-s}} + (1-\alpha)(S_{t-1} + T_{t-1})$$
$$T_t = \beta (S_t - S_{t-1}) + (1-\beta) T_{t-1}$$
$$I_t = \gamma \frac{Y_t}{S_t} + (1-\gamma) I_{t-s}$$

The firm estimates: α = 0.25; β = 0.1; & γ = 0.4. It observes Y_t = 5; last quarter's smoothed forecasts: S_{t-1} = 3, T_{t-1} = 1.2; & last year's seasonal factors: I_{t-4} = 1.1, I_{t-3} = 0.7, I_{t-2} = 1.2, & I_{t-3} = 0.8.

- Components forecasts:

$$S_t = 0.25 \frac{5}{1.1} + (1-0.25) * (3 + 1.2) = 4.2864$$

Holt-Winters (HW) ES: Multiplicative

Example (continuation):

$$S_t = 0.25 * \frac{5}{1.1} + (1-0.25) * (3 + 1.2) = 4.2864$$

$$T_t = 0.1 * (4.2864 - 3) + (1-0.1) * 1.2 = 1.2086$$

$$I_t = 0.4 * \frac{5}{4.2864} + (1-0.4) * 1.1 = 1.1266$$

The forecast for h = 1 (next quarter) is:

$$\hat{Y}_{t+1} = (4.2864 + 1.2086) * 0.7 = 4.8125$$

The forecast for h = 2 & 3 are:

$$\hat{Y}_{t+2} = (4.2864 + 2 * 1.2086) * 1.2 = 7.8475.$$
$$\hat{Y}_{t+3} = (4.2864 + 3 * 1.2086) * 0.8 = 6.1329.$$

HW ES: Initial Values

• Initial values for algorithm

- We need at least one complete season of data to determine the initial estimates of I_{t-s}.
- Initial values for *multiplicative* model:

$$S_0 = \sum_{t=1}^{s} Y_t \,/s$$

$$T_0 = \frac{1}{s}\left(\frac{Y_{s+1}-Y_1}{s} + \frac{Y_{s+2}-Y_2}{s} + \cdots + \frac{Y_{s+s}-Y_s}{s}\right)$$

$$\text{or } T_0 = \left[\left\{\sum_{t=1}^{s} Y_t/s\right\} - \left\{\sum_{t=s+1}^{2s} Y_t/s\right\}\right]/s$$

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HW ES: Initial Values

• Algorithm to compute initial values for seasonal component I_s.

Assume we have T observation and quarterly seasonality (s=4):

(1) Compute the averages of each of T years.

$$A_t = \sum_{i=1}^{4} Y_{t,i}/4\,, \quad t = 1, 2, \cdots, 6 \quad \text{(yearly averages)}$$

(2) Divide the observations by the appropriate yearly mean: $Y_{t,i}/A_t$.

(3) I_s is formed by computing the average $Y_{t,i}/A_t$ per year:

$$I_s = \sum_{i=1}^{T} Y_{t,s}/A_t \quad s = 1, 2, 3, 4$$

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HW ES: Damped Model

• We can damp the trend as the forecast horizon increases, using a parameter ϕ. For the multiplicative model we have:

$$S_t = \alpha \frac{Y_t}{I_{t-s}} + (1-\alpha)(S_{t-1} - \phi\, T_{t-1})$$
$$T_t = \beta(S_t - S_{t-1}) + (1-\beta)T_{t-1}$$
$$I_t = \gamma \frac{Y_t}{S_t} + (1-\gamma)I_{t-s}$$

• *h-step ahead* forecast:

$$\hat{Y}_t(h) = \{S_t + (1 + \phi + \phi^2 + \cdots + \phi^{2h-1})T_t\} * I_{t+h-s}$$

• This model is based on practice: It seems to work well for industrial outputs. Not a lot of theory or clear justification behind the damped trend.

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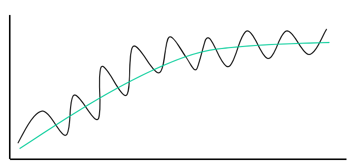
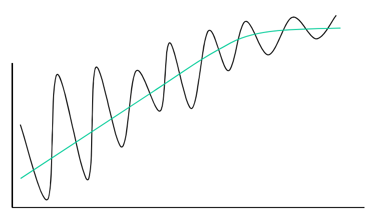
ES Models – Damped Model: Types



5. Dampened trend with additive seasonal variability.



6. Multiplicative seasonal variability and dampened trend.

• Overall, we have different models, incorporating different features:
- Trend: Additive or multiplicative, dampened or not
- Seasonal variability: Additive or multiplicative

• Q: With all these models, which one we should use? It depends on the data at hand.

HW ES: Example – Log U.S. Vehicles Sales

Example: We want to forecast log U.S. monthly vehicle sales with HW. We use the R function *HoltWinters*().

```
l_car_18 <- l_car[1:512]
l_car_ts <- ts(l_car_18, start = c(1976, 1), frequency = 12)     # convert lr_d in a ts object
hw_d_car <- HoltWinters(l_car_18, seasonal="additive")
> hw_d_car
Holt-Winters exponential smoothing with trend and additive seasonal component.

Call:
HoltWinters(x = lr_d_ts, seasonal = "additive")

Smoothing parameters:
 alpha: 0.4355244                ⇒ Estimated smoothing parameter
 beta : 0.009373815              ⇒ Estimated trend parameter ≈ 0 (no trend)
 gamma:0.3446495                 ⇒ Estimated seasonal parameter
```

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HW ES: Example – Log U.S. Vehicles Sales

Example (continuation):

```
> hw_d_car

Coefficients:
         [,1]
a    7.177857555               ⇒ forecast for level
b    0.0001100345              ⇒ forecast for trend
s1  -0.075314457               ⇒ forecast for seasonal month 1
s2  -0.084468361               ⇒ forecast for seasonal month 2
s3   0.049447067
s4  -0.273299309
s5  -0.138251757
s6  -0.026603921
s7  -0.144953062
s8   0.079214066
s9   0.037899454
s10  0.020477134
s11  0.089309775
s12 -0.012530316
```

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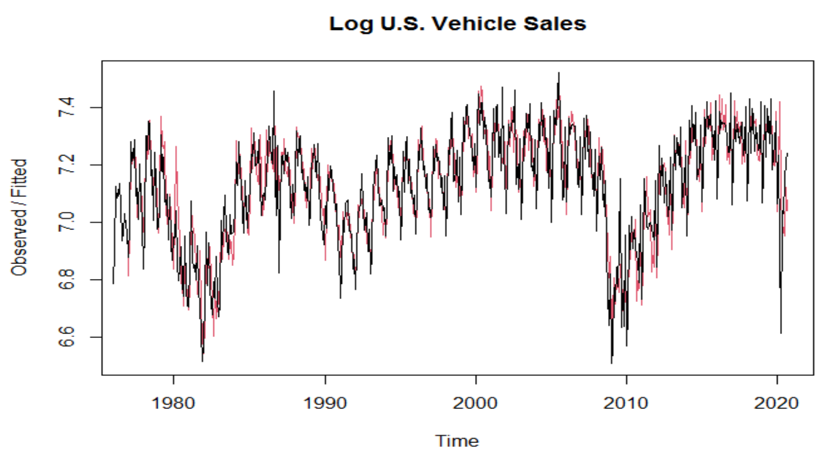
HW ES: Example – Log U.S. Vehicles Sales

Example (continuation):
plot(hw_d_car)



SES: Forecasting Log U.S. Vehicles Sales

Example (continuation): Now, we forecast one-step ahead forecasts

```
T_last <- nrow(hw_d_car$fitted)
h <- 25
ses_f_hw <- matrix(0,h,1)
 alpha <- 0.4355244
 beta <- 0.009373815
 gamma <- 0.3446495
y <- l_car
T <- length(l_car)
sm <- matrix(0,T,1)
Tr <- matrix(0,T,1)
I <- matrix(0,T,1)
T1 <- T-h+1
a <- T1
sm[a-1] <- 7.177857555
Tr[a-1] <- -0.000309358
I[501:512] <- c(-0.075314457,-0.084468361,0.049447067,-0.273299309,-0.138251757, -
0.026603921, -0.144953062,0.079214066,0.037899454,0.020477134,0.089309775,-
0.012530316)
```

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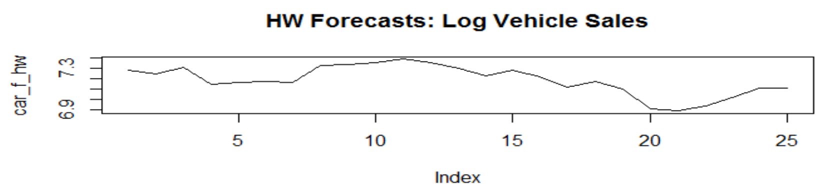
SES: Forecasting Log U.S. Vehicles Sales

Example (continuation):

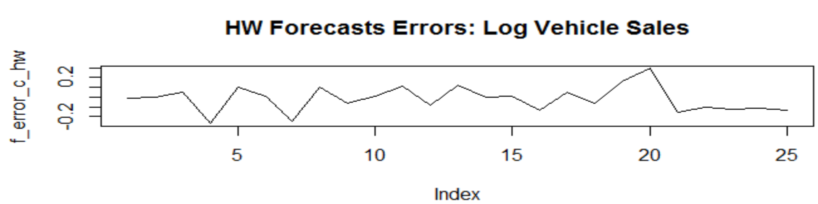
```
while (a <= T) {
        sm[a] = alpha * y[a-1] +  (1-alpha) * sm[a-1]
        Tr[a] = beta * (sm[a] - sm[a-1]) + (1 - beta) * Tr[a-1]
        I[a] = gamma * (y[a] - sm[a]) + (1 - gamma) * I[a - 12]
a <- a + 1
}
hh <- c(1:h)
car_f_hw <- sm[T1:T] + hh*Tr[T1:T] + I[T1:T]
car_f_hw
f_error_c_hw<- car_f_hw - y[T1:T]
plot(car_f_hw, type="l", main ="SES Forecasts: Log Vehicle Sales")
```



SES: Forecasting Log U.S. Vehicles Sales

Example (continuation):

```
plot(f_error_c_hw, type="l", main ="SES Forecasts Errors: Log Vehicle Sales")
```




```
MSE_hw <- sum(f_error_c_hw^2)/h
> MSE_hw
[1] 0.01655964
```

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HW ES: Remarks

• Remarks

- If a computer program selects $\gamma = 0 = \beta$, it has a lack of trend or seasonality. It implies a constant (deterministic) component. In this case, an ARIMA model with deterministic trend may be a more appropriate model.

- For HW ES, a seasonal weight near one implies that a non-seasonal model may be more appropriate.

- We can model seasonalities as multiplicative or additive:
  ⇒ Multiplicative seasonality: $\text{Forecast}_t = S_t * I_{t-s}$.
  ⇒ Additive seasonality: $\text{Forecast}_t = S_t + I_{t-s}$.

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Evaluation of forecasts – Accuracy measures

• The mean squared error (*MSE*) and mean absolute error (*MAE*) are the most popular accuracy measures:

$$\text{MSE} = \frac{1}{m}\sum_{i=T+1}^{T+m}(\hat{y}_i - y_i)^2 = \frac{1}{m}\sum_{i=T+1}^{T+m} {e_i}^2$$

$$\text{MAE} = \frac{1}{m}\sum_{i=T+1}^{T+m} |\hat{y}_i - y_i\,| = \frac{1}{m}\sum_{i=T+1}^{T+m} |e_i|$$

where *m* is the number of out-of-sample forecasts.

• But other measures are routinely used:

- Mean absolute percentage error (*MAPE*) $= \frac{100}{T-(m-1)}\sum_{i=T+1}^{T+m} |\frac{\hat{y}_i - y_i}{y_i}|$

- Absolute *MAPE (AMAPE)* $= \frac{100}{T-(m-1)}\sum_{i=T+1}^{T+m} |\frac{\hat{y}_i - y_i}{\hat{y}_i + y_i}|$

<u>Remark</u>: There is an asymmetry in MAPE, the level y_i matters.

Evaluation of forecasts – Accuracy measures

- % correct sign predictions (PCSP) = $\frac{1}{T-(m-1)}\sum_{i=T+1}^{T+m} z_i$

where z_i = 1 if $(\hat{y}_{i+l} * y_{i+l}) > 0$
= 0, otherwise.

- % correct direction change predictions (PCDP)= $\frac{1}{T-(m-1)}\sum_{i=T+1}^{T+m} z_i$

where z_i = 1 if $(\hat{y}_{i+l} - y_i) * (y_{i+l} - y_i) > 0$
= 0, otherwise.

Remark: We value forecasts with the right direction (sign) or forecast that can predict turning points. For stock investors, the sign matters!

• MSE penalizes large errors more heavily than small errors, the sign prediction criterion, like MAE, does not penalize large errors more.

Evaluation of forecasts – Accuracy measures

Example: We compute MSE and the % of correct direction change (PCDC) predictions for the one-step forecasts for U.S. monthly vehicles sales based on the SES and HW ES models.

```
> MSE_ses
[1] 0.027889

> MSE_hw
[1] 0.01655964
```

• We calculate PCDC with following script for HW & SES:

```
bb_hw <- (car_f_hw - y[(T1-1):(T-1)]) * (y[T1:T] - y[(T1-1):(T-1)])
indicator_hw <- ifelse(bb_hw > 0,1,0)     # ifelse ("if else") produces a 1 if condition is true
pcdc_hw <- sum(indicator_hw)/h
> indicator_hw
 [1] 1 1 1 0 1 1 1 1 1 1 0 1 1 1 1 0 1 1 1 1 1 1 0 0 0
> pcdc_hw
[1] 0.76
```

Evaluation of forecasts – Accuracy measures

Example (continuation):

```
bb_s <- (ses_f_c - y[(T1-1):(T-1)]) * (y[T1:T] - y[(T1-1):(T-1)])
indicator_s <- ifelse(bb_s > 0,1,0)
pcdc_s <- sum(indicator_s)/h
> indicator_s
 [1] 1 0 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 1 1 1 0 1 1 0 0 0
> pcdc_s
[1] 0.76
```

Note: Same percentage of correct direction change (PCDC) predictions, but the sequence of correct predictions is not the same.

Evaluation of forecasts – DM Test

- To determine if one model predicts better than another, we define the loss differential between two forecasts:

$$d_t = g(e_t^{M1}) - g(e_t^{M2})$$

where g(.) is the forecasting loss function. M1 and M2 are two competing sets of forecasts –could be from models or something else.

- We only need $\{e_t^{M1}\}$ & $\{e_t^{M2}\}$, not the structure of M1 or M2. In this sense, this approach is "*model-free*."

- Typical (symmetric) loss functions: $g(e_t) = e_t^2$ & $g(e_t) = |e_t|$.

- But other g(.)'s can be used: $g(e_t) = \exp(\lambda e_t^2) - \lambda e_t^2 \quad (\lambda > 0)$.

Evaluation of forecasts – DM Test

- Then, we test the null hypotheses of equal predictive accuracy:

H_0: $E[d_t] = 0$

H_1: $E[d_t] = \mu \neq 0$.

- Diebold and Mariano (1995) assume $\{e_t^{M1}\}$ & $\{e_t^{M2}\}$ is covariance stationarity and other regularity conditions (finite Var[d_t], independence of forecasts after ℓ periods) needed to apply CLT. Then,

$$\frac{\bar{d} - \mu}{\sqrt{Var[\bar{d}]/T}} \xrightarrow{d} N(0,1), \qquad \bar{d} = \frac{1}{m}\sum_{i=T+1}^{T+m} d_i$$

- Then, under H_0, the DM test is a simple *z-test*:

$$DM = \frac{\bar{d}}{\sqrt{\hat{V}ar[\bar{d}]/T}} \xrightarrow{d} N(0,1)$$

Evaluation of forecasts – DM Test

where $\hat{V}ar[\bar{d}]$ is a consistent estimator of the variance, usually based on sample autocovariances of d_t:

$$\hat{V}ar[\bar{d}] = \gamma(0) + 2\sum_{j=k}^{\ell} \gamma(j)$$

- There are some suggestion to calculate small sample modification of the DM test. For example, :

$$DM^* = DM/\{[T + 1 - 2\,\ell + \ell\,(\ell - 1)/T]/T\}^{1/2} \sim t_{T-1}.$$

where ℓ-step ahead forecast. If ARCH is suspected, replace ℓ with $[0.5\,\sqrt{(T)}] + \ell$.

Note: If $\{e_t^{M1}\}$ & $\{e_t^{M2}\}$ are perfectly correlated, the numerator and denominator of the DM test are both converging to 0 as $T \to \infty$.

⇒ Avoid DM test when this situation is suspected (say, two nested models.) Though, in small samples, it is OK.

Evaluation of forecasts – DM Test

Example: Code in R

```
 dm.test <- function (e1, e2, h = 1, power = 2) {
d <- c(abs(e1))^power - c(abs(e2))^power
  d.cov <- acf(d, na.action = na.omit, lag.max = h - 1, type = "covariance", plot = FALSE)$acf[, , 1]
  d.var <- sum(c(d.cov[1], 2 * d.cov[-1]))/length(d)
  dv <- d.var                                   #max(1e-8,d.var)
  if(dv > 0)
    STATISTIC <- mean(d, na.rm = TRUE) / sqrt(dv)
  else if(h==1)
    stop("Variance of DM statistic is zero")
  else
  {
    warning("Variance is negative, using horizon h=1")
    return(dm.test(e1,e2,alternative,h=1,power))
  }
    n <- length(d)
  k <- ((n + 1 - 2*h + (h/n) * (h-1))/n)^(1/2)
  STATISTIC <- STATISTIC * k
  names(STATISTIC) <- "DM"
}
```

Evaluation of forecasts – DM Test

Example: We compare the SES and HW forecasts for the log of U.S. monthly vehicle sales. We use the *dm.test* function, part of the forecast package.

```
library(forecast)
> dm.test(f_error_c_ses, f_error_c_hw, power=2)

        Diebold-Mariano Test

data:  f_error_c_sesf_error_c_hw
DM = 1.6756, Forecast horizon = 1, Loss function power = 2, p-value = 0.1068
alternative hypothesis: two.sided

> dm.test(f_error_c_ses,f_error_c_hw, power=1)

        Diebold-Mariano Test

data:  f_error_c_sesf_error_c_hw
DM = 1.94, Forecast horizon = 1, Loss function power = 1, p-value = 0.064
alternative hypothesis: two.sided
```

Note: Cannot reject H_0: $MSE_{SES} = MSE_{HW}$ at 5% level

Evaluation of forecasts – DM Test: Remarks

• The DM tests is routinely used. Its "model-free" approach has appeal. There are model-dependent tests, with more complicated asymptotic distributions.

• The loss function does not need to be symmetric (like MSE).

• The DM test is based on the notion of unconditional –i.e., on average over the whole sample- expected loss.

• Following Morgan, Granger and Newbold (1977), the DM statistic can be calculated by regression of d_t, on an intercept, using NW SE. But, we can also condition on variables that may explain d_t. We move from an unconditional to a conditional expected loss perspective.

Combination of Forecasts

• Idea – from Bates & Granger (*Operations Research Quarterly,* 1969):

- We have different forecasts from R models:

$$\hat{Y}_T^{M1}(\ell), \hat{Y}_T^{M2}(\ell), \ldots . \hat{Y}_T^{MR}(\ell)$$

• Q: Why not combine them?

$$\hat{Y}_T^{Comb}(\ell) = \omega_{M1}\hat{Y}_T^{M1}(\ell) + \omega_{M2}\hat{Y}_T^{M2}(\ell) + \ldots . + \omega_{MR}\hat{Y}_T^{MR}(\ell)$$

• Very common practice in economics, finance and politics, reported by the press as "consensus forecast." Usually, as a simple average.

• Q: Advantage? Lower forecast variance. Diversification argument.

Intuition: Individual forecasts are each based on partial information sets (say, private information) or models.

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Combination of Forecasts – Optimal Weights

- The variance of the forecasts is:

$$Var[\hat{Y}_T^{Comb}(\ell)] = \sum_{j=1}^{R} (\omega_{Mj})^2 Var[\hat{Y}_T^{Mj}(\ell)] +$$

$$+ 2 \sum_{j=1}^{R} \sum_{i=j+1}^{R} \omega_{Mj}\omega_{Mi} \text{Covar}[\hat{Y}_T^{Mj}(\ell)\hat{Y}_T^{Mi}(\ell)]$$

Note: Ideally, we would like to have negatively correlated forecasts.

- Assuming unbiased forecasts and uncorrelated errors,

$$Var[\hat{Y}_T^{Comb}(\ell)] = \sum_{j=1}^{R} (\omega_{Mj})^2 \sigma_j^2$$

Example: Simple average: ω_j=1/R. Then,

$$Var[\hat{Y}_T^{Comb}(\ell)] = 1/R^2 \sum_{j=1}^{R} \sigma_j^2 .$$

Combination of Forecasts – Optimal Weights

Example: We combine the SES and HW forecast of log US vehicles sales:

```
f_comb <- (ses_f_c + car_f_hw)/2
f_error_comb <- f_comb - y[T1:T]
> var(f_comb)
[1] 0.0178981
> var(car_f_hw)
[1] 0.02042458
> var(ses_f_c)
[1] 0.01823237
```

Combination of Forecasts – Optimal Weights

- We can derived optimal weights –i,e., ω_j's that minimize the variance of the forecast. Under the uncorrelated assumption:

Under the uncorrelated assumption:

$$\omega_{Mj} *= \sigma_j^{-2} \Big/ \sum_{j=1}^{R} \sigma_j^{-2}$$

- The ω_j*'s are inversely proportional to their variances.

- In general, forecasts are biased and correlated. The correlations will appear in the above formula for the optimal weights. For the two forecasts case:

$$\omega_{Mj} *= (\sigma_1^2 - \sigma_{12})/(\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}) = (\sigma_1^2 - \rho\sigma_1\sigma_2)/(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2)$$

Combination of Forecasts: Regression Weights

- In general, forecasts are biased and correlated. The correlations will appear in the above formula for the optimal weights. Ideally, we would like to have negatively correlated forecasts.

- Granger and Ramanathan(1984) used a regression method to combine forecasts.

- Regress the actual value on the forecasts. The estimated coefficients are the weights:

$$y_{T+\ell} = \beta_1 \hat{Y}_T^{M1}(\ell) + \beta_2 \hat{Y}_T^{M2}(\ell) + \ldots + \beta_R \hat{Y}_T^{MR}(\ell) + \varepsilon_{T+\ell}$$

- Should use a constrained regression

– Omit the constant

– Enforce non-negative coefficients.

– Constrain coefficients to sum to one

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Combination of Forecasts: Regression Weights

Example: We regress the SES and HW forecasts against the observed car sales to obtain optimal weights. We omit the constant

```
> lm(y[T1:T] ~ ses_f_c + car_f_hw - 1)

Call:
lm(formula = y[T1:T] ~ ses_f_c + car_f_hw - 1)

Coefficients:
 ses_f_c  car_f_hw
 -0.5426    1.5472
```

Note: Coefficients (weights) add up to 1. But, we see negative weights... In general, we use a constrained regression, forcing parameters to be between 0 and 1 (& non-negative). But, h=25 delivers not a lot of observations to do non-linear estimation.

Combination of Forecasts: Regression Weights

- Remarks:

- To get weights, we do not include a constant. Here, we are assuming unbiased forecasts. If the forecasts are biased, we include a constant.

- To account for potential correlation of errors, we can allow for ARMA residuals or include y_{T+l-1} in the regression.

- Time varying weights are also possible.

- Should weights matter? Two views:

- Simple averages outperform more complicated combination techniques.

- Sampling variability may affect weight estimates to the extent that the combination has a larger MSE.

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Forecasting: Final Comments

• Since the late 1960s, combination weights have generally been chosen to minimize a symmetric, squared-error loss function.

• But, asymmetric loss functions can also be used. More recent research work find that the optimal weights depend on higher order moments, such a skewness.

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