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# Optimal debt with unobservable investments

**Paul Povel***

**and**

**Michael Raith****

*We study financial contracting when both an entrepreneur's investment and the resulting revenue are unobservable to an outside investor. We show that a debt contract is always optimal; repayment is induced by a liquidation threat that increases with the extent of default. Moreover, when the entrepreneur's decision concerns the scale of his project, a contract that minimizes liquidation losses is optimal. When the decision concerns managerial effort or project risk, however, it may be optimal to write a contract with a greater threat of liquidation, to induce the entrepreneur to exert more effort or to choose a less risky project.*

## 1. Introduction

■ Financial contracting theory has explored a variety of settings in which debt emerges as an optimal contract between a firm and an outside investor. Most contributions have focused on the agency problem that arises when a firm's revenue is difficult to verify, and have typically found that a debt contract is the optimal way to induce a firm to repay its investor. Models that explain debt as a solution to this agency problem, however, assume that the investment to be financed is given, or contractible. (See, e.g., Townsend (1979), Gale and Hellwig (1985), Diamond (1984), Bolton and Scharfstein (1990), and Hart and Moore (1998).)

In reality, investors have little control not only over how much a firm repays, but also over how their funds are used. Firms can divert borrowed funds to other purposes; they can pursue safe or risky investment strategies; and their managers can choose how much effort to exert. Beginning with Jensen and Meckling (1976) and Myers (1977), capital-structure theory has argued that if firms incur debt, their incentives to subsequently make value-maximizing decisions may be distorted, thus increasing the cost of raising external funds (see Harris and Raviv (1991) for a survey). These costs raise the question of why firms issue debt in the first place, instead of securities that cause fewer distortions.[1]

* University of Minnesota; povel@umn.edu.

** University of Rochester; raith@simon.rochester.edu.

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[1] Managerial moral hazard may be a reason to use debt; see Aghion and Bolton (1992), Zender (1991), or Dewatripont and Tirole (1994). However, these studies abstract from repayment incentive problems.

In this article we study how financial contracts are designed if both investment and repayment are subject to moral hazard. We develop a model in which an entrepreneur obtains funds from an outside investor. The investor cannot observe how these funds are invested, nor can she observe what revenue is realized. An optimal contract then must ensure both that sufficient repayments are made and that a particular investment is chosen. In our setting, financial contracting and investment cannot be analyzed separately: the details of the contract affect the entrepreneur's investment choice and hence the distribution of revenue, which in turn affects the details of the optimal contract.

We show that under very general conditions, a debt contract is optimal. The entrepreneur promises to repay a fixed amount, and if he is unable to do so, he must pay his entire revenue and face some probability of liquidation. The debt contract induces repayment by stipulating that the probability of liquidation increases with the extent of default. Debt contracts are familiar from other models in which revenue is not verifiable (e.g., Diamond, 1984; Bolton and Scharfstein, 1990, 1996; Gertner, Scharfstein, and Stein, 1994; or Hart and Moore, 1998). We show that even when the entrepreneur's investment decision is also unobservable (a moral hazard problem that is commonly considered a cost of using debt), debt remains the optimal contract.

Debt contracts may differ in how the probability of liquidation varies with the entrepreneur's repayment. If the entrepreneur's investment were contractible, a *simple debt contract* would be optimal (see Diamond, 1984, or Bolton and Scharfstein, 1990). A simple debt contract minimizes the probability of liquidation while inducing the entrepreneur to repay his debt, or otherwise his entire revenue. The probability of liquidation is a linear function of the extent of default.

With unobservable investment, a simple debt contract provides first-best investment incentives. The probability of liquidation is chosen such that the entrepreneur's expected liquidation loss equals his shortfall in repayment. For any level of revenue, therefore, the entrepreneur's utility loss caused by his repayment and the expected liquidation loss is a constant, and hence his incentives to invest are the same as if he had sufficient own funds.

A moral hazard problem still exists because the first-best investment decision is optimal for an entrepreneur with sufficient own funds, but it is not necessarily optimal if it must be financed with debt. An alternative investment may be preferred despite its lower profitability, if the expected liquidation loss is smaller. Since a simple debt contract induces first-best incentives, it may not be the optimal contract.

We first consider a model in which the entrepreneur can consume some of the borrowed funds and invest on a smaller scale. At the contracting stage, the entrepreneur prefers to invest on a smaller scale than he would if he had sufficient funds of his own. He can credibly promise to choose that scale by borrowing exactly the amount needed, using a simple debt contract. The first-best investment incentives created by the contract ensure that he does not invest on a smaller scale, while with the amount borrowed he cannot invest more. Since the entrepreneur chooses the same investment he would choose if the investment were contractible, a simple debt contract is in fact optimal.

Nonsimple debt contracts may be optimal for other moral hazard problems. We consider two examples. In the first, the pecuniary cost of a project is fixed, but the entrepreneur can increase expected revenue by exerting unobservable effort. Since the cost of debt finance decreases as the entrepreneur's expected revenue increases, it may be optimal to induce effort above the first-best level by writing a debt contract that punishes default more strongly than would a simple debt contract. Under some standard assumptions on the distribution of revenue, we characterize the structure of the optimal contract for this case. A nonsimple debt contract is optimal if the gain from inducing higher effort outweighs the cost of punishing default more strongly. Both the gain and the cost are of first-order magnitude, however, and a simple debt contract may remain optimal.

We obtain similar results for a model in which the entrepreneur chooses between projects with different risk and return structures: with debt financing, a project with a lower expected current profit may be preferable if it is also less risky. Again, either a simple or a nonsimple debt contract may be optimal, depending on the parameters. Whenever a simple debt contract is optimal, the entrepreneur chooses risk as if the investment were financed internally. This is an unusual form of risk shifting, as, *ex ante,* the firm wants to commit *not* to choose the first-best investment. It also implies that the risk decisions of internally and debt-financed firms may be empirically indistinguishable.

To our knowledge, ours is the first study of financial contracting where neither investments nor their returns are fully contractible. Both of these moral hazard problems, however, have been studied extensively in isolation.

Diamond (1984) and Bolton and Scharfstein (1990, 1996) assume that a lender cannot observe a borrower's revenue and enforces repayment by threatening with an abstract punishment (Diamond) or with a refusal to provide additional funds (Bolton and Scharfstein). Gertner, Scharfstein, and Stein (1994), Gromb (1994), and Hart and Moore (1998) make similar assumptions. But none of these articles considers the borrower's choice of investment.[2]

A second group of articles studies financial contracting when revenue is contractible but depends on a nonverifiable decision made by the borrower. In Innes (1990), Hellwig (1994), and Biais and Casamatta (1999), a borrower makes an unobservable decision regarding his effort and/or the riskiness of revenue. Aghion and Bolton (1992), Zender (1991), and Dewatripont and Tirole (1994) focus on the allocation of control rights once a project has been started. Optimal contracts resemble debt in the sense that they shift decision rights from the entrepreneur to the investor following bad news about the firm's performance and prospects. These authors, however, are not concerned with the moral hazard problems emphasized by Jensen and Meckling (1976) and Myers (1977) that motivate our analysis.

Recently, several authors have studied optimal financial contracts in dynamic settings, each focusing on one dimension of moral hazard. In Albuquerque and Hopenhayn (2004), all information is symmetric, but the entrepreneur can default on his repayment.[3] DeMarzo and Fishman (2000), Clementi and Hopenhayn (2002), and Quadrini (2004) study models with perfect enforcement but asymmetric information: in the first two revenue is unobservable, whereas in the last one the entrepreneur's choice of investment level is unobservable.

In an extension, DeMarzo and Fishman (2000) allow for an unobservable investment decision and find that the optimal contract of their basic model creates first-best investment incentives. They point out that in this situation, first-best investment incentives, and hence the contract, are generally suboptimal. We explain the differences between our model and DeMarzo and Fishman's in more detail in Section 2.

Our article is structured as follows. Section 2 presents our basic model. Section 3 examines simple debt contracts, which are optimal when the investment is contractible. Section 4 shows that with unobservable investment, a debt contract is always optimal, although its details may differ from those of a simple debt contract. Sections 5 through 7 study the circumstances under which a simple debt contract is in fact optimal. We look at scenarios in which the entrepreneur's unobservable choice concerns the scale of investment, managerial effort, or project risk, respectively. Section 8 concludes. The proofs of most results can be found in the Appendix, and we discuss an extension of the model in the web Appendix (www.rje.org/main/sup-mat.html).

## 2. Model

■ Consider a risk-neutral entrepreneur $E$, who requires funds from a risk-neutral investor $I$ to finance a project. The timing is as follows (see Figure 1).

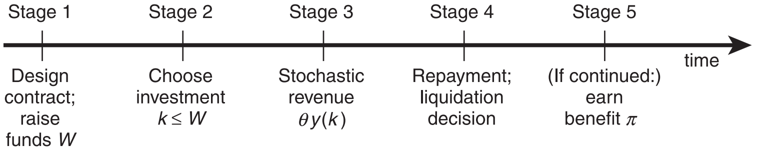
In stage 1, $E$ offers a financial contract on a take-it-or-leave-it basis to $I$, whereby $I$ is to provide an amount $W$ to $E$. $I$ accepts if her resulting expected payoff is at least $V_I$. Similarly, $E$ offers a contract only if his resulting expected payoff exceeds his reservation payoff $V_E$. At the time of contracting, $I$ and $E$ have symmetric information; in particular, $I$ knows $E$'s investment opportunities.

[2] In Diamond (1984), the borrower is a financial intermediary who invests depositors' funds. Diamond does not explicitly consider the intermediary's investment incentives.

[3] Similar setups are used in Monge-Naranjo (2001) and Cooley, Marimon, and Quadrini (2001).

FIGURE 1

TIMELINE




In stage 2, $E$ invests an amount $k$; $I$ cannot observe this amount. $E$ does not have any funds other than $W$, therefore $k \leq W$. In stage 3, the investment generates revenue $\theta y(k)$. Here, $\theta$ is the realization of a random variable $\tilde{\theta}$ that is distributed with cumulative density function $F$ and a continuous and differentiable density $f$ over the support $[0, \bar{\theta}]$ for some $\bar{\theta} > 0$. Assume that $\mathrm{E}(\tilde{\theta}) = 1$. The function $y$ is twice differentiable and satisfies $y(0) \geq 0$, $y' > 0$, and $y'' < 0$. Assume that $y'(0) > 1$ and $\lim_{k\to\infty} y'(k) < 1$. Then $\int_0^{\bar{\theta}} \theta y(k) f(\theta) \mathrm{d}\theta - k = y(k) - k$ has a unique maximum in $k$, which we denote by $k^{FB}$; the superscript stands for "first best"—an entrepreneur with unlimited own funds would choose $k^{FB}$.

In stage 4, $E$ can make a transfer to $I$, using the funds he has available. As in Diamond (1984) and Bolton and Scharfstein (1990), we assume that $I$ cannot observe $E$'s realized revenue, and that $E$ is protected by limited liability. Therefore $E$ can always claim that he invested all funds borrowed and that his revenue is zero, and make no repayment at all. At the end of stage 4, contractual provisions may call for liquidation of the project.

In stage 5, $E$ receives a nontransferable payoff $\pi > 0$ if the project was not liquidated. This payoff may represent noncontractible future earnings that the project generates, or control rents that $E$ enjoys if the project is completed. If the project is liquidated in stage 4, neither $E$ nor $I$ receives any additional payoff.[4]

The contract must ensure that $E$ chooses a certain investment and that his expected transfer to $I$ is sufficiently high. The contract may be contingent on the amount $W$ that $E$ raises and on any messages that are exchanged. We can apply the revelation principle and restrict our analysis to direct-revelation mechanisms, in which $E$ reveals his private information. This private information includes $E$'s choice of investment and his realized revenue. Since $E$'s choice of $k$ is deterministic,[5] $I$ can predict $k$ for any given contract, and $E$'s only private information is therefore his revenue, which is related one-to-one to his total funds at the end of stage 3, $R(W, k, \theta) = \theta y(k) + W - k$, where $R(W, k, \theta) \in [0, \bar{R}(W)]$ with $\bar{R}(W) = \bar{\theta} y(k^{FB}) + W - k^{FB}$. A direct mechanism asks $E$ at the end of stage 3 to announce his total funds. Contingent on his announcement $\hat{R}$, the mechanism specifies what transfers must be made and how likely liquidation is in stage 4.

We define a *contract* as a triple $(W, T, \beta)$, where $T : [0, \bar{R}(W)] \to [0, \bar{R}(W)]$ is a transfer from $E$ to $I$, and $\beta : [0, \bar{R}(W)] \to [0, 1]$ is the probability that $E$'s project is allowed to continue. Both are functions of $E$'s announced funds $\hat{R}$. A contract *satisfies limited liability* if $T(\hat{R}) \leq R$ for all $(R, \hat{R}) \in [0, \bar{R}(W)]^2$. Limited liability implies that $T(R) \leq R$; i.e., $E$'s payment cannot exceed his funds. It also implies that $E$ cannot be prevented from announcing $\hat{R} \neq R$ as long as the associated transfer does not exceed $R$. This assumption is common both in models with costly state verification (e.g., Townsend, 1979; Gale and Hellwig, 1985) and in models with unobservable cash flows (e.g., Diamond, 1984; Bolton and Scharfstein, 1990).

In contrast, DeMarzo and Fishman (2000) consider contracts that ask the entrepreneur to transfer *all* of his revenue to the investor, part of which may then be reimbursed. This corresponds to a constraint on feasible messages $\hat{R} < R$ that is more restrictive than the constraint $T(\hat{R}) < R$

[4] The results in this article are robust with respect to some generalizations: (1) A liquidation value may accrue to $I$, as long as it is strictly smaller than $\pi$ (otherwise the firm can issue fully collateralized debt, and the agency problem disappears); see Povel and Raith (forthcoming). (2) $\pi$ may be positively related to $E$'s first-period revenue; see the web Appendix.

[5] It will be obvious that $E$ cannot benefit from playing a mixed strategy.

we use here. Greater restrictions on the opportunities for lying in turn imply a less restrictive incentive-compatibility constraint, and hence a larger set of feasible contracts.

In particular, DeMarzo and Fishman's assumption allows for contracts under which a \$1 increase in revenue can lead to a greater than \$1 increase in the entrepreneur's payoff. Such a contract is not optimal in their base model but may be optimal if the entrepreneur must exert unobservable effort (as in Section 6 below). The problem with such a contract is that the entrepreneur has an incentive to borrow funds from a third party to increase his cash flow, if it increases his own payoff by a larger amount (see Innes, 1990, and Hart and Moore, 1998). This objection seems justified in particular when cash flow is unobservable and thus the lender cannot distinguish the firm's revenue from borrowed funds. Our more restrictive assumptions rule out contracts under which $E$'s payoff can increase faster than his revenue.

A contract is *incentive compatible* if $E$ has an incentive to announce $R$ truthfully, i.e.,

$$R - T(R) + \beta(R)\pi \geq R - T(\hat{R}) + \beta(\hat{R})\pi \qquad \forall\, (R, \hat{R}) \text{ such that } T(\hat{R}) \leq R. \tag{1}$$

A contract is *feasible* if it satisfies limited liability, is incentive compatible, and satisfies both $E$'s individual-rationality constraint

$$\int_0^{\bar{\theta}} \left[R(W, k^*(W, T, \beta), \theta) - T(R(W, k^*(W, T, \beta), \theta)) + \beta(R(W, k^*(W, T, \beta), \theta))\pi\right] f(\theta)\mathrm{d}\theta \geq V_E \tag{2}$$

and $I$'s individual-rationality constraint

$$\int_0^{\bar{\theta}} T(R(W, k^*(W, T, \beta), \theta)) f(\theta)\mathrm{d}\theta - W \geq V_I, \tag{3}$$

where $k^*(W, T, \beta)$ is $E$'s optimal investment in stage 2 of the game, given the contract $(W, T, \beta)$ and assuming that in stage 4 he truthfully reveals $R$:

$$k^*(W, T, \beta) = \arg\max_{k \leq W} \int_0^{\bar{\theta}} [R(W, k, \theta) - T(R(W, k, \theta)) + \beta(R(W, k, \theta))\pi]\, f(\theta)\mathrm{d}\theta. \tag{4}$$

A contract is *optimal* if it is feasible and maximizes $E$'s payoff (the left-hand side of (2)).

Every constrained Pareto-efficient contract is an optimal contract for some pair $(V_I, V_E)$. There is thus no loss of generality in assuming that $E$ makes a take-it-or-leave-it offer. Moreover, since none of our results depend on $V_I$ or $V_E$, we hereafter assume for simplicity that $V_I = V_E = 0$.

## 3. Simple debt contracts

■ With a contractible investment, the optimal contract is a debt contract for which the continuation probability $\beta$ takes a very simple form. We first characterize this contract, which we refer to as a simple debt contract. We then explore the consequences of using a simple debt contract for $E$'s investment choice when his investment his not contractible.

*Definition 1.* A contract $(W, T, \beta)$ is a simple debt contract if for some $D \in (0, \min\{\bar{\theta} y(k^*(W, T, \beta)), \pi\})$,

$$\text{for all } R \geq D, \qquad T(R) = D \text{ and } \beta(R) = 1, \tag{5}$$

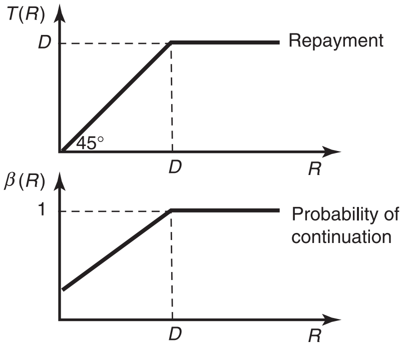
$$\text{for all } R < D, \qquad T(R) = R \text{ and } \beta(R) = \bar{\beta}(R), \tag{6}$$

where $\bar{\beta}(R) = 1 - (D - R)/\pi$.

With a simple debt contract, (1) is binding everywhere. If the investment is contractible,

FIGURE 2

REPAYMENT AND PROBABILITY OF CONTINUATION UNDER A SIMPLE DEBT CONTRACT




the only objective in choosing $T$ and $\beta$ is to minimize the probability of liquidation, subject to incentive compatibility. A simple debt contract is then the unique optimal contract; see Diamond (1984) and Bolton and Scharfstein (1990).

Figure 2 depicts $E$'s repayment $T(R)$ in the upper panel and the probability of continuation $\beta(R)$ in the lower panel. Clearly, the repayment pattern is that of a debt contract. What additionally characterizes a simple debt contract is that if $E$ defaults, the probability of liquidation is a linear function of $E$'s default $D - R$ (with slope $1/\pi$).

The existence of debt is often thought to distort an entrepreneur's incentives to invest, relative to the first best. That is not the case under a simple debt contract.

*Proposition 1.* Under a simple debt contract, $E$ has first-best incentives when choosing $k$.

*Proof.* For any $R$ given at the end of stage 3, $E$'s payoff at the end of stage 4 is $R - T(R) + \beta(R)\pi$, which under a simple debt contract simplifies to $R + \pi - D$ both for $R \geq D$ and for $R < D$. Since $D$ is already fixed when $E$ chooses $k$, the integrand in (4) equals $R(W, k, \theta) + \pi - D = \theta y(k) + W - k + \pi - D$, and so the integral in (4) reduces to $y(k) + W - k + \pi - D$, which has a unique maximum at $k^{FB}$. *Q.E.D.*

With a simple debt contract, $E$ has first-best incentives because his net loss is constant for any realized revenue: either $E$ pays $D$ and can continue with certainty, or else he pays $R < D$ and faces an expected liquidation loss of $[1 - \beta(R)]\pi = D - R$, which adds to $D$. When choosing $k$, therefore, $E$'s objective function is the same as that of an entrepreneur with unlimited funds, plus a constant.

Proposition 1 may seem to suggest that a simple contract must be optimal: it minimizes the probability of liquidation (subject to incentive compatibility), and it generates first-best investment incentives. This conclusion is premature. First, $E$ cannot choose $k^{FB}$ unless $W$ is large enough. More importantly, $E$ faces a risk of losing $\pi$, which depends on $k$ and the distribution of $\tilde{\theta} y(k)$; so $k^{FB}$ is not necessarily the second-best investment.[6] Inducing the second-best instead of the first-best investment, however, can be difficult if the investment is unobservable to $I$. We show in the following sections that in spite of these complications, a debt contract is still optimal under very general conditions. Whether or not the optimal $\beta$ is that of a simple debt contract, however, depends on the set of available investments.

## 4. Optimality of a debt contract

■ Our main result is that even if $E$'s investment is not contractible, a debt contract is optimal irrespective of any assumptions about the set of available investments:

[6] We would like to thank Peter DeMarzo and Michael Fishman for pointing this out to us.

*Proposition 2*. A contract $(W, T, \beta)$ is optimal only if there exists $D \in (0, \min\{\bar{\theta} y(k^*(W, T, \beta)), \pi\})$ such that

$$\text{for all } R \geq D, \qquad T(R) = D \text{ and } \beta(R) = 1, \tag{7}$$

$$\text{for all } R < D, \qquad T(R) = R \text{ and } \beta(R) < 1, \tag{8}$$

and for all $(R, R')$ such that $R < D$, $R' < R$,

$$\frac{\beta(R) - \beta(R')}{R - R'} \geq \frac{1}{\pi}. \tag{9}$$

*Proof*. See the Appendix.

Some basic properties must hold for any incentive-compatible contract: condition (1) implies that $E$'s payment must be a constant (namely $D$) whenever $\beta = 1$, and cannot exceed $\pi$. It also implies that $E$'s payment and the continuation probability must be positively related. With an optimal contract, in addition, we have $T(R) = R$ for $R < D$; i.e., if $E$ cannot pay $D$, he pays as much as he can.

When the investment is contractible, it is straightforward to prove that a simple debt contract is optimal, because then the only objective is to maximize $\beta(R)$ subject to (1); see Section 3.

When the investment is unobservable, however, this simple proof no longer works. The objective now is to induce $E$ both to choose a particular investment in stage 2 and to repay his loan in stage 4. Since the contract influences $E$'s choice of $k$, under an optimal contract (1) may not be binding everywhere; we analyze such cases in Sections 6 and 7.

Nevertheless, debt is optimal. The argument of the proof is as follows. Suppose a contract specifies that for some revenue levels the probability of liquidation is positive but the repayment is strictly less than the funds $E$ has available. Then for each of these revenue levels, $E$'s repayment could be increased and the liquidation probability decreased, such that $E$'s payoff is unchanged. In stage 2, $E$ would then still choose the same investment as under the original contract. The new contract would lead to a higher expected payment to $I$, making her willing to lend more or to accept a smaller repayment. It follows that the original contract cannot have been optimal.

This argument does not rely on any assumptions about how $k$ is chosen; it is the same if instead of project scale, $E$'s unobservable decision in stage 2 concerns unobservable effort provision or the choice between projects that differ in risk. A debt contract is therefore also optimal in these cases. Proposition 2 leaves the precise specification of $\beta$ open, however; it only requires that the probability increase sufficiently strongly with the extent of the default. As we show in Sections 5–7, the optimal $\beta$ does depend on the details of $E$'s project choice; in particular, it may be optimal to use a higher than minimal liquidation threat to ensure that $E$'s promise to choose a certain investment is credible.

## 5. Choice of scale

■ In this section we argue that even though $E$'s incentives after writing a debt contract differ from his incentives *ex ante,* he can commit to his *ex ante* second-best scale of investment by borrowing exactly the amount required to finance it. It then follows that a simple debt contract is optimal in this setting.

As a benchmark, consider the case of contractible investment. If $k$ is contractible and a simple debt contract is used, $E$'s maximization problem is

$$\max_{k,W,D} \quad y(k) - D + \pi + W - k \tag{10}$$

$$\text{subject to} \quad \int_0^{\frac{D}{y(k)+W-k}} [\theta y(k) + W - k] f(\theta) \mathrm{d}\theta + \left[1 - F\left(\frac{D}{y(k) + w - k}\right)\right] D \geq W, \tag{11}$$

$$k \leq W. \tag{12}$$

The left-hand side in (11) is $I$'s total expected repayment from $E$: if $\theta y(k) < D$, $E$ pays $\theta y(k)$, otherwise he pays $D$. The solution to (10)–(12) is stated in the next result.

*Proposition 3.* With contractible $k$ and a simple debt contract, it is optimal for $E$ to choose $k = W < k^{FB}$.

*Proof.* See the Appendix.

The result $k = W$ follows from the fact that $E$ has nothing to gain from borrowing any more than he wants to invest. Moreover, $E$ underinvests, i.e., chooses $k < k^{FB}$, because he stands to lose $\pi$ with some probability if his revenue falls below $D$. By borrowing less, he can reduce $D$ and thus the likelihood of default. While an entrepreneur with sufficient own funds maximizes his current-period profit by choosing $k^{FB}$, an entrepreneur relying on debt finance balances this incentive against the prospect of losing his future benefit, $\pi$, and hence prefers to invest less than $k^{FB}$.

Now consider the case of unobservable $k$. Proposition 1 and the financing constraint $k \leq W$ immediately imply the following.

*Corollary 1.* Under a simple debt contract $(W, T, \bar{\beta})$, $E$ chooses the investment $k^*(W, T, \bar{\beta}) = \min\{W, k^{FB}\}$.

Assuming use of a simple debt contract, we can then write $E$'s maximization problem as

$$\max_{k,W,D} \quad y(k) - D + \pi + W - k \tag{13}$$

$$\text{subject to} \quad \int_0^{\frac{D}{y(k)}} \theta y(k) f(\theta) \mathrm{d}\theta + \left[1 - F\left(\frac{D}{y(k)}\right)\right] D \geq W, \tag{14}$$

$$k = \min\{W, k^{FB}\}. \tag{15}$$

The only difference between (10)–(12) and (13)–(15) is that in the unobservable case, the constraint regarding the choice of $k$ is more restrictive. But since the optimal solution to (10)–(12) satisfies $k = W < k^{FB}$, the same solution also satisfies $k = \min\{W, k^{FB}\}$. That is, with unobservable investment and a simple debt contract, the outcome and payoff is the same for $E$ as with contractible investment. Then we have the following.

*Corollary 2.* With unobservable choice of scale $k$, a simple debt contract is optimal.

Intuitively, even when $k$ is not contractible, $E$ can in effect commit himself to the second-best level of $k$ given by the solution of (10)–(12) by borrowing $W = k$. *Ex post,* the financing constraint $k \leq W$ prevents upward deviations, while $E$'s first-best incentives rule out downward deviations. Given that $E$ will choose the right investment in stage 2, a simple debt contract is optimal because it minimizes the expected liquidation loss.

To conclude, in the model discussed so far, $E$ can in effect commit himself to any scale below the first-best level by limiting his borrowing. Since with contractible investment a simple debt contract is optimal and leads to a choice of scale below the first-best level, it follows that unobservability of $E$'s investment does not pose any further constraints, and that a simple debt contract is optimal. The question then is whether a simple debt contract is still optimal when the financing constraint $k \leq W$ no longer allows $E$ to commit himself to an investment at the time of borrowing. We study this question in the next two sections.

## 6. Managerial effort

■ Suppose that $E$'s unobservable decision after signing a contract with $I$ concerns not how much of the borrowed funds $W$ to invest, but instead how much effort to exert. We show that in this case, a nonsimple debt contract may be optimal.

Suppose that $E$ must raise a fixed amount $k$ from $I$ to finance a project. As before, it is trivial that $E$ will not borrow more than $k$, so let $W = k$. After investing $k$, he exerts managerial effort that is unobservable to $I$. Assume that $E$ can choose from two different effort levels, $e_h$ and $e_\ell$, with $e_h > e_\ell$. Exerting effort $e_i$ causes $E$ disutility of $e_i$ and leads to revenue $\tilde{\theta} y_i$ with $y_h > y_\ell$; the expected revenue therefore is $\int_0^{\bar{\theta}} \theta y_i f(\theta) \mathrm{d}\theta = y_i$.

To make the comparison interesting, assume that $y_\ell - e_\ell > y_h - e_h$, i.e., exerting *low* effort is first best. With a simple debt contract, $E$'s expected payoff upon choosing $e_i$ is $y_i - e_i + \pi - D$, which is analogous to (A3) in the Appendix except for one difference: while $E$ pays for the pecuniary costs of the project by borrowing $W = k$, he incurs the disutility $e_i$ directly. Proposition 1 nevertheless holds: under a simple debt contract, $E$ has first-best incentives and therefore chooses $e_\ell$.

With debt finance, however, this may not be optimal. To see this, notice that if $E$'s effort $e_i$ were contractible, $E$ and $I$ would write a simple debt contract with a promised repayment $D_i$ that solves $I$'s individual-rationality constraint (11), i.e.,

$$y_i \int_{\underline{\theta}}^{D_i/y_i} \theta f(\theta) \mathrm{d}\theta + [1 - F(D_i/y_i)] D_i - k = 0. \tag{16}$$

Since the left-hand side of (16) is increasing in both $D_i$ and $y_i$, and since $y_h > y_\ell$, we must have $D_h < D_\ell$. It is then possible to find parameters such that $y_h - e_h - D_h > y_\ell - e_\ell - D_\ell$, i.e., that if effort were contractible and $E$ and $I$ used a simple debt contract, high effort would be optimal. The question then is whether a nonsimple debt contract can induce $E$ to choose $e_h$ if effort is unobservable, and whether $E$ can gain from switching to such a contract.

Notice that if the contract induces $E$ to choose $e_h$, his debt is $D_h$ as defined by (16), which does not depend on $\beta$. Recall that under a simple debt contract and debt $D_h$, $E$'s payoff from choosing $e_i$ is $y_i - e_i + \pi - D_h$. Under a nonsimple debt contract, he therefore chooses $e_h$ if

$$\begin{aligned} &y_h - e_h + \pi - D_h - \pi \int_0^{D_h/y_h} [\bar{\beta}(\theta y_h) - \beta(\theta y_h)] f(\theta) \mathrm{d}\theta \\ &\quad \geq y_\ell - e_\ell + \pi - D_h - \pi \int_0^{D_h/y_\ell} [\bar{\beta}(\theta y_\ell) - \beta(\theta y_\ell)] f(\theta) \mathrm{d}\theta, \end{aligned}$$

which after substituting $R = \theta y_i$ in the integrals can be rearranged to

$$\int_0^{D_h} [\bar{\beta}(R) - \beta(R)] \left[ f\left(\frac{R}{y_\ell}\right) \frac{y_h}{y_\ell} - f\left(\frac{R}{y_h}\right) \right] \mathrm{d}R \geq \frac{y_h}{\pi} [y_\ell - e_\ell - (y_h - e_h)]. \tag{17}$$

The optimal contract that induces $E$ to choose $e_h$ is then characterized by the function $\beta$ that minimizes the additional liquidation probability

$$\int_0^{D_h/y_h} [\bar{\beta}(\theta y_h) - \beta(\theta y_h)] f(\theta) \mathrm{d}\theta = \int_0^{D_h} [\bar{\beta}(R) - \beta(R)] f\left(\frac{R}{y_h}\right) \frac{1}{y_h} \mathrm{d}R, \tag{18}$$

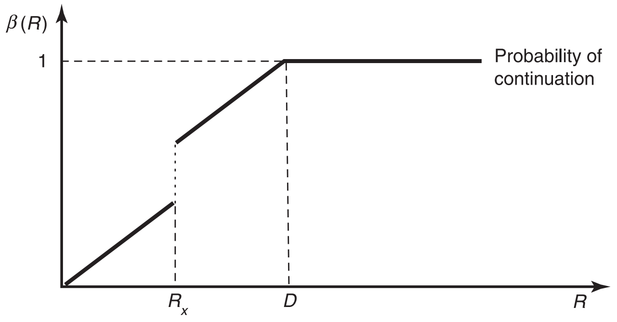
while satisfying (17) and the conditions of Proposition 2. The structure of an optimal nonsimple debt contract can be determined if we assume that the distribution of $R$ induced by $\tilde{\theta}$ and $e_i$ is log-concave and satisfies the monotone-likelihood-ratio property (MLRP). MLRP is a standard assumption in moral hazard models; log-concavity is a mild assumption that is satisfied for most commonly used distribution functions (see Bagnoli and Bergstrom, 1989). The assumptions translate into properties of $f$:

*Lemma 1.* Define $\tilde{R}_i = \tilde{\theta} y_i$ for $i \in \{\ell, h\}$, and denote by $G_i(R)$ and $g_i(R)$ the c.d.f. and density of $\tilde{R}_i$. Then

(i) $g$ is log-concave in $R$, i.e., $(\mathrm{d}g_i(R)/\mathrm{d}R)/g_i(R)$ is nonincreasing in $R$ for all $R \in [0, \bar{\theta} y_i]$ and $i \in \{h, \ell\}$, if and only if $f'(\theta)/f(\theta)$ is nonincreasing in $\theta$ for all $\theta \in [0, \bar{\theta}]$.

FIGURE 3

PROBABILITY OF CONTINUATION UNDER AN OPTIMAL NONSIMPLE DEBT CONTRACT




(ii) $g$ satisfies MLRP, i.e., $(d/dR)(g_\ell(R)/g_h(R)) \leq 0$ for all $R \in [0, \bar{\theta} y_\ell]$, if and only if $\theta f'(\theta)/f(\theta)$ is nonincreasing in $\theta$ for all $\theta \in [0, \bar{\theta}]$.

*Proof.* See the Appendix.

*Proposition 4.* Assume that $f'(\theta)/f(\theta)$ and $\theta f'(\theta)/f(\theta)$ are nonincreasing in $\theta$. Then a contract of the following form is optimal within the set of all contracts that satisfy (17):

$$\beta(R) = \begin{cases} \dfrac{R}{\pi} & \text{if } R < R_x \\[2ex] 1 - \dfrac{D_h - R}{\pi} & \text{if } R \in [R_x, D_h] \\[2ex] 1 & \text{if } R > D_h, \end{cases} \tag{19}$$

where $R_x \in [0, D_h]$ is the smallest value of $R$ such that

$$F\left(\frac{R}{y_\ell}\right) - F\left(\frac{R}{y_h}\right) \geq \frac{y_\ell - e_\ell - y_h + e_h}{\pi - D_h}. \tag{20}$$

*Proof.* See the Appendix.

A function $\beta$ satisfying (19) is depicted in Figure 3. Below some $R_x \in (0, D_h)$, $\beta(R)$ drops from its maximal to its minimal level consistent with incentive compatibility. Intuitively, the MLRP implies that the lower the realized $R$, the higher the likelihood that low effort was invested. It is then optimal to impose the maximum punishment when revenue is low, and the minimum punishment when it is higher.

A contract with the properties of Proposition 4 need not exist; i.e., it may not be possible to find any contract that induces $E$ to choose $e_h$. Moreover, even if it is possible to induce $e_h$ rather than $e_\ell$, the gain of doing so may not be worth the increase in the expected liquidation loss. $E$ prefers a nonsimple contract inducing $e_h$ over a simple debt contract inducing $e_\ell$ if and only if $y_h - e_h - D_h - F(R_x/y_h)(\pi - D_h) \geq y_\ell - e_\ell - D_\ell$, or

$$F(R_x/y_h) \leq \frac{y_h - e_h - D_h - (y_\ell - e_\ell - D_\ell)}{\pi - D_h}. \tag{21}$$

Both (20) and (21) need to be satisfied for a nonsimple contract to be optimal. Since both the gain of choosing a higher effort level and the additional liquidation loss necessary to induce it are first-order effects, it depends on the parameters of the problem whether a simple or a nonsimple contract is optimal.

To illustrate that both cases can occur, consider the following numerical example. Assume that $\theta$ is uniform on [0,2], and let $k = 2$, $\pi = 4$, $y_h = 4$, $y_\ell = 3$, and $e_\ell = 0$. Then $e_\ell$ is the first-best effort level if $e_h > 1$. Solving (16) for each level of effort leads to $D_h = 2.343$ and $D_\ell = 2.536$. There are four intervals of $e_h$ to distinguish. If $e_h \in (1, 1.048)$, an $R_x$ satisfying (20) and (21) exists, hence a nonsimple contract inducing $e_h$ is optimal. If $e_h \in (1.048, 1.162)$, an $R_x$ satisfying (20) exists, but (21) is violated. That is, a nonsimple debt contract can induce $E$ to choose $e_h$, but the additional liquidation loss required is too costly; hence a simple debt contract is optimal. If $e_h \in (1.162, 1.193)$, $E$ would still want to commit to $e_h$ if he could do so under a simple debt contract. However, no contract can induce $e_h$ if effort is unobservable, and therefore a simple debt contract is optimal. Finally, if $e_h > 1.193$, $E$ would not want to choose $e_h$ even if he could commit to it.

## 7. Choice of risk

■ When $E$ chooses between projects that differ in their risk rather than their cost, the conclusions are similar to those for the case of unobservable effort. Assume that $E$ can choose between two projects $a_h$ and $a_\ell$ that each cost $k$. The projects may differ in both the expected value and the riskiness of their revenue. The riskiness of revenue is captured by the distribution of $\tilde{\theta}$, which now depends on the project chosen; for $i = h, \ell$, denote by $F_i(\theta)$ and $f_i(\theta)$ the cumulative denstity function and density of $\tilde{\theta}$. Project $a_i$ leads to revenue $\tilde{\theta} y_i$, where $E(\tilde{\theta}) = 1$; therefore the expected revenue is given by $\int_0^{\bar{\theta}} \theta y_i f_i(\theta) \mathrm{d}\theta = y_i$.

If $E$'s project were contractible and he wanted to choose $a_i$, he would offer a simple debt contract such that his debt $D_i$ solves (16) (with $f_i$ in place of $f$). To make the comparison interesting, assume that $a_h$ has a higher expected return ($y_h > y_\ell$) and is riskier than $a_\ell$. With unobservable investment and a simple debt contract, $E$ would then choose $a_h$.

Because of the higher risk, however, it is possible that $D_h > D_\ell$ and even $y_\ell - D_\ell + \pi > y_h - D_h + \pi$, i.e., that if $E$'s project were contractible, he would choose $a_\ell$. We then have an unusual form of asset substitution (see Jensen and Meckling, 1976): $E$ would like to commit *not* to choose the first-best investment, because it is too risky, but under a simple debt contract this is not possible. $E$ and $I$ may then prefer a nonsimple debt contract.

The analysis is similar to that in Section 6. A nonsimple debt contract with continuation function $\beta$ induces $E$ to choose $a_\ell$ if

$$\int_0^{D_\ell} [\bar{\beta}(R) - \beta(R)] \left[ f_h\left(\frac{R}{y_h}\right)\frac{y_\ell}{y_h} - f_\ell\left(\frac{R}{y_\ell}\right)\right] \mathrm{d}R \geq \frac{y_\ell\,(y_h - y_\ell)}{\pi}, \tag{22}$$

(the derivation is similar to that of (17)). As before, additional assumptions about the distribution of revenue allow us to determine the optimal contract. We use two equivalence results analogous to those in Lemma 1. First, $g_\ell(R)$, the density of revenue with project $a_\ell$, is log-concave if and only if $f_\ell'(\theta)/f_\ell(\theta)$ is nonincreasing in $\theta$. Second, assume that $a_\ell$ is less risky than $a_h$ in the sense that for low $R$, $a_\ell$ is the more favorable project in the sense of MLRP (this property cannot hold for all $R$, since $y_h > y_\ell$): $(\mathrm{d}/\mathrm{d}R)(g_h(R)/g_\ell(R)) \leq 0$ for all $R \in [0, D_\ell/y_\ell]$. This holds if and only if $f_h(\theta y_\ell/y_h)/f_\ell(\theta)$ is nonincreasing in $\theta$ for all $\theta \leq D_\ell/y_\ell$.[7] We then have the following.

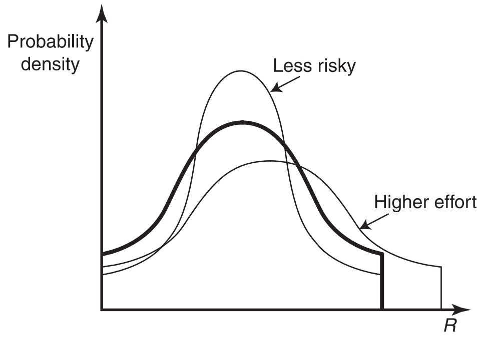
*Proposition 5*. Assume that $f_\ell'(\theta)/f_\ell(\theta)$ is nonincreasing in $\theta$ for all $\theta \in [0, \bar{\theta}]$, and that $f_h(\theta y_\ell/y_h)/f_\ell(\theta)$ is nonincreasing in $\theta$ for all $\theta \in [0, D_\ell/y_\ell]$. Then a contract of the following form is optimal within the set of all contracts that satisfy (22):

$$\beta(R) = \begin{cases} \dfrac{R}{\pi} & \text{if } R < R_x \\[2ex] 1 - \dfrac{D_\ell - R}{\pi} & \text{if } R \in [R_x, D_\ell] \\[2ex] 1 & \text{if } R > D_\ell, \end{cases} \tag{23}$$

[7] The MRLP holds if and only if $g_h(R)/g_\ell(R) = [f_h(R/y_h)/y_h]/[f_\ell(R/y_\ell)/y_\ell]$ is decreasing in $R$, or equivalently, if $f_h(R/y_h)/f_\ell(R/y_\ell)$ is decreasing in $R$. Substituting $\theta = R/y_\ell$ then leads to the stated condition.

FIGURE 4

PROBABILITY DISTRIBUTIONS WITH CHANGED RISK OR EFFORT




where $R_x \in [0, D_\ell]$ is the smallest value of $R$ such that

$$F_h\left(\frac{R}{y_h}\right) - F_\ell\left(\frac{R}{y_\ell}\right) \geq \frac{y_h - y_\ell}{\pi - D_\ell}. \tag{24}$$

*Proof.* See the Appendix.

If (22) can be satisfied, then the optimal contract according to Proposition 5 leads to a liquidation probability *in excess* of that required by a simple debt contract of $F_\ell(R_x/y_\ell)$. $E$ therefore prefers a nonsimple debt contract that induces him to choose $a_\ell$ over a simple contract that induces $a_h$ if

$$y_\ell - D_\ell + \pi - F_\ell(R_x/y_\ell)(\pi - D_\ell) \geq y_h - D_h + \pi. \tag{25}$$

To illustrate, suppose that $\tilde{\theta}_i$ has a support of [0, 2], with $f_\ell(\theta) = 3\theta(2 - \theta)/4$ and hence $F_\ell(\theta) = \theta^2(3 - \theta)/4$, and $f_h(\theta) = 1/2$ and hence $F_h(\theta) = \theta/2$. Thus, $a_h$ is the riskier project, and $f_h(\theta)/f_\ell(\theta)$ is decreasing in $\theta$ for $\theta \leq 1$. Let $k = 2$, $y_\ell = 4$, and $\pi = 4$. Using (16), we obtain $D_\ell = 2.13$, and $D_h$ is given by the solution of $D_h - D_h^2/(4y_h) = 2$. If $y_h > 4$, $a_h$ is first best. The larger is $y_h$, the more attractive is $a_h$, and therefore the larger $R_x$ must be to induce $E$ to choose $a_\ell$ instead.

There are four intervals to distinguish: if $y_h < 4.135$, there exists $R_x$ such that both (24) (and hence (22)) and (25) hold; that is, a nonsimple debt contract is optimal. If $y_h \in (4.135, 4.15)$ then an $R_x$ that satisfies (24) exists, but (25) does not hold. That is, the benefit of inducing $E$ to choose a low-risk project is not worth the additional liquidation loss, and a simple debt contract is optimal. If $y_h \in (4.15, 4.19)$, it is no longer possible to induce $a_\ell$ under a nonsimple debt contract, and a simple debt contract is optimal. However, $a_\ell$ is still better in the sense that if the project were contractible, $E$ would choose $a_\ell$. Finally, if $y_h > 4.19$, $a_h$ is optimal even if $E$ could commit to $a_\ell$ in the contract.

Figure 4 illustrates the parallels between effort and risk choice. The thick curve depicts the density of revenue if $E$ chooses the first-best investment. Alternatively, $E$ might choose a less risky project with the same expected revenue, or exert more effort and thereby stretch the distribution to the right. If $\hat{\theta}$ is small enough for the assumptions of Proposition 5 to hold (which means that $k$ must be small enough), then, given a simple debt contract, each of the two alternatives is more efficient as it leads to a lower probability of liquidation. However, the choice of a more efficient investment can be induced only at the cost of increasing the liquidation probability, which may be too high a price to pay.

## 8. Conclusion

■ The existing financial contracting literature explains how debt can emerge as an optimal contract when there is asymmetric information between a firm and an outside investor, and the firm's investment is given. Capital structure theory, in contrast, emphasizes the distortionary impact of debt on firms' investment decisions, and attributes the costs of using debt to these distortions. The properties of a debt contract are taken as given. We have combined these two approaches and have shown that contract design and a firm's incentives to invest are linked in a complex way that has previously been ignored.

The implications of our analysis differ considerably from those of models in which debt is used for exogenous reasons. Debt finance is costly, not primarily because of the distortions induced by debt, but because debt may lead to liquidation. It is therefore generally efficient for firms to "underinvest" if they have to rely on debt finance.[8] Furthermore, any distortions that do arise are distortions relative to the second-best investment, not the first best. Firms may therefore prefer contracts that *ex post* restrict their choice of investment, even though (or rather, because) the investments they would choose without such covenants are the same that internally financed firms would choose.

These contrasts with the earlier capital structure literature highlight the relevance of the financial contracting approach used here. The results of Jensen and Meckling (1976) and Myers (1977) are based on characteristics of financial contracts that are assumed rather than derived. For example, asset substitution arises if an entrepreneur is the residual claimant to his earnings in good states of the world but not in bad ones. Focusing on the repayment structure leads to an incomplete description of debt, however, because an entrepreneur has an incentive to repay only if otherwise he risks losing control over his assets. A debt relationship can be explained only by constraints on feasible contracts, and investors must be able to seize control of the firm under certain conditions for financing to be feasible in the first place. Hence we need to analyze the consequences of debt in a setting in which debt is actually optimal.

This article is only a first step in studying the interaction between optimal contracting and investment in the presence of a double moral hazard problem, and we have made a number of simplifying assumptions. It remains to be seen to what extent our results generalize when some of these assumptions are relaxed, for example if we allow for risk-averse players or embed the problem studied here in a dynamic framework.

### Appendix

■ Proofs of Lemma 1 and Propositions 2, 3, 4, and 5 follow. We discuss how our results change if we allow $\pi$ to depend on $k$ in the web Appendix, available at http://www.rje.org/main/sup-mat.html.

*Proof of Proposition 2.* The proof proceeds in four steps.

*Step 1: For all pairs* $(R, R')$, $R \neq R'$,

$$\beta(R) = \beta(R') \Longleftrightarrow T(R) = T(R') \tag{A1}$$

$$\beta(R) > \beta(R') \Longleftrightarrow T(R) > T(R'). \tag{A2}$$

To prove (A1), assume without loss of generality $R > R'$. Suppose $\beta(R) = \beta(R')$. Because of limited liability, $T(R') \leq R' < R$, and (1) implies $T(R) \leq T(R')$. We therefore have $T(R) \leq T(R') \leq R'$, and (1) implies $T(R') \leq T(R)$. It follows that $T(R) = T(R')$. Conversely, if $T(R) = T(R')$ but $\beta(R) \neq \beta(R')$, then (1) is obviously violated.

To prove (A2), consider any $(R, R')$ for which $\beta(R) > \beta(R')$. If $T(R) > R'$, then limited liability implies $T(R) > T(R')$. If $T(R) \leq R'$, then (1) implies the same. Conversely, if $T(R) > T(R')$ and therefore $T(R') < R$, (1) implies that $\beta(R) > \beta(R')$.

*Step 2:* $\sup_R \beta(R) = 1$, *and* $\sup_R T(R) = D$ *for some* $D \leq \pi$. If in contrast $\sup_R \beta(R) < 1$, construct a new contract $(W, T, \beta^1)$ such that $\beta^1(R) = \beta(R) + 1 - \sup_R \beta(R)$; this leaves $I$'s payoff unchanged and increases $E$'s. Since $\hat{R} = 0$

[8] In Povel and Raith (2002), we analyze how a firm's investment depends on its internal funds and the information asymmetry between firm and investor, and relate our results to the empirical literature (see also Hubbard, 1998).

is always a possible announcement, (1) implies $T(R) \leq \pi$ for all $R$. Step 1 implies that for all $R$ such that $\beta(R) = 1$, we have $T(R) = D$.

*Step 3: $\beta(R) < 1$ implies $T(R) = R$.* Suppose not. Then there must exist a nonempty set $\rho = \{R \mid \beta(R) < 1$ and $T(R) < R\}$, which we can decompose into the subsets

$$\rho_a = \left\{ R \,\middle|\, R \in \rho \text{ and } \beta(R) + \frac{R - T(R)}{\pi} \leq 1 \right\}$$

and

$$\rho_b = \left\{ R \,\middle|\, R \in \rho \text{ and } \beta(R) + \frac{R - T(R)}{\pi} > 1 \right\}.$$

Define a new contract $(W, T^1, \beta^1)$ by

$$T^1(R) = \min\{T(R) + [1 - \beta(R)]\pi, R\} \quad \text{and} \quad \beta^1(R) = \min\left\{1, \beta(R) + \frac{R - T(R)}{\pi}\right\}.$$

Let $u^0(R, \hat{R}) = R - T(\hat{R}) + \beta(\hat{R})\pi$ denote $E$'s expected payoff in stage 4 under $(W, T, \beta)$ if his funds are $R$ and he announces $\hat{R}$, and define $u^1(R, \hat{R})$ analogously for $(W, T^1, \beta^1)$.

For all $R \notin \rho$, $T^1(R) = T(R)$ and $\beta^1(R) = \beta(R)$, and therefore $u^1(R, \hat{R}) = u^0(R, \hat{R})$ for all $\hat{R} \notin \rho$. Moreover, it is straightforward to show that $u^1(R, \hat{R}) = u^0(R, \hat{R})$ for both $\hat{R} \in \rho_a$ and $\hat{R} \in \rho_b$. It follows that $u^1(R, \hat{R}) = u^0(R, \hat{R})$ for all $(R, \hat{R})$, i.e., $E$'s payoff is the same under both contracts for any $R$, whether he announces it truthfully or not.

Since $T^1(R) \geq T(R)$, any announcement feasible under $(W, T^1, \beta^1)$ is also feasible under $(W, T, \beta)$. Consequently, since $(W, T, \beta)$ is incentive compatible, so is $(W, T^1, \beta^1)$.

Since $u^1(R, R) = u^0(R, R)$ for all $R$ irrespective of the $k$ chosen in stage 2, $k^*(W, T^1, \beta^1) = k^*(W, T, \beta)$, and therefore $E$'s *ex ante* expected payoff is the same too. $I$'s expected payoff is strictly higher under $(W, T^1, \beta^1)$ because $E$ repays strictly more in expected terms, while increasing $\beta$ does not affect $I$'s payoff. Denote the increase in $I$'s payoff by $\delta$.

Now define a new contract $(W, T^2, \beta^2)$ by $T^2(R) = T^1(R) - \delta$ and $\beta^2(R) = \beta^1(R)$. ($T^2(R)$ may be negative, implying a payment from $I$ to $E$.) Since for any $k$ and any realized $\theta$, $E$'s payoff under $(W, T^2, \beta^2)$ exceeds his payoff under $(W, T^1, \beta^1)$ by a constant, $k^*(W, T^2, \beta^2) = k^*(W, T^1, \beta^1) = k^*(W, T, \beta)$. By construction of $\delta$, $(W, T^2, \beta^2)$ satisfies $I$'s individual-rationality constraint, while $E$'s payoff is higher than under $(W, T^1, \beta^1)$. Thus, $(W, T^2, \beta^2)$ satisfies all constraints and provides a strictly greater payoff to $E$. It follows that $(W, T, \beta)$ cannot have been optimal.

*Step 4: Conditions (7)–(9) hold.* To prove (7), suppose $R \geq D$. Then if $\beta(R) < 1$, Step 3 implies $T(R) = R \geq D$. Since $D = \sup'_R T(R')$, we must have $T(R) = D$ and, from Step 1, $\beta(R) = 1$, a contradiction. Therefore, $\beta(R) = 1$ and hence $T(R) = D$. To prove (8), suppose $R < D$. If $\beta(R) = 1$, then $T(R) = D > R$, which is not possible. Hence $\beta(R) < 1$, and Step 3 implies $T(R) = R$. Finally, for all $(R, R')$ such that $R' < R < D$, (1) implies $R - R + \beta(R)\pi \geq R - R' + \beta(R')\pi$, which is equivalent to (9). *Q.E.D.*

*Proof of Proposition 3.* (i) First, we argue that we can restrict attention to solutions of (10)–(12) that satisfy $k = W \leq k^{FB}$. To see this, consider any given $W \leq k^{FB}$. For any $k \leq W \leq k^{FB}$, both (10) and the left-hand side of (11) are increasing in $k$. It follows that $k = W$ maximizes $E$'s profit while making (11) least restrictive. For any $W > k^{FB}$, both (10) and the left-hand side of (11) are maximized at $k = k^{FB}$. Moreover, if $W$ is varied within the range $W > k^{FB}$, the smallest $D$ solving (11) varies in the same way, leaving (10) unchanged. It follows that there is nothing to gain for $E$ from borrowing $W > k^{FB}$, and thus we can restrict attention to $W \leq k^{FB}$.

Moreover, it is clear that in the solution to (10)–(12), (11) will always be binding. We can thus write $E$'s problem (10)–(12) more simply as

$$\max_{k \leq k^{FB}} y(k) - D + \pi, \tag{A3}$$

where $D$ satisfies

$$\int_0^{D/y(k)} \theta y(k) f(\theta) \mathrm{d}\theta + [1 - F(D/y(k))]D = k. \tag{A4}$$

(ii) The program (A3)–(A4) has a unique solution. For $k = 0$, $D = 0$ is the only solution. For $k > 0$, the left-hand side of (A4) is zero for $D = 0$ and is strictly increasing in $D$ (with slope $1 - F(D/y(k))$) for all $D < \bar{\theta} y(k)$. If the left-hand side is less than $k$ even for $D = \bar{\theta} y(k)$, its maximal possible value, then $k$ cannot be financed. It follows that there exists a unique solution to (A4) in $D$ for any feasible $k$; i.e., $D(k)$ is a well-defined function. Since $k^{FB}$ is well defined, (A3)–(A4) has a unique solution if $D(k)$ given by (A4) has a slope of at least one and is convex in $k$. That is the case: write (A4) as

$$\int_0^{D/y(k)} [\theta y(k) - D] f(\theta) \mathrm{d}\theta + D - k = 0. \tag{A5}$$

Equation (A5) is twice differentiable in both $k$ and $D$, and so $D(k)$ implicitly defined by (A5) exists and is twice differentiable. Differentiating (A5) twice with respect to $k$ yields (omitting arguments)

$$0 = \int_0^{D/y(k)} [\theta y'' - D''] f(\theta) \mathrm{d}\theta - f(D/y) \frac{(Dy' - D'y)^2}{y^3} + D''$$
$$= \int_0^{D/y(k)} \theta y'' f(\theta) \mathrm{d}\theta - f(D/y) \frac{(Dy' - D'y)^2}{y^3} + [1 - F(D/y)] D''. \tag{A6}$$

Since the first two terms in (A6) are negative, $D''$ must be positive. Finally, the integral in (A5) is negative because it is evaluated for $\theta \le D/y(k)$. (A5) therefore implies that $D > k$ for any $k > 0$. Since $D(k)$ is strictly convex and since $D(0) = 0$, we have $D'(k) > D(k)/k > 1$ for all $k > 0$.

(iii) The Lagrangian corresponding to the program (A3)–(A5) is

$$\mathcal{L}(k, D, \lambda) = y(k) - D + \pi + \lambda \left\{ y(k) \int_0^{D/y(k)} \theta f(\theta) \mathrm{d}\theta + \left[ 1 - F\left( \frac{D}{y(k)} \right) \right] D - k \right\},$$

which leads to the first-order conditions

$$\frac{\partial \mathcal{L}}{\partial k} = y'(k) + \lambda \left[ y'(k) \int_0^{D/y(k)} \theta f(\theta) \mathrm{d}\theta - 1 \right] = 0 \tag{A7}$$

and

$$\frac{\partial \mathcal{L}}{\partial D} = -1 + \lambda \left[ 1 - F\left( \frac{D}{y(k)} \right) \right] = 0. \tag{A8}$$

Eliminate $\lambda$ in (A7) using (A8); rearrange to obtain

$$[y'(k) - 1] + \frac{\int_0^{D/y(k)} [y'(k)\theta - 1] f(\theta) \mathrm{d}\theta}{1 - F(D/y(k))} = 0. \tag{A9}$$

Since $y'(k^{FB}) = 1$, the first term is nonpositive for $k \ge k^{FB}$, and the integrand in the second term then is strictly negative as the integral is taken over low values of $\theta$. It follows that (A9) is negative for all $k \ge k^{FB}$, and so we must have $k < k^{FB}$. *Q.E.D.*

*Proof of Lemma 1.* Given the definition of $\tilde{R}$, we have $G_i(R) = F(R/y_i)/y_i$, $g_i(R) = f(R/y_i)/y_i$, and $\mathrm{d}g_i(R)/\mathrm{d}R = f'(R/y_i)/[y_i]^2$. Part (i) then follows immediately. The MLRP holds if and only if $g_\ell(R)/g_h(R) = (y_h/y_\ell) f(R/y_\ell)/f(R/y_h)$ is nonincreasing in $R$, or

$$f'\left(\frac{R}{y_\ell}\right) \frac{1}{y_\ell} f\left(\frac{R}{y_h}\right) \le f'\left(\frac{R}{y_h}\right) \frac{1}{y_h} f\left(\frac{R}{y_\ell}\right) \quad \Longleftrightarrow \quad \frac{f'\left(\frac{R}{y_\ell}\right)}{f\left(\frac{R}{y_\ell}\right)} \frac{R}{y_\ell} \le \frac{f'\left(\frac{R}{y_h}\right)}{f\left(\frac{R}{y_h}\right)} \frac{R}{y_h},$$

which, since $y_h > y_\ell$, holds $\forall R \in [0, \bar{\theta} y_\ell]$ if and only if $\theta f'(\theta)/f(\theta)$ is nonincreasing in $\theta$ over $[0, \theta]$. *Q.E.D.*

*Proof of Proposition 4.* Suppose $(W, T, \beta)$ satisfies (17) but not (19). Without loss of generality, we can restrict attention to piecewise differentiable $\beta$. The contract can alternatively be described by the function $b$ with $b(R) = \bar{\beta}(R) - \beta(R)$. From Proposition 2, we must have $b(R) \in [0, 1 - D_h/\pi]$, and $b$ must be nonincreasing. Let $\rho = \{R \mid b(R) \in (0, 1 - D_h/\pi)\}$, which because of the properties of $b$ must be a nonempty interval. Suppose that $\rho$ has positive measure; otherwise the violation of (19) would be inconsequential.

Three cases can occur: either (1) $b$ is a step function over $\rho$, and $f$ is not uniform in the relevant range of $\theta$; or (2) $b$ is strictly decreasing over a subinterval of $\rho$, and $f$ is not uniform in the relevant range; or (3) (since $f$ is continuous) $f$ must be uniform over $\rho$. We show that in cases 1 and 2, $(W, T, \beta)$ is strictly suboptimal, while in case 3, $(W, T, \beta)$ can be replaced by a contract satisfying (19) that leads to the same payoff for $E$.

*Case 1.* There exist $R_0, R_1 \in \rho$ with $R_1 > R_0$, and $\delta_0, \delta_1 > 0$, such that $b(R) = b_0 \in (0, 1 - D_h/\pi)$ for all $R \in (R_0, R_1)$, $b(R) > b_0 + \delta_0$ for all $R < R_0$, and $b(R) < b_0 - \delta_1$ for all $R > R_1$, and such that $f'(R/y_h) \not\equiv 0 \forall R \in [R_0, R_1]$. Let $\Delta = (R_1 - R_0)/2$, and define $\rho_0 = \{R \in [R_0, R_0 + \Delta) \mid b(R) = b_0\}$ and $\rho_1 = \{R \in [R_0 + \Delta, R_1] \mid b(R) = b_0\}$, which may be open or closed at $R_0$ and $R_1$. Define, for $\varepsilon > 0$,

$$\tilde{b}(R) = \begin{cases} b(R) + \varepsilon & \text{for all } R \in \rho_0 \\ b(R) - \varepsilon \dfrac{f\left(\dfrac{R-\Delta}{y_h}\right)}{f\left(\dfrac{R}{y_h}\right)} & \text{for all } R \in \rho_1 \\ b(R) & \text{for all } R \notin \rho, \end{cases} \tag{A10}$$

with $\varepsilon$ small enough that $\tilde{b}(R) \leq b_0 + \delta_0 \; \forall R \in \rho_0$ and $\tilde{b}(R) \geq b_1 - \delta_1 \; \forall R \in \rho_1$. Then $\tilde{b}$ is nonincreasing over $\rho_1$ (and hence throughout) if for $x > 0$, $f(\theta - x)/f(\theta)$ is increasing in $\theta$, or

$$f'(\theta - x) f(\theta) - f'(\theta) f(\theta - x) \geq 0 \iff \frac{f'(\theta - x)}{f(\theta - x)} \geq \frac{f'(\theta)}{f(\theta)},$$

which is true because $f$ is log-concave. $E$ is indifferent between $\tilde{b}$ and $b$: in switching from $b$ to $\tilde{b}$, the probability of liquidation (see (18)) increases by

$$\int_0^{D_h} [\tilde{b}(R) - b(R)] f\left(\frac{R}{y_h}\right) \frac{1}{y_h} \mathrm{d}R = \int_{R \in \rho_0} \varepsilon f\left(\frac{R}{y_h}\right) \frac{1}{y_h} \mathrm{d}R - \int_{R \in \rho_1} \varepsilon f\left(\frac{R - \Delta}{y_h}\right) \frac{1}{y_h} \mathrm{d}R = 0, \tag{A11}$$

where the second equality follows after changing variables from $R \in \rho_1$ to $R' = R - \Delta \in \rho_0$ in the last integral. We now show that switching from $b$ to $\tilde{b}$ relaxes the incentive constraint (17), or equivalently, that

$$\int_0^{D_h} [\tilde{b}(R) - b(R)] \left[ f\left(\frac{R}{y_\ell}\right) \frac{y_h}{y_\ell} - f\left(\frac{R}{y_h}\right) \right] \mathrm{d}R \tag{A12}$$

is positive. Since $\tilde{b}$ and $b$ differ only between $R_0$ and $R_1$, (A12) equals

$$\varepsilon \int_{R \in \rho_0} \left[ f\left(\frac{R}{y_\ell}\right) \frac{y_h}{y_\ell} - f\left(\frac{R}{y_h}\right) \right] \mathrm{d}R - \varepsilon \int_{R \in \rho_1} \frac{f\left(\frac{R-\Delta}{y_h}\right)}{f\left(\frac{R}{y_h}\right)} \left[ f\left(\frac{R}{y_\ell}\right) \frac{y_h}{y_\ell} - f\left(\frac{R}{y_h}\right) \right] \mathrm{d}R. \tag{A13}$$

Substituting $R' = R - \Delta \in \rho_0$ for $R \in \rho_1$ in the second integral, (A13) simplifies to

$$\int_{R \in \rho_0} \varepsilon f\left(\frac{R}{y_h}\right) \left[ \frac{f\left(\frac{R}{y_h} \frac{y_h}{y_\ell}\right)}{f\left(\frac{R}{y_h}\right)} - \frac{f\left(\frac{R+\Delta}{y_h} \frac{y_h}{y_\ell}\right)}{f\left(\frac{R+\Delta}{y_h}\right)} \right] \mathrm{d}R. \tag{A14}$$

Since $\theta f'(\theta)/f(\theta)$ is nonincreasing in $\theta$, we have, for $a > 1$,

$$\frac{f'(a\theta) a\theta}{f(a\theta)} \leq \frac{f'(\theta)\theta}{f(\theta)} \iff a f'(a\theta) f(\theta) - f'(\theta) f(a\theta) \leq 0, \tag{A15}$$

which means that $f(a\theta)/f(\theta)$ must be decreasing in $\theta$. By assumption (for this case) $f$ is not uniform on $[R_0/y_h, R_1/y_h]$, so (A14) is strictly positive. Since with $\tilde{b}$, (17) is not binding, we can construct a function $\hat{b}$, such that $\hat{b} \leq \tilde{b}$ for all $R$ and $\hat{b} < \tilde{b}$ for a subset of $\rho$ with positive measure, and such that (17) and the conditions of Proposition 2 are satisfied. $I$'s payoff is unaffected, but $E$'s increases; so the original contract cannot have been optimal.

*Case 2.* There exist $R_0, R_1 \in \rho$ with $R_1 > R_0$ such that for all $R \in (R_0, R_1)$, $b$ is differentiable with $b'(R) < 0$; $f(R/y_h)$ and $f'(R/Y_h)$ are finite; and $f'(\theta) \neq 0$ for all $\theta \in [R_0/y_h, R_1/y_h]$. Let $\Delta = (R_1 - R_0)/2$, $\rho_0 = [R_0, R_0 + \Delta)$, and $\rho_1 = [R_0 + \Delta, R_1]$. Define, for $\varepsilon > 0$,

$$\tilde{b}(R) = \begin{cases} b(R) + \varepsilon[b(R_0) - b(R)] & \text{for all } R \in \rho_0 \\ b(R) - \varepsilon[b(R_0) - b(R - \Delta)] \dfrac{f\left(\dfrac{R-\Delta}{y_h}\right)}{f\left(\dfrac{R}{y_h}\right)} & \text{for all } R \in \rho_1 \\ b(R) & \text{for all } R \notin \rho, \end{cases} \tag{A16}$$

with $\varepsilon$ small enough such that $\tilde{b}$ is nonincreasing over $(R_0, R_1)$. $E$ is again indifferent between $\tilde{b}$ and $b$; we omit the details. Substituting $\tilde{b}$ into (A12) leads to

$$\int_{R\in\rho_0} \varepsilon[b(R_0) - b(R)] f\left(\frac{R}{y_h}\right)\left[\frac{f\left(\frac{R}{y_h}\frac{y_h}{y_\ell}\right)}{f\left(\frac{R}{y_h}\right)} - \frac{f\left(\frac{R+\Delta}{y_h}\frac{y_h}{y_\ell}\right)}{f\left(\frac{R+\Delta}{y_h}\right)}\right] \mathrm{d}R, \tag{A17}$$

which is strictly positive; see the discussion of (A14) in case 1.

*Case 3.* There exist no $R_0, R_1 \in \rho$ such that $f'(\theta) \neq 0$ for all $\theta \in [R_0/y_h, R_1/y_h]$. Choose $R_x$ such that

$$\int_{R\in\rho} b(R) f\left(\frac{R}{y_h}\right) \mathrm{d}R = \left(1 - \frac{D_h}{\pi}\right)\left[F\left(\frac{R_x}{y_h}\right) - F\left(\frac{\inf\rho}{y_h}\right)\right],$$

and define $\tilde{\beta}$ by

$$\tilde{\beta}(R) = \begin{cases} R/\pi & \text{for all } R \in \rho,\ R < R_x \\ 1 - (D_h - R)/\pi & \text{for all } R \in \rho,\ R \geq R_x \\ \beta(R) & \text{for all } R \notin \rho. \end{cases}$$

Then $\tilde{\beta}$ satisfies (19). By construction of $R_x$, (18) assumes the same value for $\tilde{\beta}$ as for $\beta$. Since (17) holds for $\beta$ and $f$ is uniform, (17) also holds for $\tilde{\beta}$. Thus, a contract satisfying (19) performs as well as the original contract.

*In all three cases*, a contract satisfying (19) for some $R_x \in [0, D_h]$ is optimal. The left-hand side of (17) then simplifies to

$$\int_0^{R_x} \frac{\pi - D_h}{\pi} y_h \left[f\left(\frac{R}{y_\ell}\right)\frac{1}{y_\ell} - f\left(\frac{R}{y_h}\right)\frac{1}{y_h}\right]\mathrm{d}R = \frac{\pi - D_h}{\pi} y_h \left[F\left(\frac{R_x}{y_\ell}\right) - F\left(\frac{R_x}{y_h}\right)\right],$$

and therefore (17) is equivalent to (20) for $R_x = R$. The optimal $R_x$ is hence the smallest $R$ for which (20) holds. *Q.E.D.*

*Proof of Proposition 5.* The proof is almost identical to that of Proposition 4; we therefore only explain the necessary modifications. In case 1, let

$$\tilde{b}(R) = b(R) - \varepsilon \frac{f_\ell\left(\dfrac{R - \Delta}{y_\ell}\right)}{f_\ell\left(\dfrac{R}{y_\ell}\right)} \qquad \text{for all } R \in \rho_1, \tag{A18}$$

and similarly in case 2. As in the proof of Proposition 4, log-concavity of $f_\ell$ is sufficient to ensure that $\tilde{b}$ is nonincreasing, and $E$ is indifferent between $b$ and $\tilde{b}$. Similarly, the MLRP assumption ensures that switching from $b$ to $\tilde{b}$ relaxes the incentive constraint (22). *Q.E.D.*

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