Asset Valuation with known cash flows

– Annuities and Perpetuities

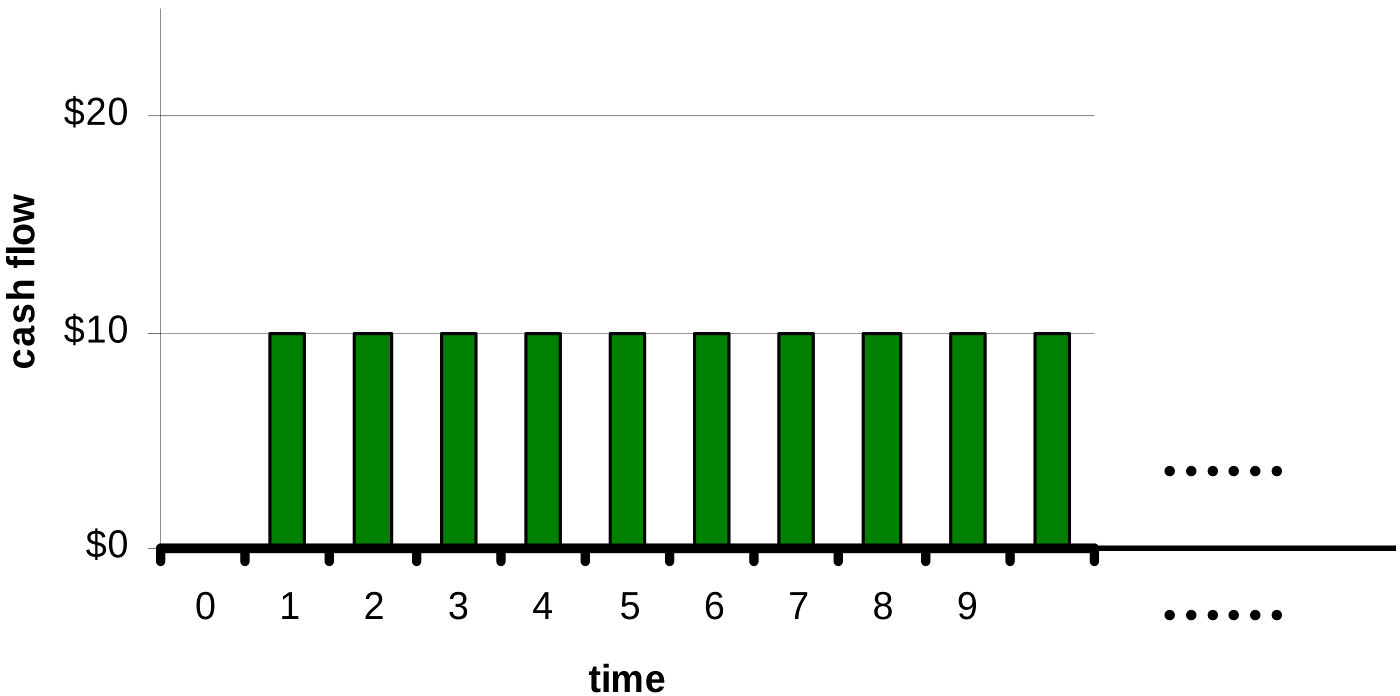
– care loan, saving for retirement, mortgage

Simple Perpetuity

A perpetuity is a stream of cash flows each of the amount of “CF” dollars, that are received at the end of each period forever

- Note:
  - Cash flows are the **same** over time
  - There is no cash flow today (i.e. you receive the first cash flow one period from now)

Simple Perpetuity



Valuing a perpetuity

The PV of a perpetuity is,

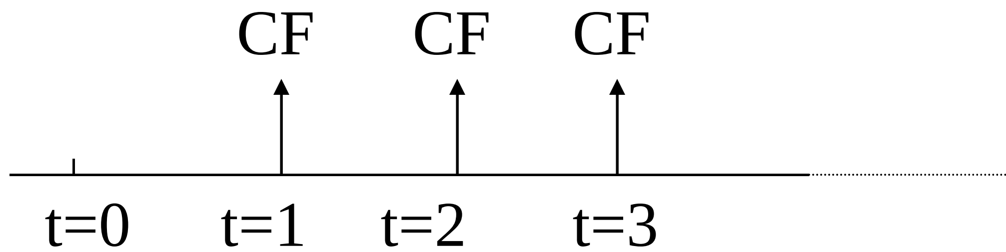
$$\mathrm{PV} = \frac{\mathrm{CF}}{1+\mathrm{r}} + \frac{\mathrm{CF}}{(1+\mathrm{r})^2} + \frac{\mathrm{CF}}{(1+\mathrm{r})^3} + \ldots$$

$$= \sum_{i=1}^{\infty} \frac{\mathrm{CF}}{(1+\mathrm{r})^{\mathrm{i}}} = \frac{CF}{r}$$

Valuing a perpetuity (cont'd)

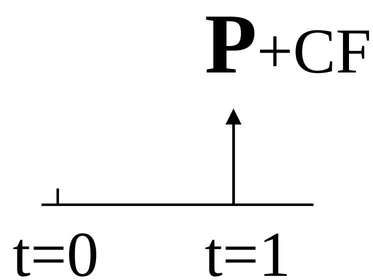
- *perpetuity*

P =



- *notice that…*

P =



$$P = \frac{P + CF}{1 + r} \Rightarrow P = \frac{CF}{r}$$

Example: You will receive $100 forever beginning the next year. The annual interest rate is 10%. Find PV.

PV = $100/0.1 = **$1,000**

Check:

If we invest $1,000 then we should be able to "replicate" the stream of cash flows generated by the perpetuity. That is by investing $1,000 today we should receive a payment of $100 each year forever.

<u>This is how we can do this:</u>

1) invest $1,000 today for one year
2) accumulate $1,000(1.1) = $1,100 in our bank account at time 1
3) Withdraw $100 from our bank account (we're left with $1000)
4) Invest the remaining $1,000 at time 1 for one additional year.

Do the same year after year…

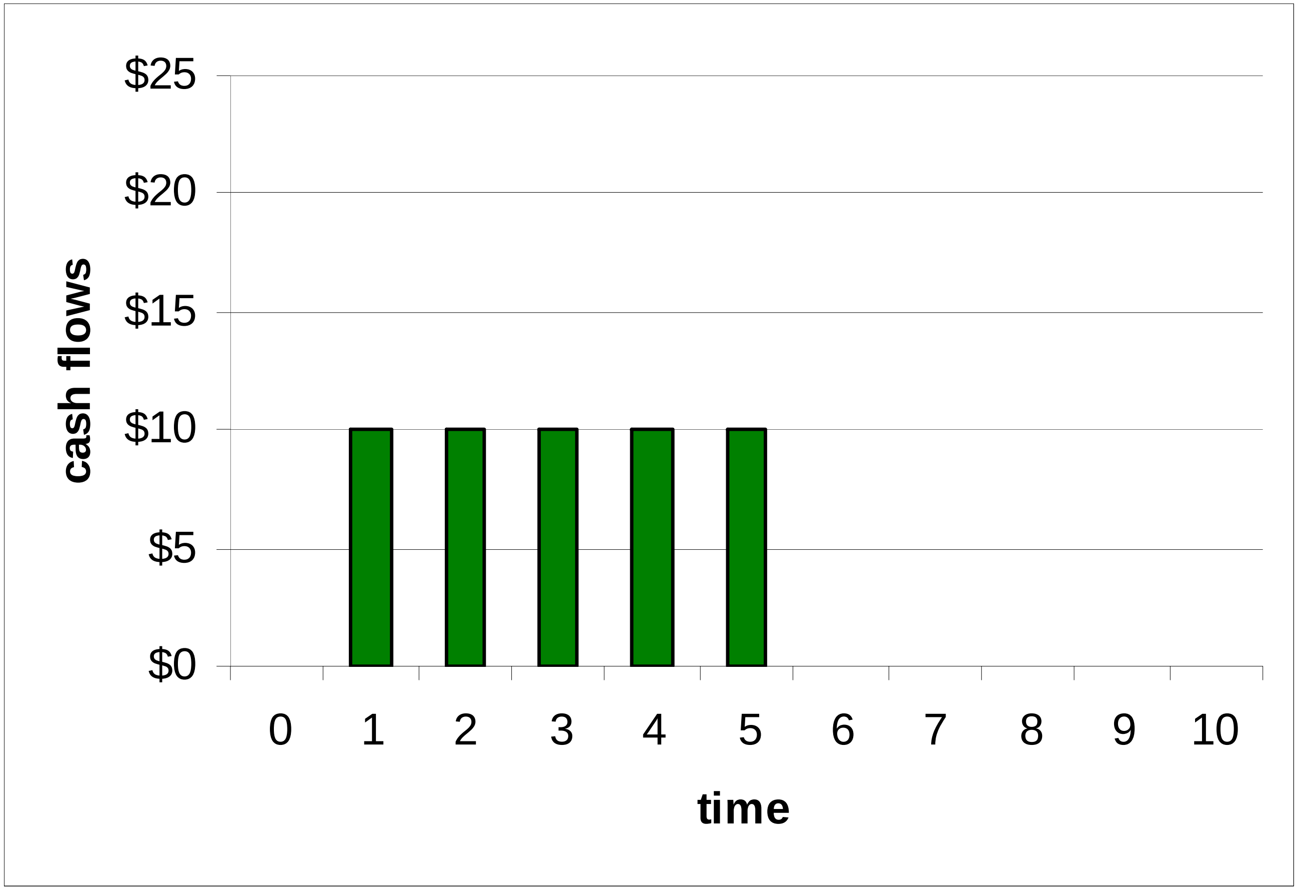
Simple Annuity

An annuity is a stream of cash flows each of the amount of "CF" dollars, that are received at the end of each period for the duration of "n" periods

Note:

- Cash flows are the **same** over time
- There is no cash flow today (i.e. you receive the first cash flow one period from now)

Simple five year Annuity



Simple annuity formula

The PV of an annuity for n years is,

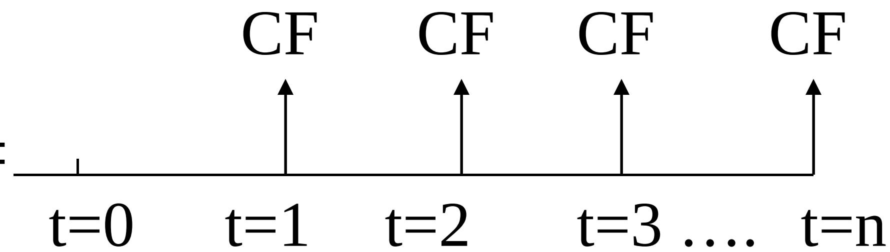
$$PV = \frac{CF}{1+r} + \frac{CF}{(1+r)^2} + \frac{CF}{(1+r)^3} + ... + \frac{CF}{(1+r)^n}$$

$$= \sum_{i=1}^{n} \frac{CF}{(1+r)^i} = \frac{CF}{r}\left(1 - \frac{1}{(1+r)^n}\right)$$

Valuing a simple annuity (cont'd)

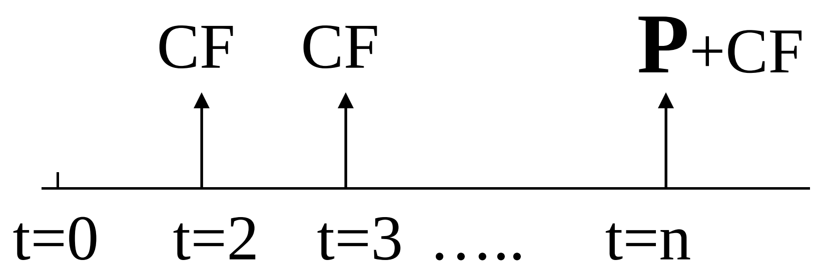
- *annuity*

a(n) =



- *notice that…*

P =



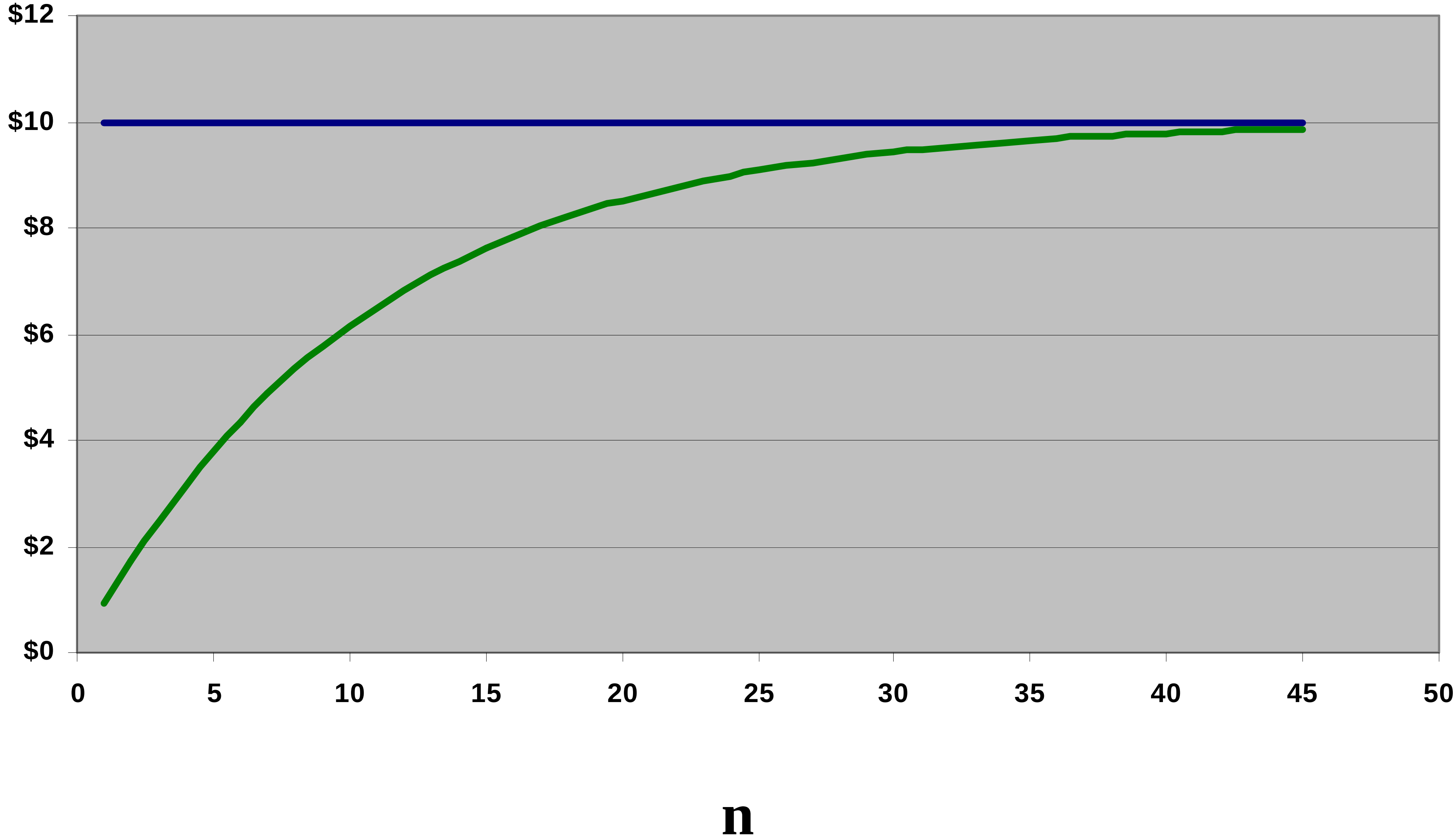
$$P = a(n) + \frac{P}{(1+r)^n} \Rightarrow a(n) = P\left(1 - \frac{1}{(1+r)^n}\right) = \frac{CF}{r}\left(1 - \frac{1}{(1+r)^n}\right)$$

Example: Find the present value of an annuity that pays $500 for the duration of 7 years (beginning at the end of the first year). The annual interest rate is 5%.

$$PV = \frac{CF}{r}\left(1 - \frac{1}{(1+r)^n}\right) =$$

$$= \frac{\$500}{0.05}\left(1 - \frac{1}{1.05^7}\right) = \$2,893.17$$

“n” year annuity versus perpetuity when r=10%

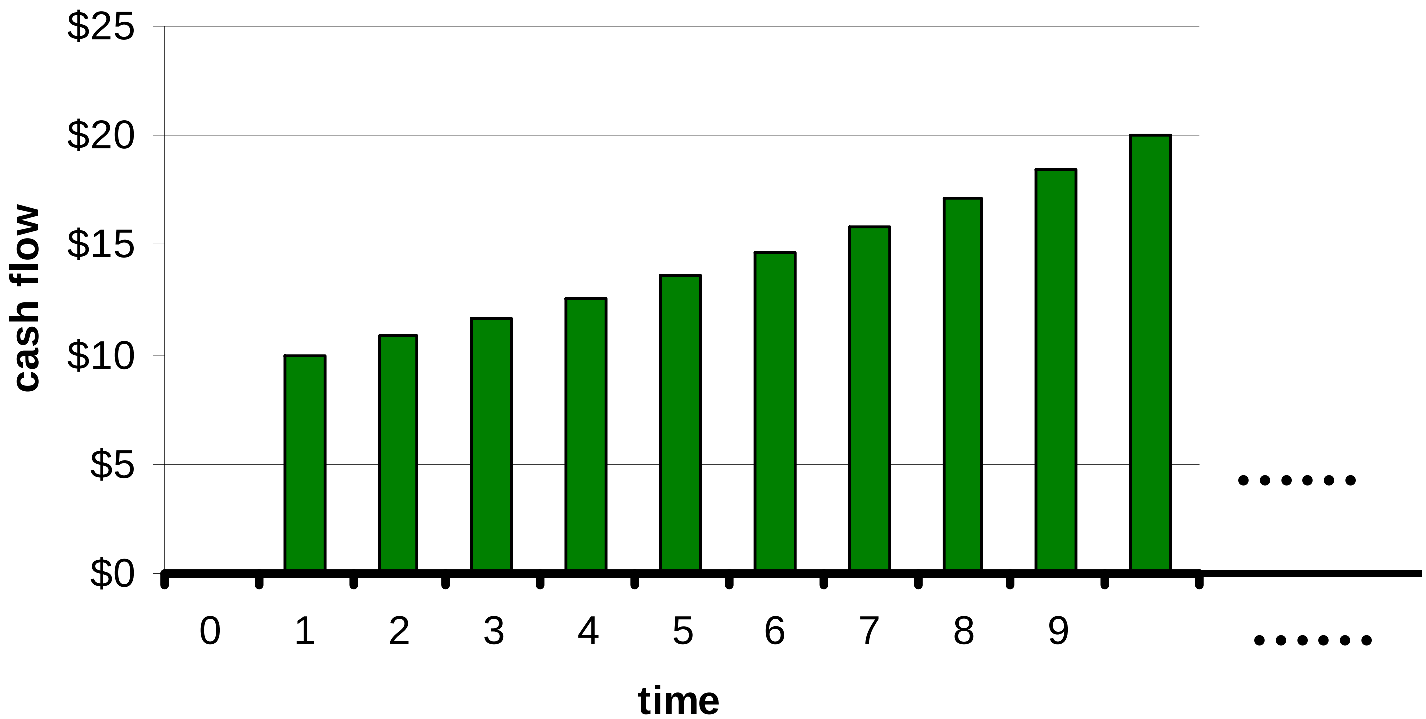


Growing perpetuity

A growing perpetuity is a stream of cash flows that grows over time with growth rate "g" where cash flows are received at the end of each period forever

- Note:
  - Cash flows grow over time with rate "g"
  - There is no cash flow today (i.e. you receive the first cash flow one period from now)

Growing perpetuity with growth rate of 8%



Growing perpetuity formula

- The first cash flow "CF" is received at the end of the first period and is growing at rate "g" afterwards

In particular, cash flows look like:

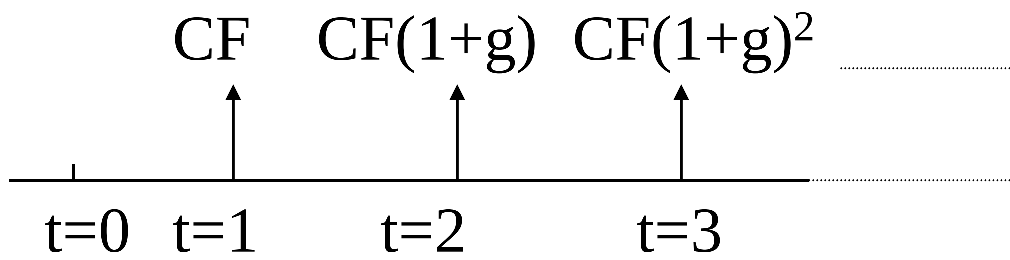
t=0	t=1	t=2	t=3	…..	t=n	…..
	CF	CF(1+g)	CF(1+g)^2	…..	CF(1+g)$^{n-1}$	…..

$$PV = \frac{CF}{1+r} + \frac{CF(1+g)}{(1+r)^2} + \frac{CF(1+g)^2}{(1+r)^3} + \ldots$$

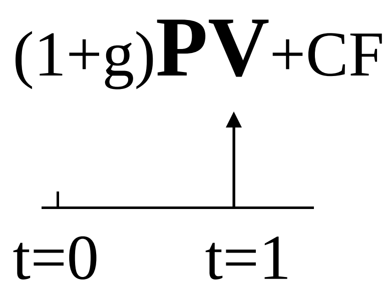
$$= \sum_{i=1}^{\infty} \frac{CF(1+g)^{i-1}}{(1+r)^i} = \frac{CF}{r-g}$$

Valuing a growing perpetuity (cont'd)

- *perpetuity*

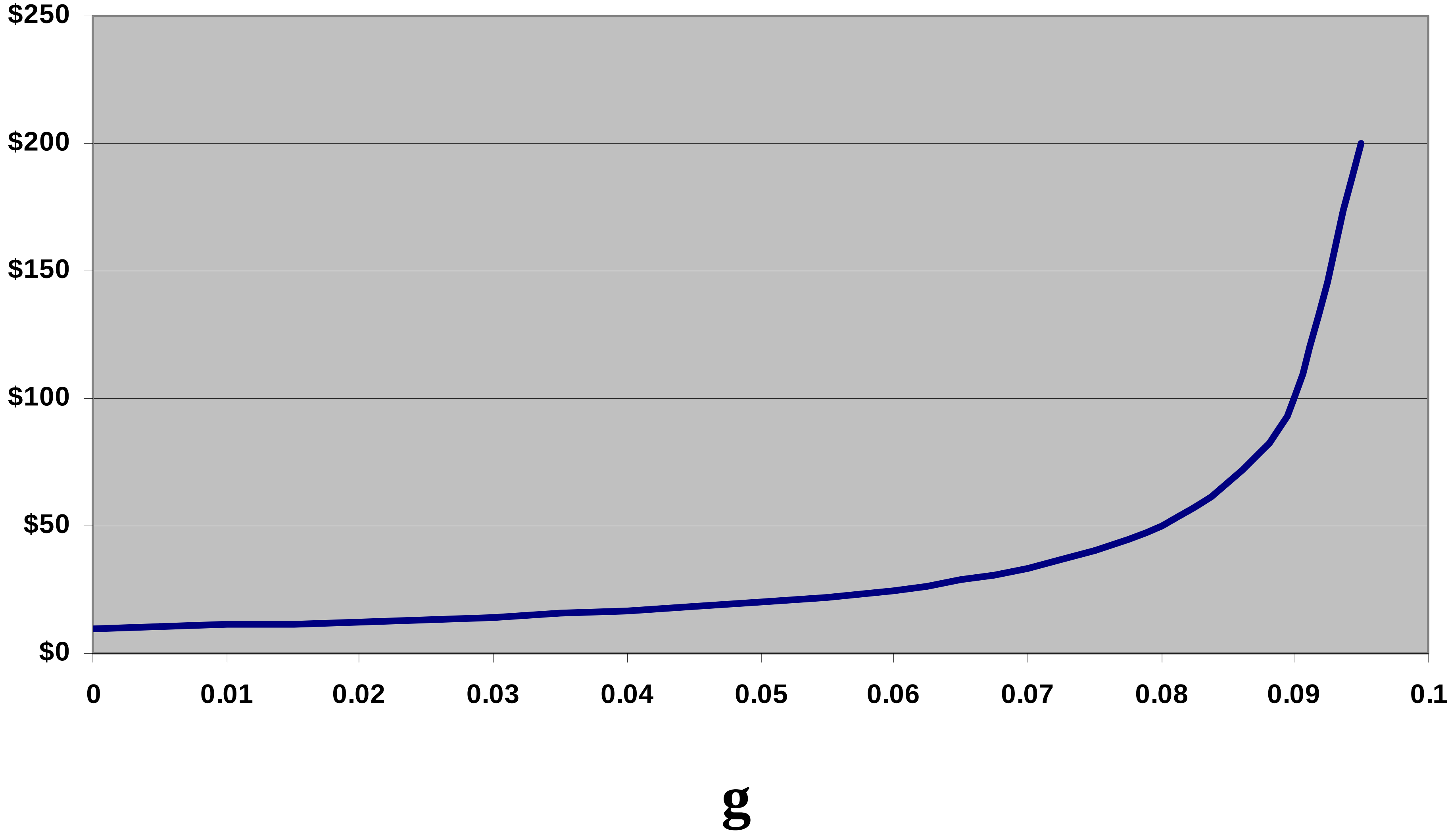
PV = 

- *notice that…*

PV = 

$$PV = \frac{(1+g)PV + CF}{1+r} \Rightarrow PV = \frac{CF}{r-g}$$

Growing perpetuity with growth rate "g" and interest rate r=10%

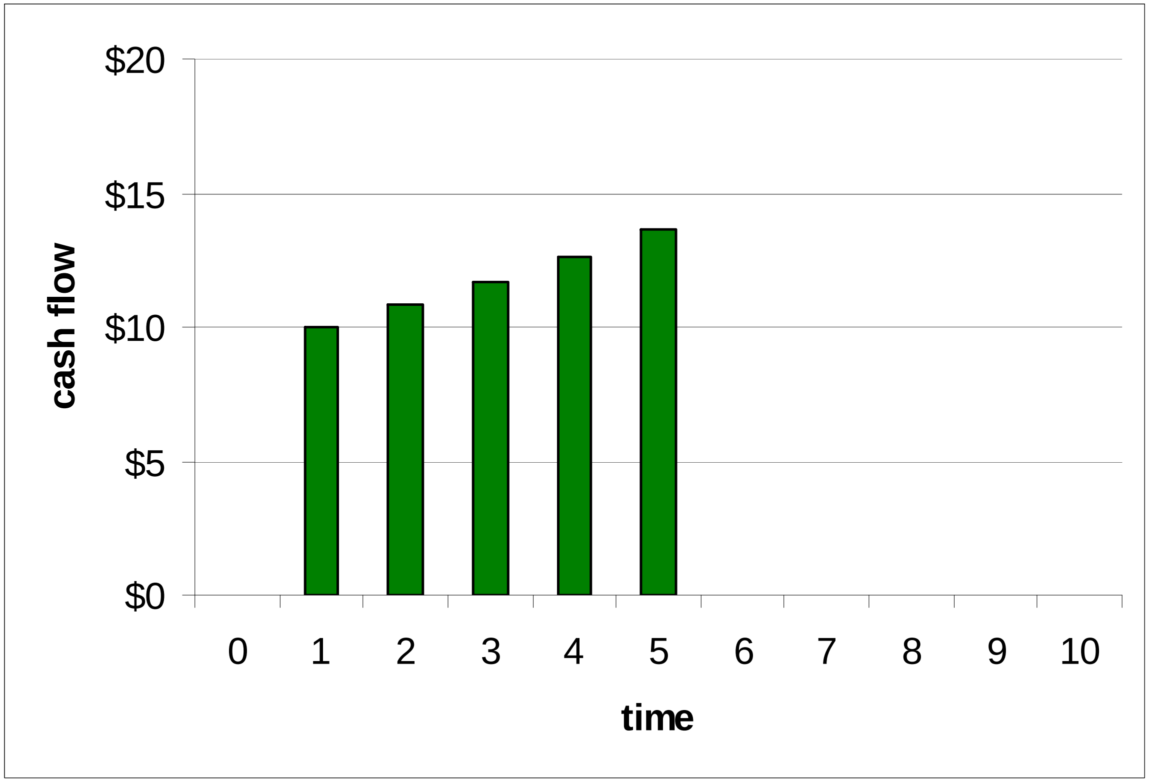


Growing Annuity

A growing annuity is a stream of cash flows that grows over time with growth rate "g" where cash flows are received at the end of each period for the duration of "n" years.

- Note:
  - Cash flows grow over time with rate "g"
  - There is no cash flow today (i.e. you receive the first cash flow one period from now)

Five year growing Annuity with growth rate of 8%



Growing annuity formula

The PV of a growing annuity for n years is,

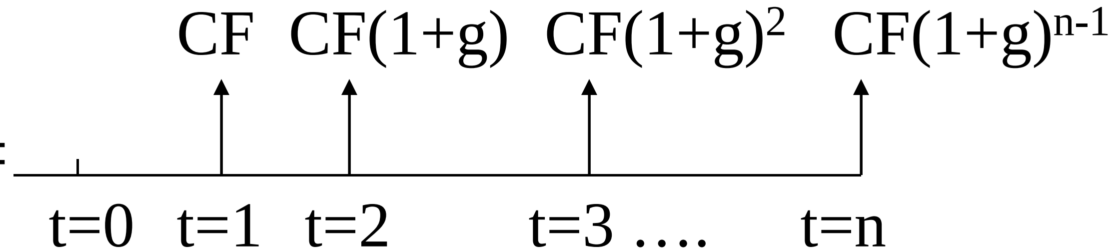
$$PV = \frac{CF}{1+r} + \frac{CF(1+g)}{(1+r)^2} + \frac{CF(1+g)^2}{(1+r)^3} + \ldots + \frac{CF(1+g)^{n-1}}{(1+r)^n}$$

$$= \sum_{i=1}^{n} \frac{CF(1+g)^{i-1}}{(1+r)^i} = \frac{CF}{r - g}\left(1 - \frac{(1+g)^n}{(1+r)^n}\right)$$

Valuing a growing annuity (cont'd)

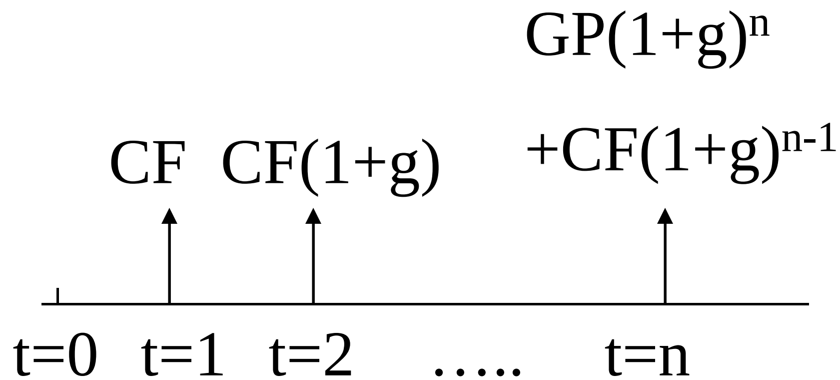
- *annuity*

a(n) =



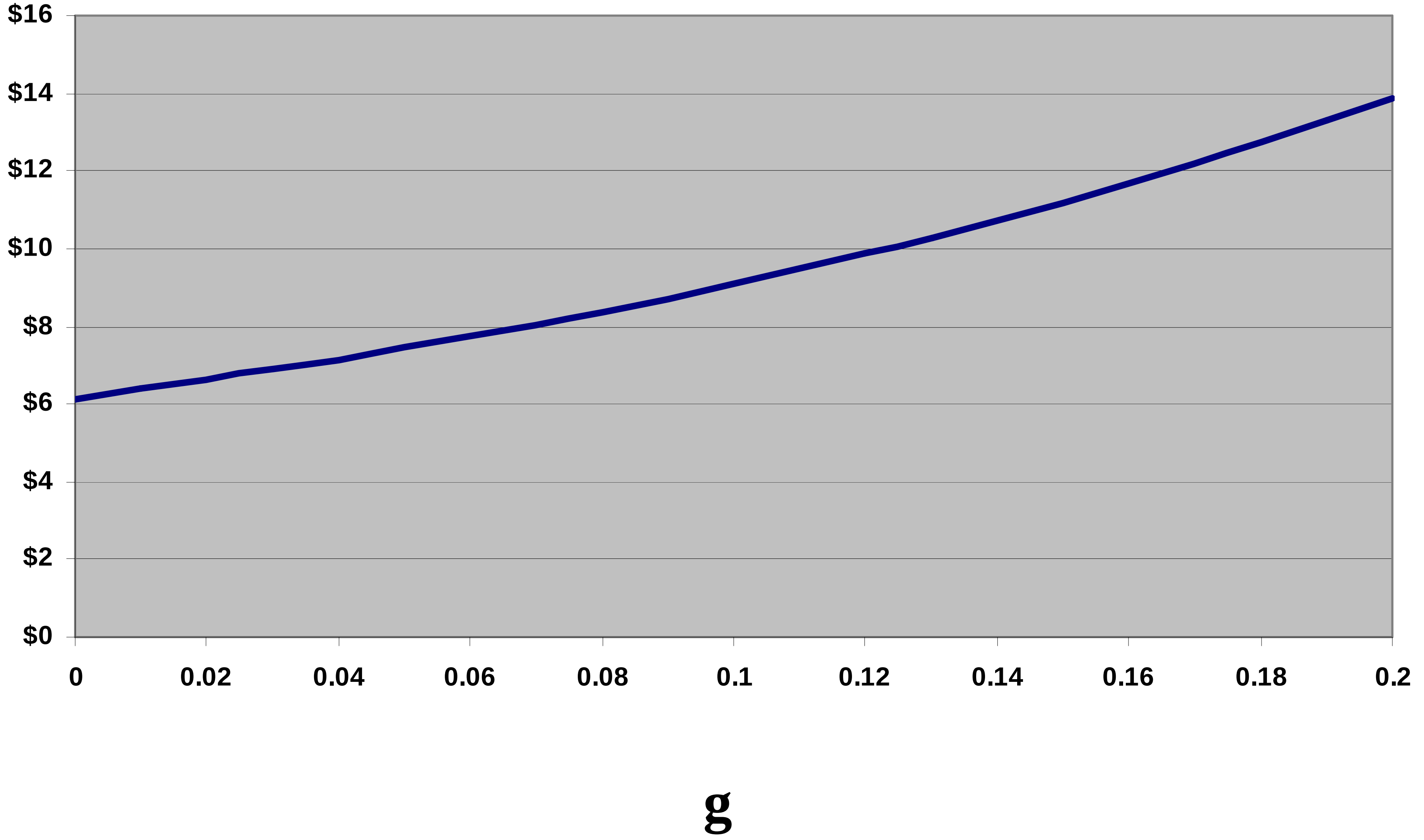
- *notice that…*

GP =



$$\mathrm{GP} = \mathrm{a(n)} + \frac{\mathrm{GP}(1+\mathrm{g})^{\mathrm{n}}}{(1+\mathrm{r})^{\mathrm{n}}} \Rightarrow \mathrm{a(n)} = \frac{\mathrm{CF}}{\mathrm{r-g}}\left(1 - \frac{(1+g)^{\mathrm{n}}}{(1+\mathrm{r})^{\mathrm{n}}}\right)$$

Growing annuity with growth rate "g" and interest rate r=10%



Growing annuity formula for r=g

$$PV = \frac{CF}{1+r} + \frac{CF(1+g)}{(1+r)^2} + \frac{CF(1+g)^2}{(1+r)^3} + \ldots + \frac{CF(1+g)^{n-1}}{(1+r)^n}$$

$$= \frac{CF}{1+r} + \frac{CF}{(1+r)} + \frac{CF}{(1+r)} + \ldots + \frac{CF}{(1+r)}$$

$$= \frac{n \cdot CF}{1+r}$$

- **<u>Example 1</u>**: if you save $1,000 each year for 35 years, how much will you have in your bank account after 35 years if the interest rate is 10%?

$$PV = \frac{\$1{,}000}{10\%}\left(1 - \frac{1}{(1.10)^{35}}\right) = 9{,}644 \quad \Rightarrow \quad \mathrm{FV} = \mathrm{PV}(1.10)^{35} = \$271{,}024$$

- How much would you need to save each year in order to accumulate $300,000 after 35 years?

$$\frac{\$\mathrm{X}}{10\%}\left(1 - \frac{1}{(1.10)^{35}}\right)(1.10)^{35} = \$300{,}000 \Rightarrow \mathrm{X} = \$1{,}107$$

$$\frac{\$\mathrm{Y}}{10\%}\left(1 - \frac{1}{(1.10)^{35}}\right)(1.10)^{35} = 300{,}000 - 271{,}024 = 28{,}976 \Rightarrow Y = \$107$$

$$\frac{\$\mathrm{X}}{\$1000} = \frac{\$300{,}000}{\$271{,}024} = 1.107 \Rightarrow \mathrm{X} = \$1{,}107$$

- What is the present value of your installments if you save $1,000 each year for (a) 35 years and (b) forever?

$$(a)\ \$9{,}644 \qquad (b)\frac{\$1{,}000}{10\%}=\$10{,}000$$

- What is the present value of your installments if the interest rate changes to 9%?

$$PV=\frac{\$1{,}000}{9\%}\left(1-\frac{1}{(1.09)^{35}}\right)=10{,}566$$

- What is the future value of your installments if the interest rate changes to 9%?

$$FV=\frac{\$1{,}000}{9\%}\left(1-\frac{1}{(1.09)^{35}}\right)(1.09)^{35}=\$215{,}711$$

- **<u>Example 2</u>**: You want to rent an apartment in Houston for one year. The landlord is not willing to reduce the monthly rent of \$1,000 but offers the first month for no charge. You can also stay in your old apartment and pay rent of \$915 (at the beginning of each month). What should you do? Assume an interest rate of 1% per month.

PV(current rent payments) = $\$915 + \frac{\$915}{1\%}\left(1 - \frac{1}{(1.01)^{11}}\right) = \$10{,}402$

PV(alternative rent payments) = $\frac{\$1{,}000}{1\%}\left(1 - \frac{1}{(1.01)^{11}}\right) = \$10{,}368$

Would your choice be the same if you got the last month free?

$$\$1{,}000 + \frac{\$1{,}000}{1\%}\left(1 - \frac{1}{(1.01)^{10}}\right) = \$10{,}471 > \$10{,}402$$

- **Example 3**: You need a parking space for the period of two years. You can either buy a parking space for \$10,000 and then sell it in two years for \$10,500, or rent a parking space for the period of 2 years. The monthly rent is currently \$75 and is expected to rise by 0.5% each month (starting from the next). What should you do? Assume an interest rate of 1% per month.

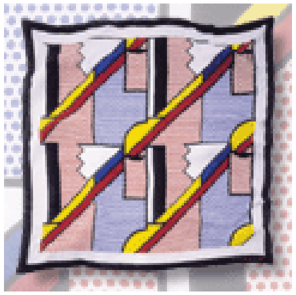
PV(buy parking space) = $$\$10{,}000 - \frac{\$10{,}500}{(1.01)^{24}} = \$1{,}731$$

PV(rent parking space) = $$\$75 + \frac{\$75(1.005)}{1\% - 0.5\%}\left(1 - \left(\frac{1.005}{1.01}\right)^{23}\right) = \$1{,}701$$

- **Example 4**: You have just earned a Federal tax return and are thinking to donate \$2,000 to the Museum of Contemporary Art in Houston. In return Museum offers free annual membership (\$100 per year paid at the beginning of the year) forever or a growing perpetuity of \$70 with growth rate of 3% per year (the first payment of \$70 is in one year). What should you do? Assume an interest rate of 7% per year.

PV(free membership offer) = $$\$100 + \frac{\$100}{7\%} = \$1{,}529$$

PV(growing perpetuity) = $$\frac{\$70}{7\% - 3\%} = \$1{,}750$$



- **<u>Example 5</u>**: 30 years ago, André François Raffray agreed to pay the 90 year old Jeanne Calment 2,500 francs ($500) per month (end) until she dies. In return he will receive her apartment when she dies. The apartment is worth $184,000. Suppose the monthly interest rate is 1%. Assuming M. Raffray thought this was a good deal, how long did he think Jeanne Calment would live?

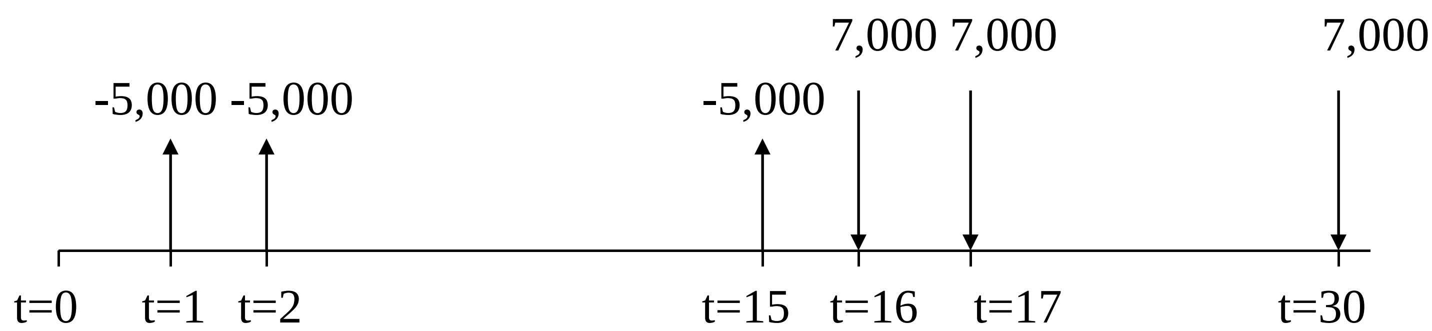
Mr. Raffray will break even if Jeanne Clament lives less than n additional months

$$184{,}000 = \frac{500}{0.01}\left(1 - \frac{1}{(1.01)^n}\right) \times 1.01^n \Rightarrow 1 + \frac{184{,}000 \times 0.01}{500} = 1.01^n$$

This implies that $$\ln\left(1 + \frac{184{,}000 \times 0.01}{500}\right) = n \ln(1.01)$$

- So... $n = 155.1$, which implies an age of 103 years old.

<u>Example 6:</u> An insurance agent offers you the following contract: you pay $5,000 per year (end) for the next 15 years and in return you will receive $7,000 a year (end) for the following 15 years. Suppose interest rates are 9%. Should you buy this contract?



$$-\frac{5000}{1.09}\left(1-\frac{1}{1.09^{15}}\right)+\frac{1}{1.09^{15}}\times\frac{7000}{1.09}\left(1-\frac{1}{1.09^{15}}\right)=-24{,}813$$

- <u>Example cont'd</u>: suppose that the insurance agent sweetens the deal and says that the payments that you receive will grow at 3% per year. Would you take the contract now?

$$-\frac{5000}{0.09}\left(1-\frac{1}{1.09^{15}}\right)+\frac{1}{1.09^{15}}\times\frac{7000}{0.09-0.03}\left(1-\left(\frac{1.03}{1.09}\right)^{15}\right)=-21{,}973$$