- Risk & Return
  - – Opportunity Cost of Capital
  - – Historical Evidence on Risk and Return
  - – The Risk of a Single Risky Asset
  - – The Risk and Return of a Portfolio of a Risky & Risk-Free assets

Opportunity Cost of Capital

How to determine the Cost of Capital?

- The **opportunity cost of capital** for a project is the expected return on an investment with similar risk
  - We will define "**similar risk**"

- How do investors decide how much risk they want in their portfolio?
- What portfolio provides the optimal tradeoff between risk and return?

Let's look how investors' attitudes toward risk are manifested in the history of security returns…

Historical Evidence – Risk and Return

- This table shows the Real Returns (Inflation adjusted) from 1925 through 2000:

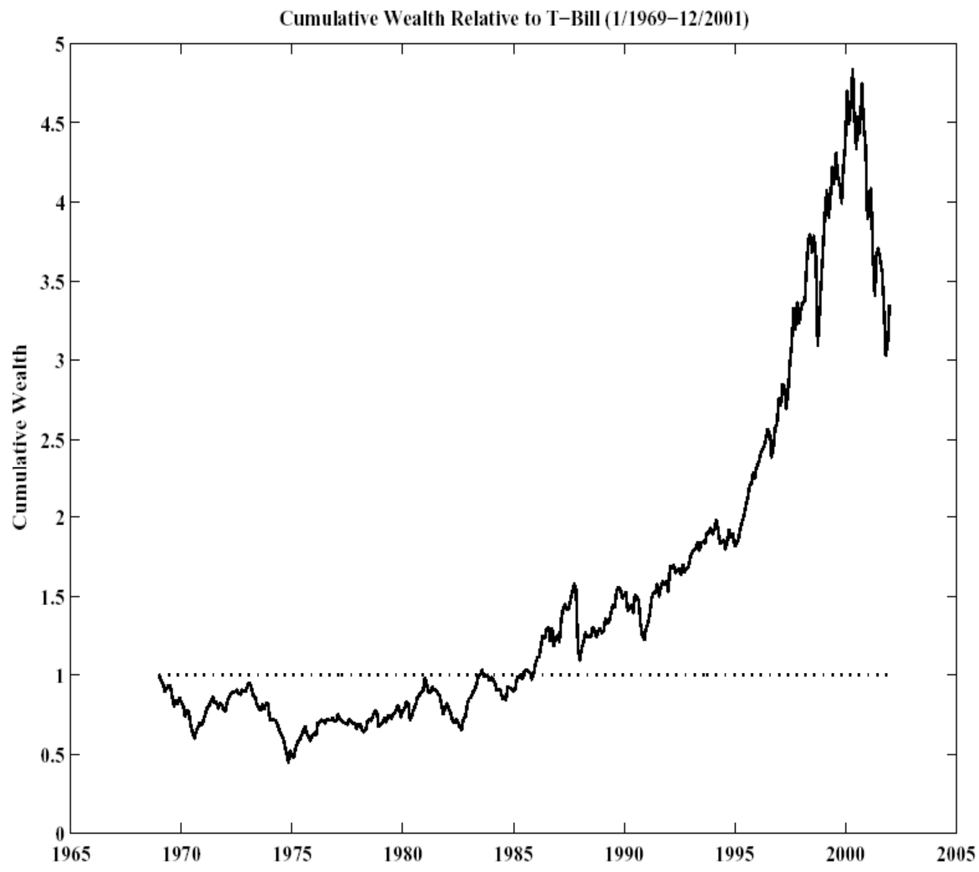
Security Class	Average Annual Returns & Standard Deviations(in %)			Total Real Return (per $ inv.)
	Nominal	Real	σ	
Small Stocks	17.3	13.8	33.4	659.6
S&P500	13.0	9.7	20.2	266.5
Corporate Bonds	6.0	3.0	8.7	6.6
Treasury Bonds	5.7	2.7	9.4	5.0
T-Bills	3.9	0.8	3.2	1.7

- This shows that over long periods of time there has been a tradeoff between risk and expected return.

Stocks have historically had much higher returns but they are also substantially riskier!

The U.S. Stock Market, 1968 – 2002

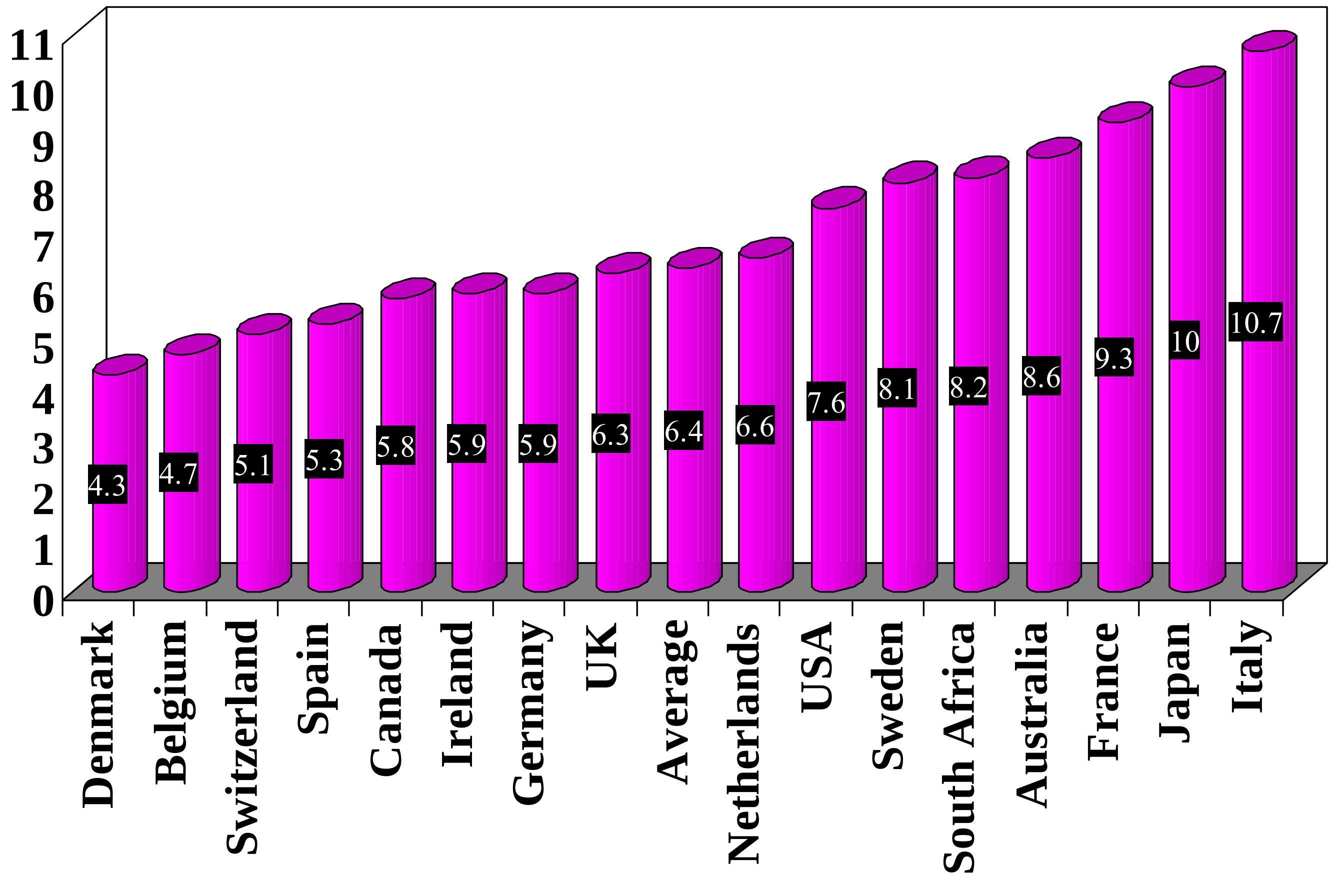
- ***Stocks versus Bonds**. The table shows the gains from investing in Stocks relative to Bonds.*



If you invest in stocks you realize high gains as well as high losses

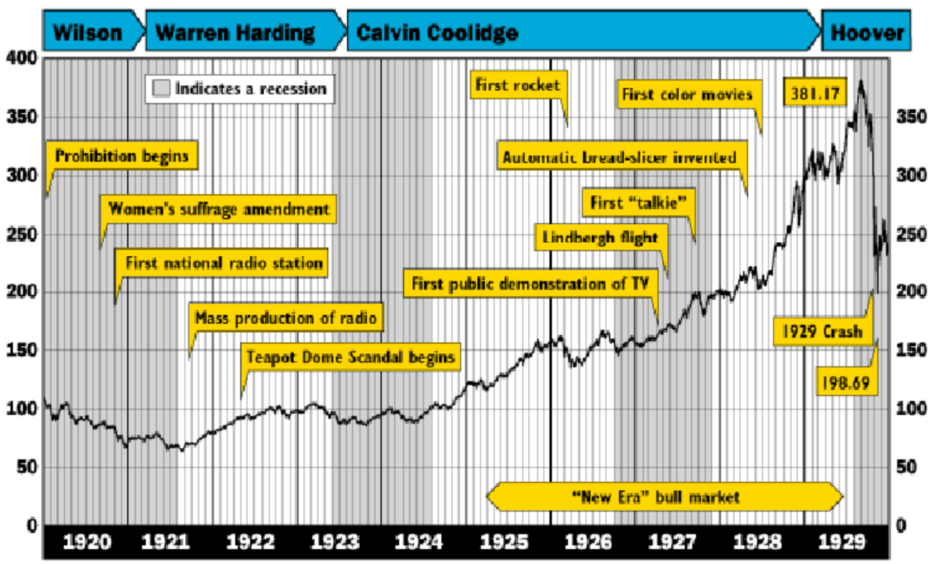
Risk Premium over the world 1900-2003

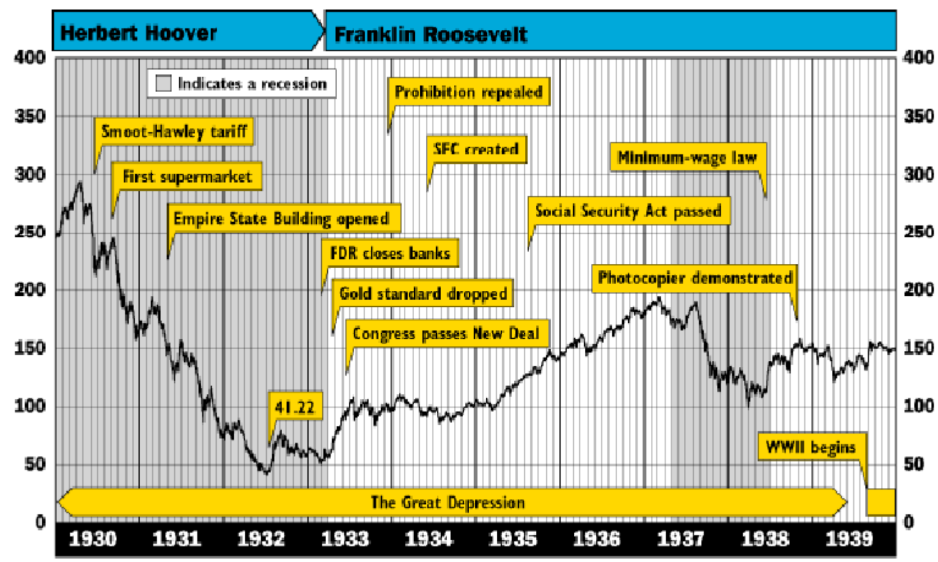
return on market = risk free rate + ***risk premium***



The U.S. Stock Market 1920-1939

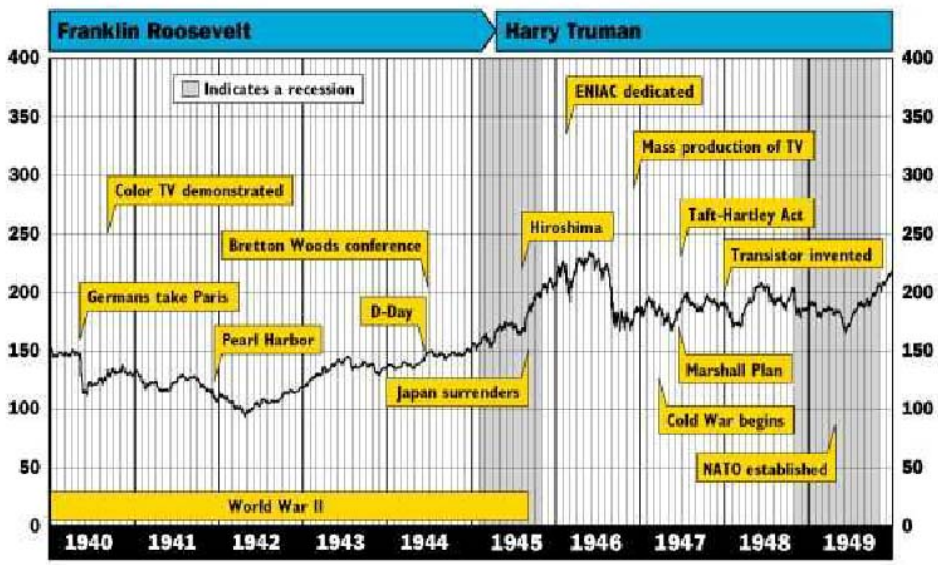
- *The Dow Jones Industrial Average fell from a high of 381.17 in 1929 to a low of 41.22 in 1932, a fall of 89.2%*

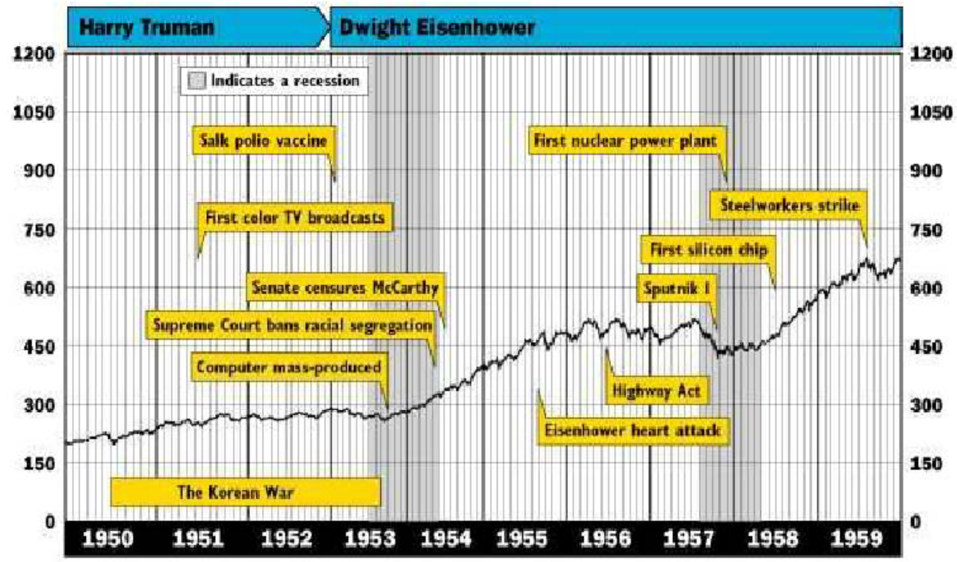




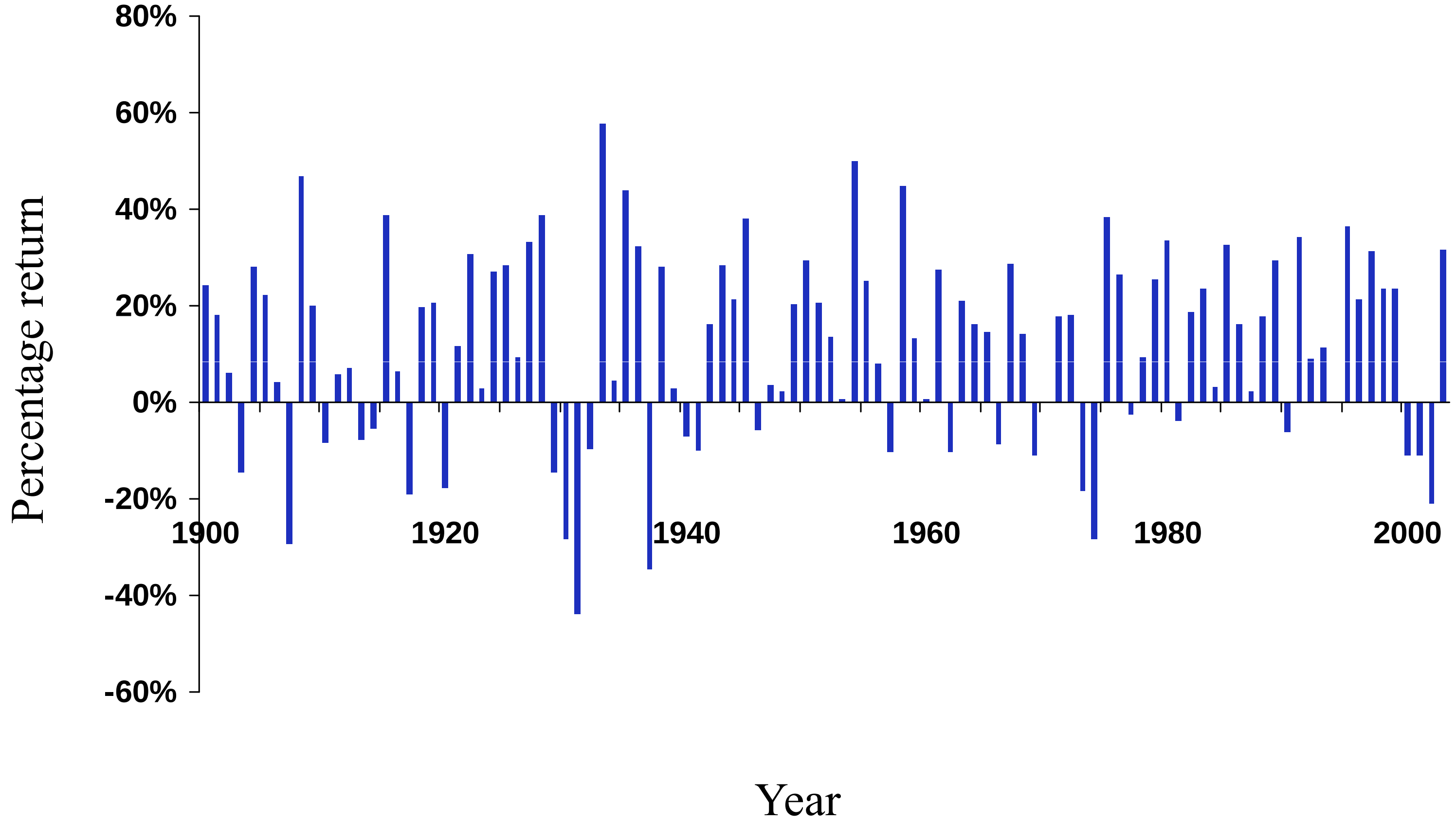
The U.S. Stock Market 1920-1959

- *The Dow Jones Industrial Average did not reach its 1929 high of 381 again until late 1954, over 25 years later.*

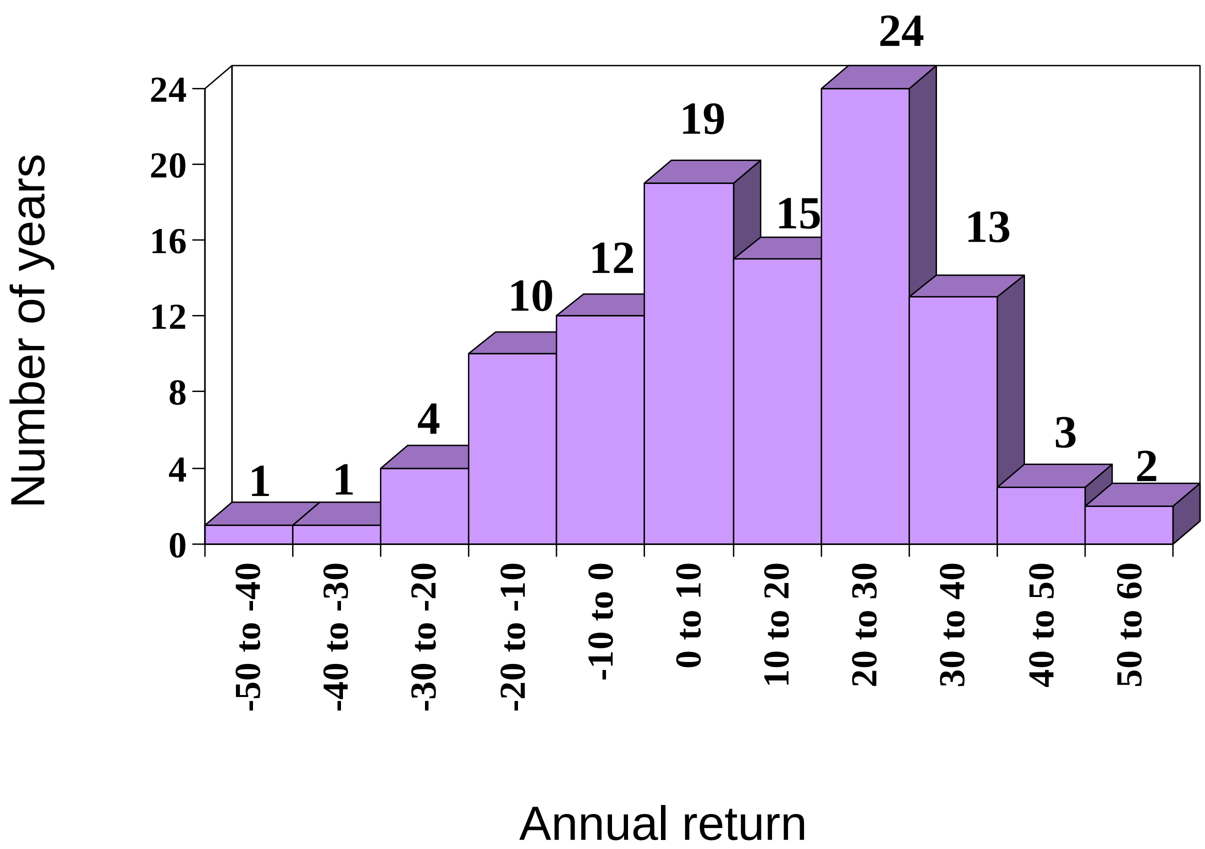




Annual rates of return for 1900-2003

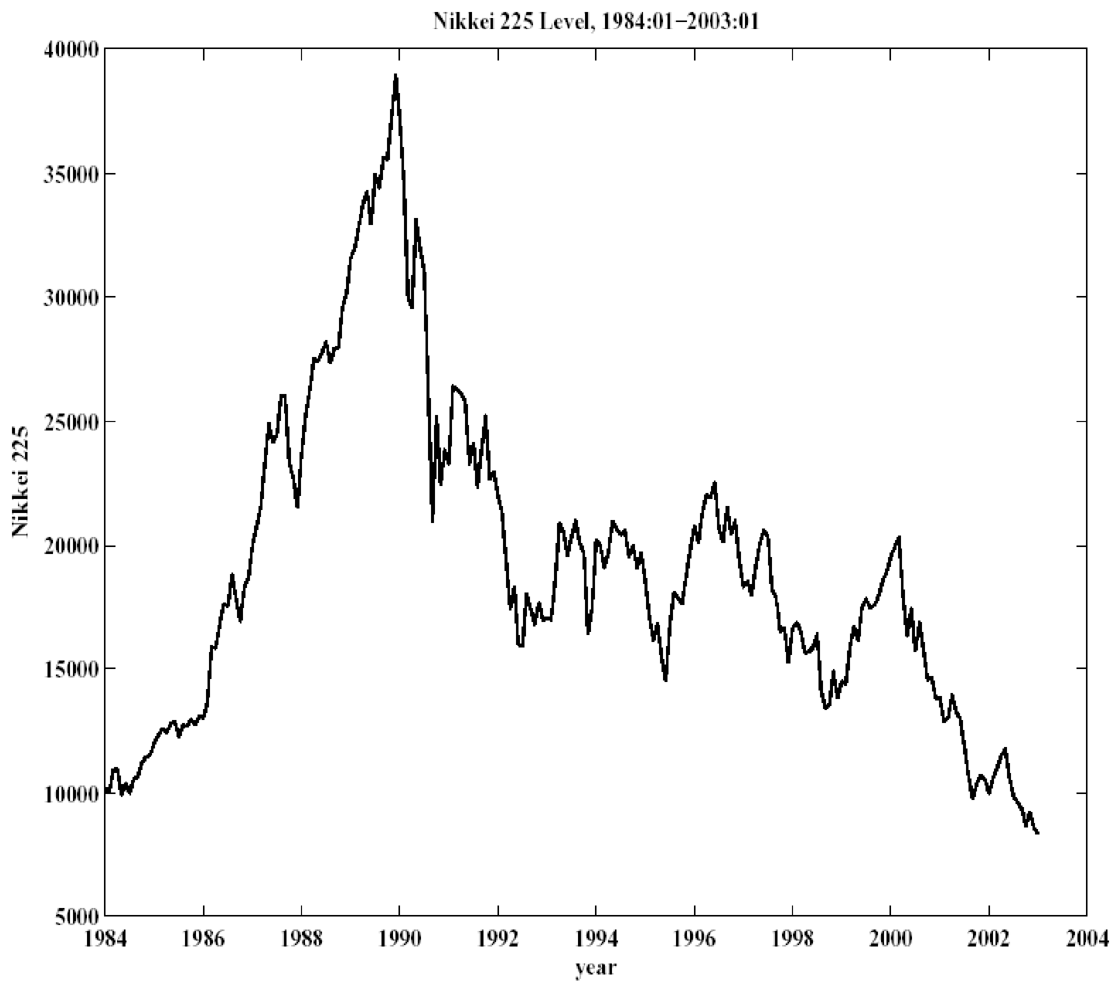


Histogram of annual rates of returns 1900-2003



The Japanese Stock Market, 1984-2003

- *The Nikkei 225 peaked at 38,957 on December 29, 1989. On February 11, 2003, roughly 13 years later, it closed at 8,485.*



Quantifying Risk and Return

<u>Terminology</u>

1. **Realized Return:** The return investors in a security actually earn over the period, measured at the end of the period.

2. **Expected Return:** The return investors in a security expect to earn over a period, measured at the beginning of the period.

We calculate the expected return "E(r)" by summing across the possible realized returns (possible events "s") times the probability of these events "P_s"

$$E(\tilde{r}) = \sum_{s=1}^{S} P_s \times r_s$$

Quantifying Risk and Return

Terminology

3. Return Variance: The expected squared deviation from the mean over a period, measured at the beginning of the period.

We calculate the return variance "V(r)" by summing across the possible realized square deviations times the probabilities.

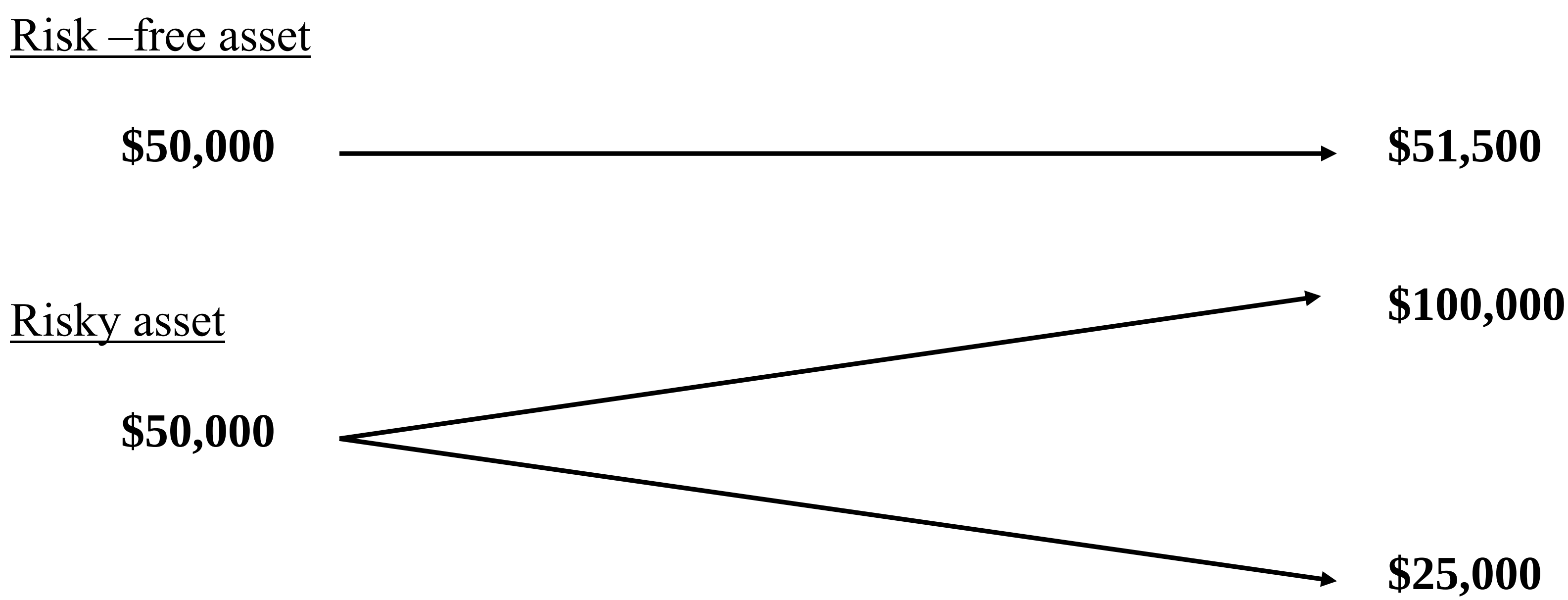
$$V(\tilde{r}) = \sigma_r^2 = \sum_{s=1}^{S} P_s \times \left[r_s - E(\tilde{r})\right]^2 = E(\tilde{r}^2) - [E(\tilde{r})]^2$$

4. Return Standard Deviation: The positive square root of the variance:

$$\sigma_r = \sqrt{\sigma_r^2}$$

Quantifying Risk and Return - Example

Consider an investor who has $50,000 to invest. She has the choice to invest either in a risk-free asset that pays 3% or in Stock A. Stock A will either go down by 50% or go up by 100% with equal probabilities. Calculate the expected return, expected excess return and return standard deviation of the portfolio which is fully invested in Stock A.



Risky asset

The two possible realized returns are:

$$r_1 = \frac{100,000 - 50,000}{50,000} = 100\%, \quad r_2 = \frac{25,000 - 50,000}{50,000} = -50\%$$

The expected return is: $E(\tilde{r}) = \frac{1}{2} \times 100\% + \frac{1}{2} \times (-50\%) = 25\%$

Risk-Free asset

The only possible return is: $r = \frac{51,500 - 50,000}{50,000} = 3\%$

Stock A's **Excess Return** is defined as its expected return minus the risk-free return:

$$r_{\text{STOCK A}} - r_{\text{RISK FREE}} = 25\% - 3\% = 22\%$$

Risky asset

The variance of the return is:

$$\sigma_r^2 = \frac{1}{2} \times (1 - 0.25)^2 + \frac{1}{2} \times (-0.5 - 0.25)^2 = 0.56$$

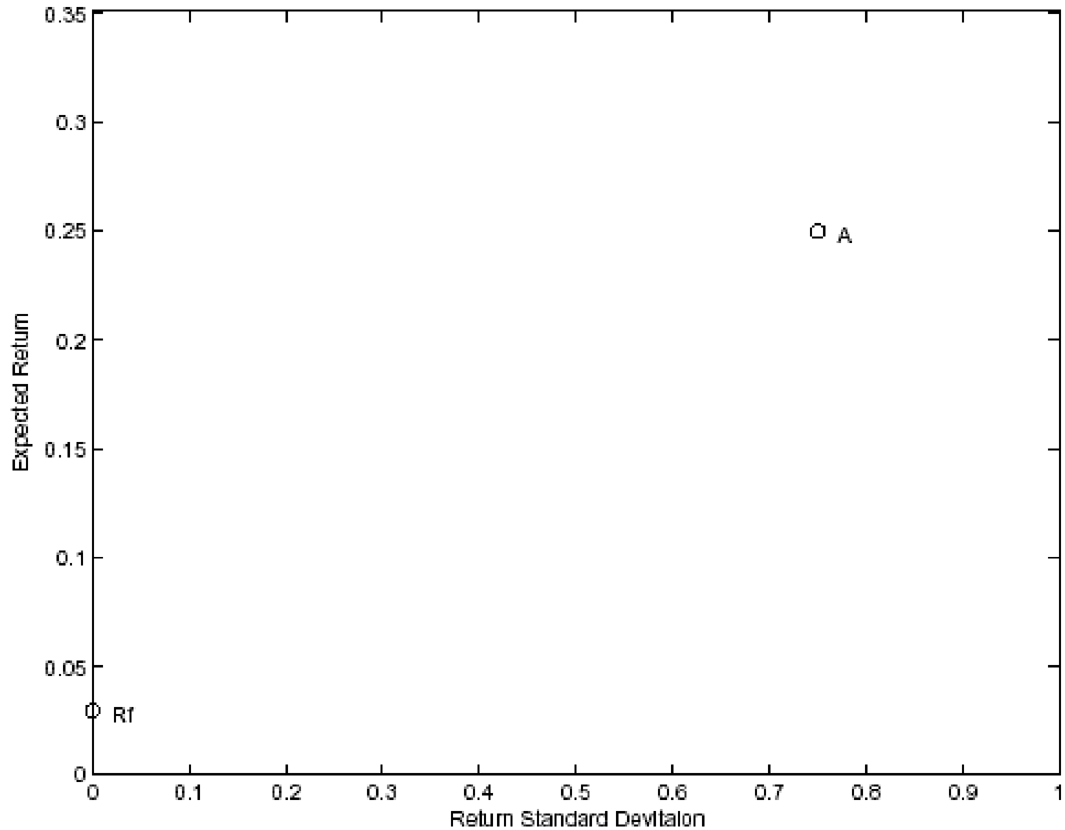
The standard deviation is: $\sigma_r = \sqrt{\sigma_r^2} = \sqrt{0.56} = 0.75 = 75\%$

The standard deviation tells us roughly how far above and below the mean we expect to be each period

What is the variance and standard deviation of the return on the Risk-Free asset?

Portfolio Choice: Risky and Risk-Free Assets

The plot below shows the “location” of the two assets



Which investment should our investor pick?

Are there other possible investments which our investor might prefer?

Risk and Return of Portfolios

<u>Example (continued):</u> Suppose that our investor invests $25K in the risk free asset and $25K in asset A? What returns could the investor earn? What is the portfolio's expected return, variance of return and standard deviation of return?

Notation

r_A = return on stock A

$E(\tilde{r}_A)$ = expected risky rate of return (stock A)

σ_A = return standard deviation (stock A)

r_f = risk free rate (of return)

w_A = fraction of the portfolio invested in risky asset (stock A)

The **return of a portfolio**

$$\tilde{r}_p = w \times \tilde{r}_A + (1 - w) \times r_f = 0.5 \times \tilde{r}_A + 0.5 \times 0.03$$

$$= \begin{cases} 0.5 \times 1 + 0.015 = 0.515 & \text{(if A goes up)} \\ 0.5 \times (-0.5) + 0.015 = -0.235 & \text{(if A goes down)} \end{cases}$$

The **portfolio's expected return**

$$E(\tilde{r}_p) = w \times E(\tilde{r}_A) + (1 - w) \times r_f$$

$$= 0.5 \times 0.25 + 0.5 \times 0.03 = 0.14$$

To calculate the **portfolio's return variance** recall the following formulas from statistics

$$\mathrm{var(x)}\,[\equiv \sigma_{\mathrm{x}}^2] = \mathrm{E(x^2)} - (\mathrm{E(x)})^2$$

$$\mathrm{var(a \cdot x)} = \mathrm{a^2 var(x)}$$

$$\mathrm{var(x+y)} = \mathrm{var(x)} + \mathrm{var(y)} + 2\mathrm{cov(x,y)}$$

$$\mathrm{cov(x,y)}\,[\equiv \sigma_{\mathrm{x,y}}] = \mathrm{E(xy)} - \mathrm{E(x)E(y)} = \sigma_{\mathrm{x}} \cdot \sigma_{\mathrm{y}} \cdot \rho_{\mathrm{x,y}}$$

$$\mathrm{cov(a \cdot x, b \cdot y)} = \mathrm{a \cdot b \cdot cov(x,y)}$$

$$\mathrm{cov}(x,x) = \sigma_{\mathrm{x}}^2$$

$$\rho_{\mathrm{x,y}} = \frac{\mathrm{cov(x,y)}}{\sqrt{\mathrm{var(x)}}\sqrt{\mathrm{var}(y)}}$$

The **portfolio's return variance**

$$\sigma_p^2 = \operatorname{var}(\tilde{r}_p) = \operatorname{var}(w \cdot \tilde{r}_A + (1-w) \cdot r_f)$$

$$= \operatorname{var}(w \cdot \tilde{r}_A) + \operatorname{var}((1-w) \cdot r_f) + 2\operatorname{cov}(w \cdot \tilde{r}_A, (1-w) \cdot r_f)$$

$$= w^2 \cdot \operatorname{var}(\tilde{r}_A)$$

$$= 0.5^2 \cdot 0.75^2 = 0.140625$$

$$\Rightarrow \sigma_p = \sqrt{\sigma_p^2} = \sqrt{0.140625} = 0.5 \cdot 0.75 = 0.375$$

Notice that the standard deviation is proportional to the fraction of her portfolio she invests in the risky asset

The Capital Allocation Line

The **Capital Allocation Line** (CAL) represents all the possible combinations of "risk" and "return" that can be generated from holding a portfolio of the risky asset and the risk free asset.

If we invest "w" in the risky asset then we have,

$$E(\tilde{r}_p) = w \cdot E(\tilde{r}_A) + (1 - w) \cdot r_f$$

$$\sigma_p = w\sigma_A \Rightarrow w = \frac{\sigma_p}{\sigma_A}$$

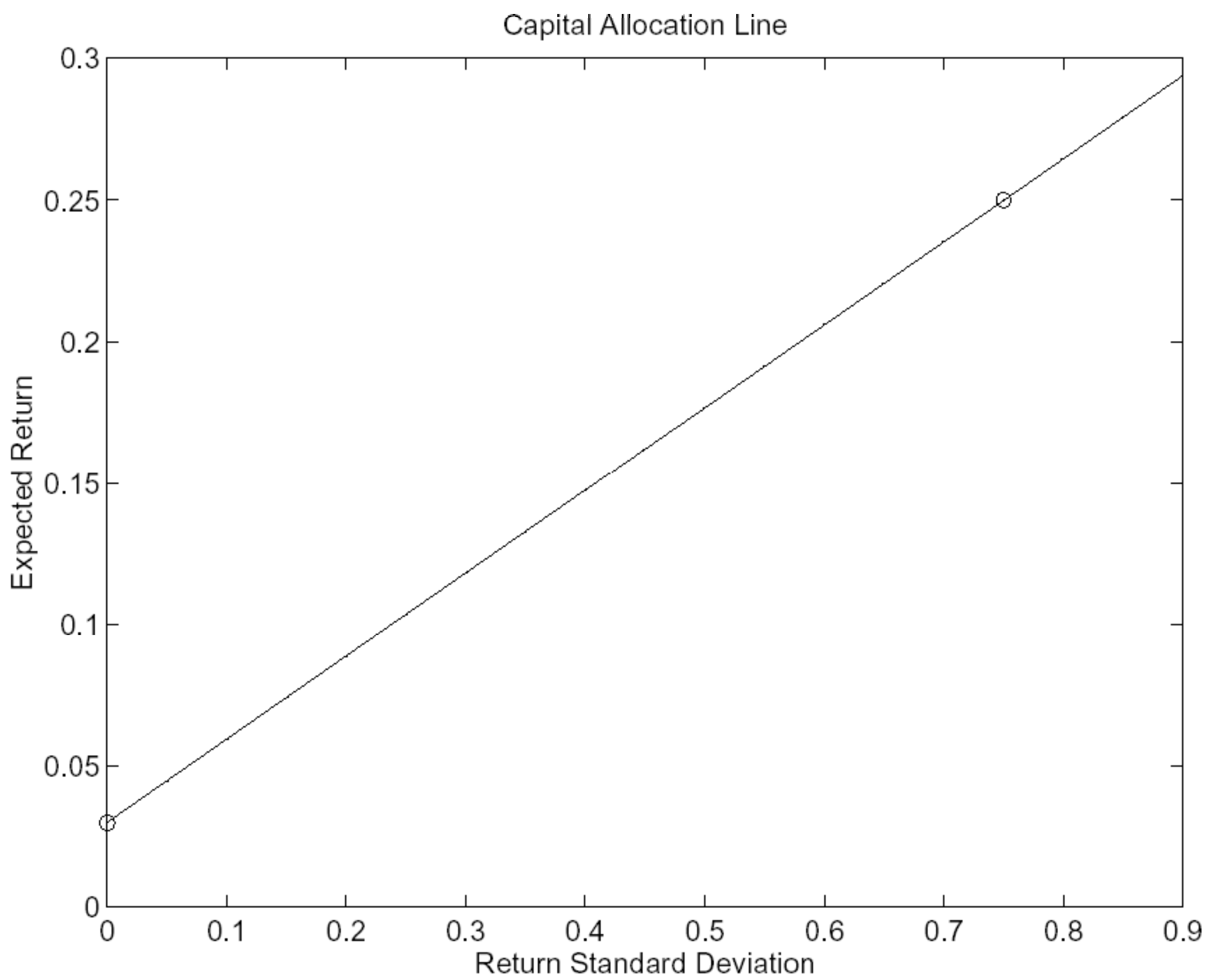
We can now substitute for "w" and get the CAL

$$E(\tilde{r}_p) = \frac{\sigma_p}{\sigma_A} E(\tilde{r}_A) + \left(1 - \frac{\sigma_p}{\sigma_A}\right) \cdot r_f = r_f + \sigma_p \left(\frac{E(\tilde{r}_A) - r_f}{\sigma_A}\right)$$

$$\text{Expected return} = \text{Risk free rate} + \text{Risk} \times \left(\frac{\text{Reward}}{\text{Risk}}\right)$$

The Capital Allocation Line

For our example: $$E(\tilde{r}_p) = 0.03 + \sigma_p\left(\frac{0.25 - 0.03}{0.75}\right) = 0.03 + 0.293 \times \sigma_p$$



Return for Portfolios with Multiple Assets

Example: You have a $1M portfolio with $200K invested in Microsoft and $800K in GM.

- If you expect (annual) returns of 10% for Microsoft and 15% for GM over the next year, then what is the expected return on the portfolio?

$$w_{\text{Microsoft}} \equiv w = \frac{200{,}000}{1{,}000{,}000} = 0.2 \Rightarrow \quad w_{\text{GM}} = (1-w) = 0.8$$

$$E(\tilde{r}_p) = w \cdot E(r_{\text{Microsoft}}) + (1-w) \cdot E(r_{\text{GM}})$$

$$= 0.2(10\%) + 0.8(15\%) = 14\%$$

Return for Portfolios with Multiple Assets

A year later, it turned out that the realized (annual) return on Microsoft was actually 12% and on GM it was -5%.

- What then is your realized return on the portfolio?

$$\tilde{r}_p = w \cdot \tilde{r}_{Microsoft} + (1 - w) \cdot \tilde{r}_{GM}$$
$$= 0.2(12\%) + 0.8(-5\%) = -1.6\%$$

- How much did you earn on your investment in Microsoft?

$$\$200K \times r_{Microsoft} = \$200K \times 12\% = \$24K$$

- How much did you earn on your portfolio?

$$\$1M \times r_p = \$1M \times (-0.016) = -\$16K$$

Risk of Portfolios with Multiple Assets

Example (continued): Suppose that the (annual) return standard deviation of these stocks over the following year will be is 40%, and the correlation between the return on Microsoft and GM is 0.3.

- What is the standard deviation of the return on your portfolio?

$$V(\tilde{r}_p) = V\left(w \cdot \tilde{r}_{Microsoft} + (1-w) \cdot r_{GM}\right)$$

$$= V\left(w \cdot \tilde{r}_{Microsoft}\right) + V\left((1-w) \cdot r_{GM}\right) + 2cov\left(w \cdot \tilde{r}_{Microsoft}, (1-w) \cdot r_{GM}\right)$$

$$= w^2 V\left(\tilde{r}_{Microsoft}\right) + (1-w)^2 V\left(r_{GM}\right) + 2w(1-w)cov\left(\tilde{r}_{Microsoft}, r_{GM}\right)$$

$$= w^2 V\left(\tilde{r}_{Microsoft}\right) + (1-w)^2 V\left(r_{GM}\right) + 2w(1-w)[\rho_{Microsoft,GM}\sigma_{Microsoft}\sigma_{GM}]$$

$$= 0.2^2(0.4)^2 + 0.8^2(0.4)^2 + 2 \times 0.2 \times 0.8[0.3 \times 0.4 \times 0.4] = 0.124$$

$$\Rightarrow \sigma_p = \sqrt{0.124} = 0.35$$