## Further Reading

The earliest known published work that introduces the ideas in this chapter was in 1202 by the famous Italian mathematician Fibonacci (or Leonardo of Pisa) in Liber Abaci (recently translated into English by Laurence Sigler, *Fibonacci's Liber Abaci*, *A Translation into Modern English of Leonardo Pisano's Book of Calculation*, Springer-Verlag, 2002). In this book, Fibonacci provides examples demonstrating the rules of time travel for cash flows.

Students who are interested in the early origins of finance and the historical development of the annuity formula will be interested in reading M. Rubinstein, *A History of the Theory of Investments*: *My Annotated Bibliography* (John Wiley and Sons, 2006) and W. Goetzmann and K. Rouwenhorst, eds., *Origins of Value*: *Innovations in the History of Finance* (Oxford University Press, 2005).

The material in this chapter should provide the foundation you need to understand the time value of money. For assistance using Excel, other spreadsheet programs, or financial calculators to compute present values, consult available help files and user manuals for additional information and examples.

Students in the lucky position of having to decide how to receive lottery winnings may consult A. Atkins and E. Dyl, "The Lotto Jackpot: The Lump Sum versus the Annuity," *Financial Practice and Education* (Fall/Winter 1995): 107–11.

## Problems

*All problems are available in* MyFinanceLab. *An asterisk (*) indicates problems with a higher level of difficulty.*

### The Timeline

1. You have just taken out a five-year loan from a bank to buy an engagement ring. The ring costs $5000. You plan to put down $1000 and borrow $4000. You will need to make annual payments of $1000 at the end of each year. Show the timeline of the loan from your perspective. How would the timeline differ if you created it from the bank's perspective?
2. You currently have a four-year-old mortgage outstanding on your house. You make monthly payments of $1500. You have just made a payment. The mortgage has 26 years to go (i.e., it had an original term of 30 years). Show the timeline from your perspective. How would the timeline differ if you created it from the bank's perspective?

### The Three Rules of Time Travel

3. Calculate the future value of $2000 in
   a. Five years at an interest rate of 5% per year.
   b. Ten years at an interest rate of 5% per year.
   c. Five years at an interest rate of 10% per year.
   d. Why is the amount of interest earned in part (a) less than half the amount of interest earned in part (b)?
4. What is the present value of $10,000 received
   a. Twelve years from today when the interest rate is 4% per year?
   b. Twenty years from today when the interest rate is 8% per year?
   c. Six years from today when the interest rate is 2% per year?
5. Your brother has offered to give you either $5000 today or $10,000 in 10 years. If the interest rate is 7% per year, which option is preferable?
6. Consider the following alternatives:
   i. $100 received in one year
   ii. $200 received in five years
   iii. $300 received in ten years

a. Rank the alternatives from most valuable to least valuable if the interest rate is 10% per year.
b. What is your ranking if the interest rate is only 5% per year?
c. What is your ranking if the interest rate is 20% per year?

**7.** Suppose you invest $1000 in an account paying 8% interest per year.
a. What is the balance in the account after 3 years? How much of this balance corresponds to "interest on interest"?
b. What is the balance in the account after 25 years? How much of this balance corresponds to interest on interest?

**8.** Your daughter is currently eight years old. You anticipate that she will be going to college in 10 years. You would like to have $100,000 in a savings account to fund her education at that time. If the account promises to pay a fixed interest rate of 3% per year, how much money do you need to put into the account today to ensure that you will have $100,000 in 10 years?

**9.** You are thinking of retiring. Your retirement plan will pay you either $250,000 immediately on retirement or $350,000 five years after the date of your retirement. Which alternative should you choose if the interest rate is
a. 0% per year?
b. 8% per year?
c. 20% per year?

**10.** Your grandfather put some money in an account for you on the day you were born. You are now 18 years old and are allowed to withdraw the money for the first time. The account currently has $3996 in it and pays an 8% interest rate.
a. How much money would be in the account if you left the money there until your 25th birthday?
b. What if you left the money until your 65th birthday?
c. How much money did your grandfather originally put in the account?

### Valuing a Stream of Cash Flows

**11.** Suppose you receive $100 at the end of each year for the next three years.
a. If the interest rate is 8%, what is the present value of these cash flows?
b. What is the future value in three years of the present value you computed in (a)?
c. Suppose you deposit the cash flows in a bank account that pays 8% interest per year. What is the balance in the account at the end of each of the next three years (after your deposit is made)? How does the final bank balance compare with your answer in (b)?

**12.** You have just received a windfall from an investment you made in a friend's business. He will be paying you $10,000 at the end of this year, $20,000 at the end of the following year, and $30,000 at the end of the year after that (three years from today). The interest rate is 3.5% per year.
a. What is the present value of your windfall?
b. What is the future value of your windfall in three years (on the date of the last payment)?

**13.** You have a loan outstanding. It requires making three annual payments at the end of the next three years of $1000 each. Your bank has offered to allow you to skip making the next two payments in lieu of making one large payment at the end of the loan's term in three years. If the interest rate on the loan is 5%, what final payment will the bank require you to make so that it is indifferent between the two forms of payment?

### Calculating the Net Present Value

**14.** You have been offered a unique investment opportunity. If you invest $10,000 today, you will receive $500 one year from now, $1500 two years from now, and $10,000 ten years from now.
a. What is the NPV of the opportunity if the interest rate is 6% per year? Should you take the opportunity?
b. What is the NPV of the opportunity if the interest rate is 2% per year? Should you take it now?

**15.** Marian Plunket owns her own business and is considering an investment. If she undertakes the investment, it will pay $4000 at the end of each of the next three years. The opportunity requires an initial investment of $1000 plus an additional investment at the end of the second year of $5000. What is the NPV of this opportunity if the interest rate is 2% per year? Should Marian take it?

## Perpetuities and Annuities

**16.** Your buddy in mechanical engineering has invented a money machine. The main drawback of the machine is that it is slow. It takes one year to manufacture $100. However, once built, the machine will last forever and will require no maintenance. The machine can be built immediately, but it will cost $1000 to build. Your buddy wants to know if he should invest the money to construct it. If the interest rate is 9.5% per year, what should your buddy do?

**17.** How would your answer to Problem 16 change if the machine takes one year to build?

**18.** The British government has a consol bond outstanding paying £100 per year forever. Assume the current interest rate is 4% per year.
a. What is the value of the bond immediately after a payment is made?
b. What is the value of the bond immediately before a payment is made?

**19.** What is the present value of $1000 paid at the end of each of the next 100 years if the interest rate is 7% per year?

***20.** You are head of the Schwartz Family Endowment for the Arts. You have decided to fund an arts school in the San Francisco Bay area in perpetuity. Every five years, you will give the school $1 million. The first payment will occur five years from today. If the interest rate is 8% per year, what is the present value of your gift?

***21.** When you purchased your house, you took out a 30-year annual-payment mortgage with an interest rate of 6% per year. The annual payment on the mortgage is $12,000. You have just made a payment and have now decided to pay the mortgage off by repaying the outstanding balance. What is the payoff amount if
a. You have lived in the house for 12 years (so there are 18 years left on the mortgage)?
b. You have lived in the house for 20 years (so there are 10 years left on the mortgage)?
c. You have lived in the house for 12 years (so there are 18 years left on the mortgage) and you decide to pay off the mortgage immediately *before* the twelfth payment is due?

**22.** You are 25 years old and decide to start saving for your retirement. You plan to save $5000 at the end of each year (so the first deposit will be one year from now), and will make the last deposit when you retire at age 65. Suppose you earn 8% per year on your retirement savings.
a. How much will you have saved for retirement?
b. How much will you have saved if you wait until age 35 to start saving (again, with your first deposit at the end of the year)?

**23.** Your grandmother has been putting $1000 into a savings account on every birthday since your first (that is, when you turned 1). The account pays an interest rate of 3%. How much money will be in the account on your 18th birthday immediately after your grandmother makes the deposit on that birthday?

**24.** A rich relative has bequeathed you a growing perpetuity. The first payment will occur in a year and will be $1000. Each year after that, on the anniversary of the last payment you will receive a payment that is 8% larger than the last payment. This pattern of payments will go on forever. If the interest rate is 12% per year,
a. What is today's value of the bequest?
b. What is the value of the bequest immediately after the first payment is made?

***25.** You are thinking of building a new machine that will save you $1000 in the first year. The machine will then begin to wear out so that the savings *decline* at a rate of 2% per year forever. What is the present value of the savings if the interest rate is 5% per year?

**26.** You work for a pharmaceutical company that has developed a new drug. The patent on the drug will last 17 years. You expect that the drug's profits will be $2 million in its first year and that this amount will grow at a rate of 5% per year for the next 17 years. Once the patent expires, other pharmaceutical companies will be able to produce the same drug and competition will likely drive profits to zero. What is the present value of the new drug if the interest rate is 10% per year?

**27.** Your oldest daughter is about to start kindergarten at a private school. Tuition is $10,000 per year, payable at the *beginning* of the school year. You expect to keep your daughter in private school through high school. You expect tuition to increase at a rate of 5% per year over the 13 years of her schooling. What is the present value of the tuition payments if the interest rate is 5% per year? How much would you need to have in the bank now to fund all 13 years of tuition?

**28.** A rich aunt has promised you $5000 one year from today. In addition, each year after that, she has promised you a payment (on the anniversary of the last payment) that is 5% larger than the last payment. She will continue to show this generosity for 20 years, giving a total of 20 payments. If the interest rate is 5%, what is her promise worth today?

**29.** You are running a hot Internet company. Analysts predict that its earnings will grow at 30% per year for the next five years. After that, as competition increases, earnings growth is expected to slow to 2% per year and continue at that level forever. Your company has just announced earnings of $1,000,000. What is the present value of all future earnings if the interest rate is 8%? (Assume all cash flows occur at the end of the year.)

***30.** Your brother has offered to give you $100, starting next year, and after that growing at 3% for the next 20 years. You would like to calculate the value of this offer by calculating how much money you would need to deposit in the local bank so that the account will generate the same cash flows as he is offering you. Your local bank will guarantee a 6% annual interest rate so long as you have money in the account.

a. How much money will you need to deposit into the account today?
b. Using an Excel spreadsheet, show explicitly that you can deposit this amount of money into the account, and every year withdraw what your brother has promised, leaving the account with nothing after the last withdrawal.

### Non-Annual Cash Flows

**31.** Suppose you currently have $5000 in your savings account, and your bank pays interest at a rate of 0.5% per month. If you make no further deposits or withdrawals, how much will you have in the account in five years?

**32.** Your firm spends $5000 every month on printing and mailing costs, sending statements to customers. If the interest rate is 0.5% per month, what is the present value of eliminating this cost by sending the statements electronically?

**33.** You have just entered an MBA program and have decided to pay for your living expenses using a credit card that has no minimum monthly payment. You intend to charge $1000 per month on the card for the next 21 months. The card carries a monthly interest rate of 1%. How much money will you owe on the card 22 months from now, when you receive your first statement post-graduation?

***34.** Your credit card charges an interest rate of 2% per month. You have a current balance of $1000, and want to pay it off. Suppose you can afford to pay off $100 per month. What will your balance be at the end of one year?

### Solving for the Cash Payments

**35.** You have decided to buy a perpetuity. The bond makes one payment at the end of every year forever and has an interest rate of 5%. If you initially put $1000 into the bond, what is the payment every year?

**36.** You are thinking of purchasing a house. The house costs $350,000. You have $50,000 in cash that you can use as a down payment on the house, but you need to borrow the rest of the purchase price. The bank is offering a 30-year mortgage that requires annual payments and has an interest rate of 7% per year. What will your annual payment be if you sign up for this mortgage?

***37.** You would like to buy the house and take the mortgage described in Problem 36. You can afford to pay only $23,500 per year. The bank agrees to allow you to pay this amount each year, yet still borrow $300,000. At the end of the mortgage (in 30 years), you must make a *balloon* payment; that is, you must repay the remaining balance on the mortgage. How much will this balloon payment be?

**38.** You have just made an offer on a new home and are seeking a mortgage. You need to borrow $600,000.

a. The bank offers a 30-year mortgage with fixed monthly payments and an interest rate of 0.5% per month. What is the amount of your monthly payment if you take this loan?

b. Alternatively, you can get a 15-year mortgage with fixed monthly payments and an interest rate of 0.4% per month. How much would your monthly payments be if you take this loan instead?

**39.** Suppose you take the 30-year mortgage described in Problem 38, part (a). How much will you still owe on the mortgage after 15 years?

***40.** You are thinking about buying a piece of art that costs $50,000. The art dealer is proposing the following deal: He will lend you the money, and you will repay the loan by making the same payment every two years for the next 20 years (i.e., a total of 10 payments). If the interest rate is 4% per year, how much will you have to pay every two years?

**41.** You are saving for retirement. To live comfortably, you decide you will need to save $2 million by the time you are 65. Today is your 30th birthday, and you decide, starting today and continuing on every birthday up to and including your 65th birthday, that you will put the same amount into a savings account. If the interest rate is 5%, how much must you set aside each year to make sure that you will have $2 million in the account on your 65th birthday?

***42.** You realize that the plan in Problem 41 has a flaw. Because your income will increase over your lifetime, it would be more realistic to save less now and more later. Instead of putting the same amount aside each year, you decide to let the amount that you set aside grow by 3% per year. Under this plan, how much will you put into the account today? (Recall that you are planning to make the first contribution to the account today.)

***43.** You are 35 years old, and decide to save $5000 each year (with the first deposit one year from now), in an account paying 8% interest per year. You will make your last deposit 30 years from now when you retire at age 65. During retirement, you plan to withdraw funds from the account at the end of each year (so your first withdrawal is at age 66). What constant amount will you be able to withdraw each year if you want the funds to last until you are 90?

***44.** You have just turned 30 years old, have just received your MBA, and have accepted your first job. Now you must decide how much money to put into your retirement plan. The plan works as follows: Every dollar in the plan earns 7% per year. You cannot make withdrawals until you retire on your sixty-fifth birthday. After that point, you can make withdrawals as you see fit. You decide that you will plan to live to 100 and work until you turn 65. You estimate that to live comfortably in retirement, you will need $100,000 per year starting at the end of the first year of retirement and ending on your 100th birthday. You will contribute the same amount to the plan at the end of every year that you work. How much do you need to contribute each year to fund your retirement?

***45.** Problem 44 is not very realistic because most retirement plans do not allow you to specify a fixed amount to contribute every year. Instead, you are required to specify a fixed percentage of your salary that you want to contribute. Assume that your starting salary is $75,000 per year

and it will grow 2% per year until you retire. Assuming everything else stays the same as in Problem 44, what percentage of your income do you need to contribute to the plan every year to fund the same retirement income

### The Internal Rate of Return

**46.** You have an investment opportunity that requires an initial investment of $5000 today and will pay $6000 in one year. What is the IRR of this opportunity?

**47.** Suppose you invest $2000 today and receive $10,000 in five years.
a. What is the IRR of this opportunity?
b. Suppose another investment opportunity also requires $2000 upfront, but pays an equal amount at the end of each year for the next five years. If this investment has the same IRR as the first one, what is the amount you will receive each year?

**48.** You are shopping for a car and read the following advertisement in the newspaper: "Own a new Spitfire! No money down. Four annual payments of just $10,000." You have shopped around and know that you can buy a Spitfire for cash for $32,500. What is the interest rate the dealer is advertising (what is the IRR of the loan in the advertisement)? Assume that you must make the annual payments at the end of each year.

**49.** A local bank is running the following advertisement in the newspaper: "For just $1000 we will pay you $100 forever!" The fine print in the ad says that for a $1000 deposit, the bank will pay $100 every year in perpetuity, starting one year after the deposit is made. What interest rate is the bank advertising (what is the IRR of this investment)?

**50.** You are considering purchasing a warehouse. The cost to purchase the warehouse is $500,000. Renting the equivalent space costs $20,000 per year. If the annual interest rate is 6%, at what rate must rental cost increase each year to make the cost of renting comparable to purchasing?

***51.** The Tillamook County Creamery Association manufactures Tillamook Cheddar Cheese. It markets this cheese in four varieties: aged 2 months, 9 months, 15 months, and 2 years. At the shop in the dairy, it sells 2 pounds of each variety for the following prices: $7.95, $9.49, $10.95, and $11.95, respectively. Consider the cheese maker's decision whether to continue to age a particular 2-pound block of cheese. At 2 months, he can either sell the cheese immediately or let it age further. If he sells it now, he will receive $7.95 immediately. If he ages the cheese, he must give up the $7.95 today to receive a higher amount in the future. What is the IRR (expressed in percent per month) of the investment of giving up $79.50 today by choosing to store 20 pounds of cheese that is currently 2 months old and instead selling 10 pounds of this cheese when it has aged 9 months, 6 pounds when it has aged 15 months, and the remaining 4 pounds when it has aged 2 years?

## Data Case

Assume today is June 1, 2009. Natasha Kingery is 30 years old and has a Bachelor of Science degree in computer science. She is currently employed as a Tier 2 field service representative for a telephony corporation located in Seattle, Washington, and earns $38,000 a year that she anticipates will grow at 3% per year. Natasha hopes to retire at age 65 and has just begun to think about the future.

Natasha has $75,000 that she recently inherited from her aunt. She invested this money in 10-year Treasury Bonds. She is considering whether she should further her education and would use her inheritance to pay for it.[9]

[9]If Natasha lacked the cash to pay for her tuition up front, she could borrow the money. More intriguingly, she could sell a fraction of her future earnings, an idea that has received attention from researchers and entrepreneurs; see Miguel Palacios, *Investing in Human Capital: A Capital Markets Approach to Student Funding*, Cambridge University Press, 2004.

She has investigated a couple of options and is asking for your help as a financial planning intern to determine the financial consequences associated with each option. Natasha has already been accepted to both of these programs, and could start either one soon.

One alternative that Natasha is considering is attaining a certification in network design. This certification would automatically promote her to a Tier 3 field service representative in her company. The base salary for a Tier 3 representative is $10,000 more than what she currently earns and she anticipates that this salary differential will grow at a rate of 3% a year as long as she keeps working. The certification program requires the completion of 20 Web-based courses and a score of 80% or better on an exam at the end of the course work. She has learned that the average amount of time necessary to finish the program is one year. The total cost of the program is $5000, due when she enrolls in the program. Because she will do all the work for the certification on her own time, Natasha does not expect to lose any income during the certification.

Another option is going back to school for an MBA degree. With an MBA degree, Natasha expects to be promoted to a managerial position in her current firm. The managerial position pays $20,000 a year more than her current position. She expects that this salary differential will also grow at a rate of 3% per year for as long as she keeps working. The evening program, which will take three years to complete, costs $25,000 per year, due at the beginning of each of her three years in school. Because she will attend classes in the evening, Natasha doesn't expect to lose any income while she is earning her MBA if she chooses to undertake the MBA.

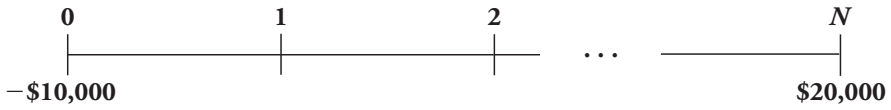
1. Determine the interest rate she is currently earning on her inheritance by going to Yahoo! Finance (http://finance.yahoo.com) and clicking on the 10-year bond link in the market summary. Then go to "Historical Prices" and enter the appropriate date, June 1, 2009, to obtain the closing yield or interest rate that she is earning. Use this interest rate as the discount rate for the remainder of this problem.
2. Create a timeline in Excel for her current situation, as well as the certification program and MBA degree options, using the following assumptions:
   - Salaries for the year are paid only once, at the end of the year.
   - The salary increase becomes effective immediately upon graduating from the MBA program or being certified. That is, because the increases become effective immediately but salaries are paid at the end of the year, the first salary increase will be paid exactly one year after graduation or certification.
3. Calculate the present value of the salary differential for completing the certification program. Subtract the cost of the program to get the NPV of undertaking the certification program.
4. Calculate the present value of the salary differential for completing the MBA degree. Calculate the present value of the cost of the MBA program. Based on your calculations, determine the NPV of undertaking the MBA.
5. Based on your answers to Questions 3 and 4, what advice would you give to Natasha? What if the two programs are mutually exclusive? That is, if Natasha undertakes one of the programs there is no further benefit to undertaking the other program. Would your advice be different?

CHAPTER 4 APPENDIX

# Solving for the Number of Periods

In addition to solving for cash flows or the interest rate, we can solve for the amount of time it will take a sum of money to grow to a known value. In this case, the interest rate, present value, and future value are all known. We need to compute how long it will take for the present value to grow to the future value.

Suppose we invest $10,000 in an account paying 10% interest, and we want to know how long it will take for the amount to grow to $20,000.




We want to determine $N$.

In terms of our formulas, we need to find $N$ so that the future value of our investment equals $20,000:

$$FV = \$10{,}000 \times 1.10^{N} = \$20{,}000 \qquad (4A.1)$$

One approach is to use trial and error to find $N$, as with the IRR. For example, with $N = 7$ years, $FV = \$19{,}487$, so it will take longer than seven years. With $N = 8$ years, $FV = \$21{,}436$, so it will take between seven and eight years. Alternatively, this problem can be solved on the annuity spreadsheet. In this case, we solve for $N$:

| | NPER | RATE | PV | PMT | FV | Excel Formula |
|---|---|---|---|---|---|---|
| **Given** | | 10.00% | −10,000 | 0 | 20,000 | |
| **Solve for NPER** | **7.27** | | | | | =NPER(0.10,0,−10000,20000) |

It will take about 7.3 years for our savings to grow to $20,000.

Finally, this problem can be solved mathematically. Dividing both sides of Eq. 4A.1 by $10,000, we have

$$1.10^{N} = 20{,}000/10{,}000 = 2$$

To solve for an exponent, we take the logarithm of both sides, and use the fact that $\ln(x^{y}) = y \ln(x)$:

$$N \ln(1.10) = \ln(2)$$

$$N = \ln(2)/\ln(1.10) = 0.6931/0.0953 \approx 7.3 \text{ years}$$

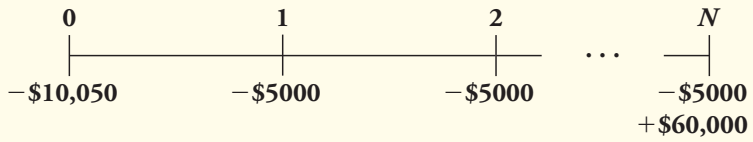
## EXAMPLE 4A.1 Solving for the Number of Periods in a Savings Plan

**Problem**

You are saving for a down payment on a house. You have $10,050 saved already, and you can afford to save an additional $5000 per year at the end of each year. If you earn 7.25% per year on your savings, how long will it take you to save $60,000?

**Solution**

The timeline for this problem is




We need to find $N$ so that the future value of our current savings plus the future value of our planned additional savings (which is an annuity) equals our desired amount:

$$10{,}050 \times 1.0725^N + 5000 \times \frac{1}{0.0725}(1.0725^N - 1) = 60{,}000$$

To solve mathematically, rearrange the equation to

$$1.0725^N = \frac{60{,}000 \times 0.0725 + 5000}{10{,}050 \times 0.0725 + 5000} = 1.632$$

We can then solve for $N$:

$$N = \frac{\ln(1.632)}{\ln(1.0725)} = 7.0 \text{ years}$$

It will take seven years to save the down payment. We can also solve this problem using the annuity spreadsheet:

| | NPER | RATE | PV | PMT | FV | Excel Formula |
|---|---|---|---|---|---|---|
| **Given** | | 7.25% | −10,050 | −5000 | 60,000 | |
| **Solve for N** | **7.00** | | | | | =NPER(0.0725,−5000,−10050,60000) |

# Problems

*All problems are available in* MyFinanceLab. *An asterisk (*) indicates problems with a higher level of difficulty.*

***A.1.** Your grandmother bought an annuity from Rock Solid Life Insurance Company for $200,000 when she retired. In exchange for the $200,000, Rock Solid will pay her $25,000 per year until she dies. The interest rate is 5%. How long must she live after the day she retired to come out ahead (that is, to get more in *value* than what she paid in)?

***A.2.** You are thinking of making an investment in a new plant. The plant will generate revenues of $1 million per year for as long as you maintain it. You expect that the maintenance cost will start at $50,000 per year and will increase 5% per year thereafter. Assume that all revenue and maintenance costs occur at the end of the year. You intend to run the plant as long as it continues to make a positive cash flow (as long as the cash generated by the plant exceeds the maintenance costs). The plant can be built and become operational immediately. If the plant costs $10 million to build, and the interest rate is 6% per year, should you invest in the plant?