Risk and Return

– Portfolios of Two Risky Assets

– Portfolios of Two Risky Assets and the Risk Free Asset

– Diversification

Introduction

- We showed last time that an investor can choose any point on the Capital Allocation Line (CAL), depending on her preferences for risk.
- There was no "optimal" combination of the risk free asset and the risky asset.

What is the optimal combination of risky assets?

- The goal of this lecture is to understand "*asset – allocation*" between risky and risk free assets based on a "*mean – variance portfolio analysis*"

…all the eggs should not be placed in the same basket…

(what we now call "*diversification*")

- Harry Markowitz developed the first theory of optimal investment (1959) while taking both risk and return into account. He won the 1990 Nobel Prize in Economics for this work.

Selecting the Optimal Risky Portfolio

- Like last time…we can calculate the expected return and return standard deviation of a portfolio of two risky assets (A & B)

$$E(\tilde{r}_p) = w_A \cdot E(\tilde{r}_A) + w_B \cdot E(\tilde{r}_B)$$

$$\sigma_p^2 = w_A^2 \cdot \sigma_A^2 + w_B^2 \cdot \sigma_B^2 + 2w_A w_B \operatorname{cov}(\tilde{r}_A, \tilde{r}_B)$$

Where $w_A = 1 - w_B$

Or using the correlation, $\rho_{A,B}$

$$\sigma_p^2 = w_A^2 \cdot \sigma_A^2 + w_B^2 \cdot \sigma_B^2 + 2w_A w_B \rho_{A,B} \sigma_A \sigma_B$$

Remember that $-1 \le \rho \le 1$

The Minimum – Variance Frontier

The **Minimum – Variance frontier** is the set of portfolios with the lowest variance for a given expected return.

For the case of two risky assets (and some constant C),

$$\underset{w_A, w_B}{\text{Min}}\ \sigma_p^2$$
$$\text{s.t} \quad E(\tilde{r}_p) = C$$

Where, $w_A = 1 - w_B$

$$E(\tilde{r}_p) = w_A \cdot E(\tilde{r}_A) + w_B \cdot E(\tilde{r}_B)$$

$$\sigma_p^2 = w_A^2 \cdot \sigma_A^2 + w_B^2 \cdot \sigma_B^2 + 2 w_A w_B \rho_{A,B} \sigma_A \sigma_B$$

Correlation and the MVF

- We will consider three cases:

$\rho_{A,B} = 1$

$\rho_{A,B} = 0$

$\rho_{A,B} = -1$

- Consider the following assets A and B:

Asset	Expected return	Return standard deviation
A	25%	75%
B	10%	25%

Consider first $\rho_{A,B} = 1$

Fraction "w" is invested in A.

- Portfolio's **expected return** is

$$E(r_p) = wE(r_A) + (1-w)E(r_B) = 0.25w + 0.1(1-w) = 0.1 + 0.15w$$

- Portfolio's **return variance** is

$$V(r_p) = w^2 V(r_A) + (1-w)^2 V(r_B) + 2w(1-w)\rho_{A,B}\sigma_A\sigma_B$$
$$= \left(w\sigma_A + (1-w)\sigma_B\right)^2$$

$$\Rightarrow \sigma_p = \left|w\sigma_A + (1-w)\sigma_B\right| = \left|0.75w + 0.25(1-w)\right| = \left|0.25 + 0.5w\right|$$

Risk/Return relation when $\rho_{A,B} = 1$

We know that $E(r_p) = 0.1 + 0.15w, \quad \sigma_p = 0.25 + 0.5w \text{ (if } w \geq -0.5)$

Substitute for "w" and get,

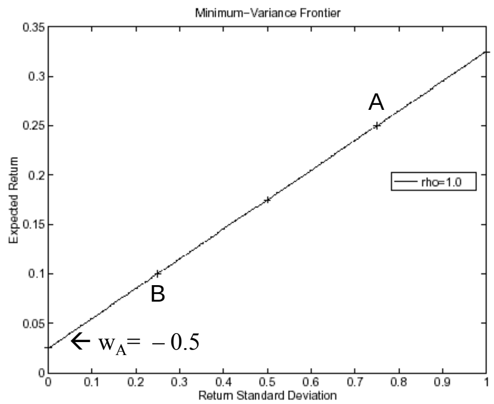
$$E(r_p) = 0.1 + 0.15\left(\frac{\sigma_p - 0.25}{0.5}\right) = 0.025 + 0.3 \times \sigma_p$$

What is the risk free rate? $V(r_f) = 0 \Rightarrow r_f = 0.025 + 0.3 \times 0 = 0.025$

How can we form a portfolio with a standard deviation of zero?

$$0 = \sigma_p = 0.25 + 0.5w \Rightarrow w = -0.5$$

What does a negative w_A mean?



Where are A and B on this plot?
Where can investors be on this plot?
What would you do if the risk free rate were 1%? 5%?

Consider $\rho_{A,B} = -1$

Fraction "w" is invested in A.

- Portfolio's **expected return** is

$$E(r_p) = wE(r_A) + (1-w)E(r_B) = 0.1 + 0.15w$$

- Portfolio's **return variance** is

$$V(r_p) = w^2 V(r_A) + (1-w)^2 V(r_B) + 2w(1-w)\rho_{A,B}\sigma_A\sigma_B$$
$$= \left(w\sigma_A - (1-w)\sigma_B\right)^2$$
$$\Rightarrow \sigma_p = \left|w\sigma_A - (1-w)\sigma_B\right| = \left|0.75w - 0.25(1-w)\right| = \left|w - 0.25\right|$$

Risk/Return relation when $\rho_{A,B} = -1$

We know that $E(r_p) = 0.1 + 0.15w$, $\sigma_p = \begin{cases} 0.25 - w, & w < 0.25 \\ w - 0.25, & w \geq 0.25 \end{cases}$

For the case (w<0.25) substitute for "w" and get,

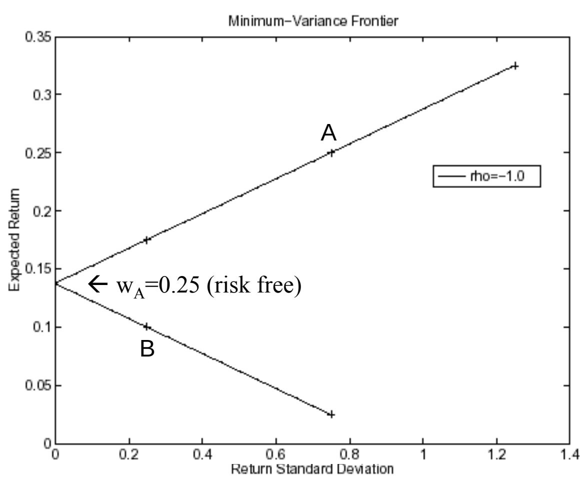
$$E(r_p) = 0.1 + 0.15(0.25 - \sigma_p) = 0.1375 - 0.15 \times \sigma_p$$

For the case (w>0.25) substitute for "w" and get,

$$E(r_p) = 0.1 + 0.15(\sigma_p + 0.25) = 0.1375 + 0.15 \times \sigma_p$$

What is the risk free rate? $r_f = 0.1375 + 0.15 \times 0 = 0.1375$

How to implement? $0 = \sigma_p = |0.25 - w| \Rightarrow w = 0.25$



Where are A and B on this plot?
Where can investors be on this plot?
Where would investors want to be on this plot?
What are the portfolio weights at which the portfolio risk is zero?

Consider $-1 < \rho_{A,B} < 1$

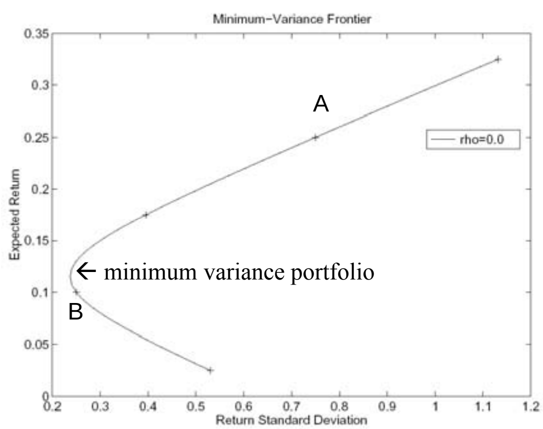
Fraction "w" is invested in A.

- Portfolio's **expected return** is

$$E(r_p) = wE(r_A) + (1-w)E(r_B) = 0.1 + 0.15w$$

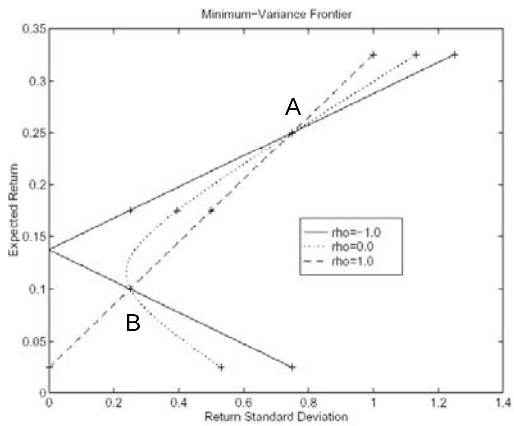
- Portfolio's **return variance** is

$$V(r_p) = w^2 V(r_A) + (1-w)^2 V(r_B) + 2w(1-w)\rho_{A,B}\sigma_A\sigma_B$$



Where are A and B on this plot?
Where would investors want to be on this plot?

All three cases together

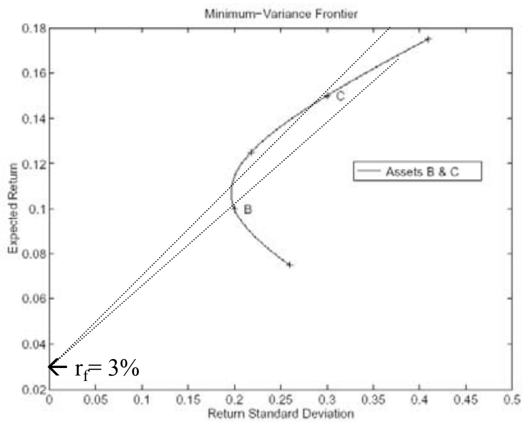


The Optimal Portfolio with Two Risky Assets and a Risk Free Asset

- Consider a risk free asset with annual rate of r_f=3% and risky assets B and C,

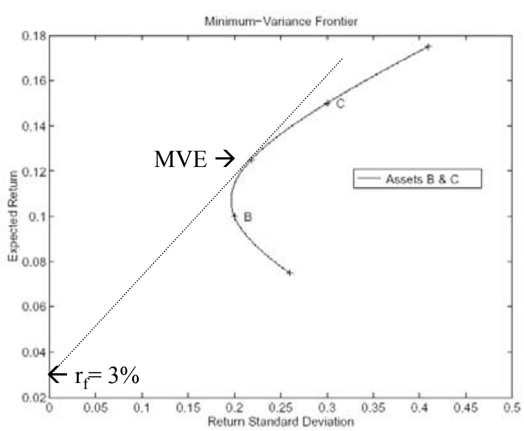
Asset	Expected return	Return standard deviation
risk free	3%	0%
B	10%	20%
C	15%	30%

- We can calculate the minimum – variance frontier as before.



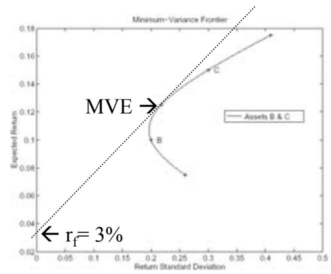
• Suppose we can construct portfolios of the risk free asset and **either** B or C. What are the two possible Capital Allocation Lines?

What if we do not want to restrict ourselves to holding B or C alone. What is the optimal portfolio of risky assets to combine with the risk free asset?



The **Mean Variance Efficient** (MVE) portfolio is given by the tangency point.

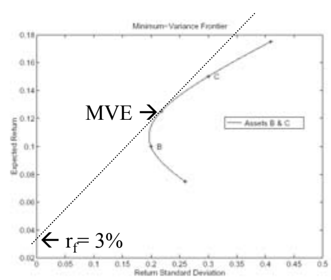
It is optimal to hold both the risk free asset and the MVE portfolio of risky assets.



The same analysis can be done for the case of more than 2 risky assets.

We can conclude that…

1. It is optimal for all investors to hold the MVE portfolio and the risk free asset.
2. Investors with different risk preferences will differ in the proportions they invest in the MVE portfolio.



Given that all investors should hold the same portfolio of risky assets,

The MVE portfolio can be thought of as the market portfolio

– *why?*

Diversification

We can define two types of risk:

Systematic: risk associated with economy wide fluctuations

- Business cycles
- Announcements: Federal Reserve Chairman Alan Greenspan's interest rate announcements, election announcements
- Wars and peace treaties
- Technological inventions
- Natural disasters and diseases

Idiosyncratic: risk associated with firm specific events $\tilde{\varepsilon}_{\text{firm i}}$

- Quality of management team
- Success of business plan

*We can diversify **idiosyncratic** risk by holding a large number of assets in our portfolio.*

Diversification

Example: Suppose that there are two possible states of nature in the economy: "boom" and "bust", with probability 0.5 each. There are many risky assets (firms), with the potential to earn expected return of 20% in a "boom" and 10% in a "bust". In either case, due to additional firm specific risk, each firm can loose or gain an additional 5% in value, with equal probabilities. The return on asset "i", r_i, has the following distribution:

$$\tilde{r}_i = \begin{cases} 0.20 + 0.05 = 0.25 & \text{w.p. } 0.25 \\ 0.20 - 0.05 = 0.15 & \text{w.p. } 0.25 \\ 0.10 + 0.05 = 0.15 & \text{w.p. } 0.25 \\ 0.10 - 0.05 = 0.05 & \text{w.p. } 0.25 \end{cases}$$

Diversification

To see this, lets define the random variables, r_{market} for the systimatic risk component and , $\tilde{\varepsilon}_{firm\,i}$ for the idiosyncratic risk component.

We can write the return of firm i "r_i" as,

$$\tilde{r}_i = \tilde{r}_{market} + \tilde{\varepsilon}_{firm\,i}$$

Where,

$$\tilde{r}_{market} = \begin{cases} 0.2 & \text{w.p. } 0.5 \\ 0.1 & \text{w.p. } 0.5 \end{cases} \qquad \tilde{\varepsilon}_{firm\,i} = \begin{cases} +0.05 & \text{w.p. } 0.5 \\ -0.05 & \text{w.p. } 0.5 \end{cases}$$

Diversification

Suppose we hold N assets, each with weight 1/N.

The return on the portfolio is,

$$r_p = \sum_{i=1}^{N} w_i \times \tilde{r}_i = \sum_{i=1}^{N} \frac{1}{N} \times \left(\tilde{r}_{market} + \varepsilon_{firm\,i}\right)$$

$$= \frac{1}{N} \times \sum_{i=1}^{N} \tilde{r}_{market} + \frac{1}{N} \times \sum_{i=1}^{N} \varepsilon_{firm\,i}$$

$$= \tilde{r}_{market} + \frac{1}{N} \times \sum_{i=1}^{N} \varepsilon_{firm\,i}$$

$$V(r_p) = V(\tilde{r}_{market}) + V\left(\frac{1}{N} \times \sum_{i=1}^{N} \varepsilon_{firm\,i}\right) + \mathrm{cov}\left(\tilde{r}_{market}, \frac{1}{N} \times \sum_{i=1}^{N} \varepsilon_{firm\,i}\right)$$

Diversification

Firm i's specific risk is not correlated with the market or other firms' outcomes, so the variance of our portfolio of "N" firms can be simplified as follows.

$$V(r_p) = V(\tilde{r}_{market}) + V\left(\frac{1}{N} \times \sum_{i=1}^{N} \varepsilon_{firm\,i}\right) + cov\left(\tilde{r}_{market}, \frac{1}{N} \times \sum_{i=1}^{N} \varepsilon_{firm\,i}\right)$$

$$= V(\tilde{r}_{market}) + V\left(\frac{1}{N} \times \sum_{i=1}^{N} \varepsilon_{firm\,i}\right) + 0$$

$$= V(\tilde{r}_{market}) + \frac{1}{N^2} \times \sum_{i=1}^{N} V(\varepsilon_{firm\,i})$$

$$= V(\tilde{r}_{market}) + \frac{N \times V(\varepsilon_{firm\,i})}{N^2} \xrightarrow[N \to \infty]{} V(\tilde{r}_{market})$$

The plot shows how the standard deviation of a portfolio of average NYSE stocks decreases as we include more stocks in our portfolio.