Optimal financing for growth firms ☆

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Abstract

We analyze the optimal contract to finance the series of investments of a growing firm. The analysis is based on the need to repeatedly raise funds when informed insiders can expropriate outside investors. The optimal contract can be implemented by a sequence of one-period debt contracts and equity ownership by outsiders. Debt is optimal, as it reduces the expected cost of auditing, while partial equity ownership by insiders is optimal, as it mitigates the need for auditing in the presence of valuable growth opportunities. The model yields time-series implications regarding capital structure, investment and its fraction financed externally, and profitability.
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1. Introduction

In the last two decades, a great deal of private and public external financing took the form of equity. Frank and Goyal (2003) find that aggregate net public equity issues as a fraction of aggregate net assets more than doubled over the period 1971–1998. Also, the amount of private equity financing supplied by venture capitalists increased from about half a billion dollars in 1979 to $10 billion in 1997 (Gompers and Lerner, 1999) and has reached much higher levels more recently (Denis, 2004). Equity financing appears to be especially important for small and high-growth firms. In particular, Frank and Goyal (2003) find that public equity issues track the

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☆ This paper is based on the main chapter of my dissertation. An earlier version circulated under the title "Growth, Investor Protection and Security Design".

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financing deficit more closely than do debt issues for the average firm and that debt issues track the financing deficit better for their sub-sample of large firms. Similarly, Mayer and Sussman (2004) find that investment "spikes" are predominantly financed with debt among large firms and with equity among small firms.[1] Private firms that receive venture capital are likely to be in early stages of development and offer high potential (Sahlman, 1990; Kaplan and Strömberg, 2003), are likely to go public (Gompers and Lerner, 1999), and are more likely to be "innovator" than "imitator" firms (Hellmann and Puri, 2000).

Several security design analyses point out the advantages of debt financing when information asymmetries are acute, e.g., to mitigate adverse selection (Myers and Majluf, 1984; Nachman and Noe, 1994), reduce monitoring costs (Townsend, 1979; Gale and Hellwig, 1985), mitigate moral hazard (Innes, 1990), and prevent early liquidation of firm assets (Diamond, 1984; Bolton and Scharfstein, 1990).[2] To explain the reliance of young and high-growth firms on external equity financing, we analyze a dynamic model of financial contracting with information asymmetries.

In the model, a risk neutral entrepreneur (the "agent") who has limited funds of her own is endowed with an (inalienable) investment technology that spans several periods. Each period, the agent approaches a competitive financial market (the "investor") to raise funds, investment occurs, and cash flows are realized. Funds are raised with a sequence of one-period financial contracts.[3] Financing constraints are present, as realized cash flows are observable to the agent only and can be diverted at the expense of investors. Cash flows can be falsified by the agent (Lacker and Weinberg, 1989) or verified by the investor (Townsend, 1979; Gale and Hellwig, 1985; Diamond, 1984). Funds diverted by the agent can be wasted on company perks (Jensen and Meckling, 1976) or used to finance future investments (Hart and Moore, 1998) subject to the cost of legitimization. Intuitively, investment of diverted funds requires some legitimization, as its value to the agent comes from the returns later realized, while for consumption there is less need for legitimization as it is consumed. Leading examples of such costly legitimization efforts on behalf of the agent are the hiring of more and/or better lawyers, time spent in evidence production/fabrication, and even bribes paid to court officials (Bond, 2007a, 2007b).[4] We, therefore, assume that the agent can at most invest a fraction of diverted funds.[5]

The presence of future investment opportunities plays a role in alleviating financial constraints. In order to induce truthful information, investors can deploy disciplinary mechanisms, such as the threat of verification (Gale and Hellwig, 1985). We show that in a dynamic setting this need not be necessary, as diversion of funds reduces the amount of funds available for investment in future periods and entails a cost for the agent. The more valuable the future investment oppor-

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[1] Similarly, Goyal et al. (2002) find that firms increase their debt in years of reduced growth, and Helwege and Liang (1996) find that firms that issue debt are much larger and grow more slowly, relative to those that issue equity or convertible debt.

[2] Studies that address the optimality of equity financing include Aghion and Bolton (1992), Chang (1992), Dewatripont and Tirole (1994) and Ravid and Spiegel (1997).

[3] In the model, the agent is free to approach a competitive financial market prior to each investment, and the investor cannot control future productivity of the agent. We show in Section 2.3 that this leads to a sequence of one-period financial contracts (as in Diamond, 1989, 1991a). For discussion regarding long-term contracts see Section 5.3.

[4] See Bond (2007a, 2007b) for a discussion of the costs associated with legitimization and persuasion of courts and a discussion of the empirical evidence.

[5] Introducing a cost associated with consuming diverted funds would not qualitatively change the main results of the paper, provided that relative to consumption, investing diverted funds imposes a higher cost on the agent. We abstract from this, however, in order to keep the analysis as simple as possible and focus on the issue that is most important for our dynamic considerations, i.e., the possibility to invest diverted funds.

tunities, the higher the cost. Thus, the presence of future investment opportunities *disciplines*, so to speak, the agent to reveal information truthfully and repay investors.

The optimal financial contract resembles a combination of (risky) debt and outside equity. In particular, up to a threshold (i.e., the face value of debt) all cash flows are allocated to the investor, and above this threshold any excess cash flows are proportionally divided between the agent and the investor (i.e., external equity). In equilibrium, the agent is indifferent to either paying a fraction of (excess) cash flows to investors or diverting funds at the cost of reducing future investments. The fraction of equity allocated to the agent is just sufficiently large enough to provide her with the incentive to repay the investor without the threat of verification, as is determined by the value of future investment opportunities. Outside equity emerges as an optimal security entirely as a consequence of the *dynamics* of the problem. In particular, in a one-period version of the model, the optimal security is debt, which is optimal as it minimizes expected verification costs.

The analysis yields several empirical predictions. In particular, it relates firms' relative reliance on debt and external equity financing to the value of future investment opportunities. High growth firms are predicted to allocate a higher fraction of firm equity to outside investors and finance a larger portion of investment with external equity, relative to debt. Intuitively, when growth opportunities are valuable, diversion of funds implies a costly reduction in future investment, and a lower allocation of firm equity to the agent suffices to induce truth-telling. The model also yields testable predictions regarding capital structure, investment, and profitability over the firm's growth cycle. In particular, young firms are predicted to rely more on external equity financing, relative to debt, and are predicted to issue new equity more frequently, relative to mature firms. Also, profitability is predicted to be non-monotonic over the firm's growth cycle, increasing at first when the firm is young but decreasing afterwards. Interestingly, the equilibrium relation between leverage and profitability for mature firms is consistent with the well-known empirical regularity that better performing firms (e.g., higher profits) have lower leverage ratios, while the equilibrium relation is reversed for young firms.[6]

In the model, incentives for expropriation by the agent are mitigated by the possibility of verification and the inability to freely invest diverted funds in future periods. As an extension of our analysis, we explore the implications of a more efficient auditing technology on the optimal financing contract (e.g., due to relationship lending) and the implications of higher costs associated with legitimizing investment of diverted funds (e.g., due to more strict legal *enforcement*). Finally, we discuss the implications of long-term contracts for our results.

Our study is related to the literature that studies security design in a dynamic setting. Bolton and Scharfstein (1990), Gromb (1999) and DeMarzo and Fishman (2007) study a dynamic environment in which the agent privately observes cash flows but can be induced to pay investors via the threat of the loss of control of the project. In DeMarzo and Fishman (2007), the optimal contract can be implemented by a combination of equity, long-term debt and a line of credit. Biais et al. (2007) and DeMarzo and Sannikov (2006) extend the discrete-time model of DeMarzo and Fishman (2007) to continuous-time.[7] The above studies follow Lacker and Weinberg (1989) and assume that diversion of funds is costly, as only a fraction of diverted funds can convert into con-

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[6] See Harris and Raviv (1992) for a thorough discussion of the empirical evidence.

[7] Biais et al. (2007) examine a stationary version of the DeMarzo and Fishman (2007) model, in which cash flows are assumed to follow a stationary binomial process, and analyze the continuous-time limit of the model obtained when the length of each period goes to zero. DeMarzo and Sannikov (2006) use the methodology developed in Sannikov (in press) and directly analyze a continuous-time version of the DeMarzo and Fishman (2007) model.

sumption. Moreover, they show that the fraction of equity allocated to investors under the optimal contract equals this fraction of diverted funds they assume cannot convert into consumption. By considering a series of investment periods, however, we show that the "cost" of diverting funds is endogenously determined by the value of future investment opportunities. Consequently, after deriving the optimal contract, we link the optimal capital structure (i.e., mix of debt and external equity) to the value of future investment opportunities and offer unique empirical predictions relating to capital structure and firm growth.[8]

Exploring the optimal long-term contract when the agent privately observes cash flows but the investor can verify cash flows at a cost, Wang (2005), and Monnet and Quintin (2005) show that low payments made by the agent are followed by higher future payments and/or verification with some probability by the investor. While we also consider a multi-period model with costly-state-verification we do not analyze the long-term financial contract but the sequence of short-term contracts. The agent's ability to revisit financial markets, prior to each investment, in our model, mitigates the agent's incentives to divert funds and brings about external equity as part of the optimal contract.[9]

The paper is organized as follows. In Section 2 we introduce the model. In Section 3 we derive the optimal contract and discuss its implementation. In Section 4 we analyze the relation between the optimal financial contract and firm growth, the relation between profitability and the firm's reliance on debt financing, and we analyze a growth cycle version of the model. In Section 5 we explore comparative statics with respect to the cost of monitoring and the fraction of diverted funds that can be reinvested, and discuss the implications of long-term contracts. In Section 6 we conclude. All proofs appear in Appendix A.

2. The model

A risk neutral agent who has limited initial funds, w_1, is endowed with a constant-returns-to-scale investment technology that spans N periods. Specifically, in period $n \in \{1, N\}$ an investment of $\theta_n \geqslant 0$ yields random cash flows $x_n \equiv \theta_n x_n^1$, where x_n^1 has c.d.f. F_n, p.d.f. f_n, support $[0, \bar{x}] \equiv \mathcal{X}$ and $E(x_n^1) > 1$. This production technology is inalienable, as in Hart and Moore (1994).[10] Each period, the agent with wealth w_n can approach a competitive financial market (the "investor") to raise funds $\theta_n - w_n$, to finance investment θ_n. At the end of each period, realized cash flows x_n are observable to the agent only and can be diverted at the expense of investors. The agent's utility is given by total consumption throughout the N periods, with a discount factor normalized to 1.

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[8] In Lacker and Weinberg (1989) the cost of diversion is a function of the amount diverted and the realized cash flow. Under the optimal contract, a fraction of cash flows are allocated to the investor, while this fraction depends on the properties of the assumed cost function. Intuitively, under the (no-falsification) optimal contract, the marginal benefit to the agent from diverting funds (net of diversion costs) equals the marginal benefit from reporting truthfully. In our model diversion is costly, as it leads to lower future investment under the optimal contract.

[9] Another branch of the literature focuses on the long-term contract in an environment with limited enforcement. In Hart and Moore (1994, 1998), Hart (1995) and Albuquerque and Hopenhayn (2004), payments to the entrepreneur suffice to prevent her from pursuing her outside option. Similarly, in Fluck (1998) and Myers (2000), investors' share of cash flows suffices to prevent the replacement of the manager. As these analyses consider symmetric information, our approach is quite different.

[10] That is, the investor cannot control future productivity of the agent. While *complete* inalienability is somewhat problematic, the assumption helps keep the analysis tractable and focused. In Section 5.3 we discuss the possibility of relaxing this assumption.

2.1. Auditing and manipulation

While the investor must rely on the agent's report, t_n, regarding the privately observed cash flow, x_n, she can attempt to verify its realization by auditing/monitoring the agent (e.g., Townsend, 1979; Gale and Hellwig, 1985). Auditing effort by the investor $a_n \in [0, \infty]$ (e.g., visiting the firm's facility, investigating financial statements, and sitting on the board of directors) results in a public signal s_n. Auditing is costly to the investor, and in particular, effort a_n results in cost δa_n ($\delta > 0$). In an attempt to divert funds to her own benefit, the agent can under-report cash flows (i.e., choose $t_n < x_n$) and falsify the state of the world (e.g., Lacker and Weinberg, 1989).[11] Falsification by the agent includes misrepresentation of accounting data, falsification of transactions, and opportunistic earnings management. The agent's falsification strategy is denoted by $\mathbf{f}_n \in \{0, 1\}$ where 0 stands for "not-falsify" and 1 for "falsify." If the agent chooses to falsify, i.e., $\mathbf{f}_n = 1$, then she bears a (non-pecuniary) falsification cost that equals the investor's auditing effort a_n. Thus, by auditing with effort a_n the investor bears the cost of δa_n but imposes a falsification cost of a_n on the agent, if the latter chooses to falsify the state of the world.

The public signal regarding realized cash flows (or *publicly observed* funds), s_n, may or may not reveal the true state of the world, x_n, depending on whether falsification takes place. The public signal reveals the true cash flow (i.e., $s_n = x_n$) when $\mathbf{f}_n = 0$ but has no informational value (i.e., $s_n = t_n$) when $\mathbf{f}_n = 1$. The signal s_n is formalized in (1):

$$s_n = s_n(x_n, t_n, \mathbf{f}_n) = \begin{cases} x_n & \text{if } \mathbf{f}_n = 0, \\ t_n & \text{if } \mathbf{f}_n = 1. \end{cases} \tag{1}$$

2.2. The link between periods

Funds (whether diverted or not) can be consumed or saved for future investment. Intuitively, and as discussed earlier, diverted funds invested require some sort of legitimization in order for the agent to benefit from the future returns they generate, while for consumption of diverted funds in the form of company perks there is less need for legitimization, as it is consumed. This is captured in the model by the assumption that only a portion of diverted funds can be used to finance future investments. Formally, one dollar of diverted funds saved converts to $(1 - \rho)$ in the following period, $\rho \in [0, 1]$.[12]

As the agent can save diverted funds, it is useful to distinguish between *legitimate* and *non-legitimate* funds. Legitimate funds are defined with respect to publicly observed funds s_n. In particular, following a given public signal s_n and payment of d_n to the investor (we discuss payments below), the amount of *legitimate* funds possessed by the agent at the end of the period equals $s_n - d_n$. The amount of *non-legitimate* funds (privately observed by the agent), then, equals $x_n - s_n$. The sum of legitimate and non-legitimate funds equals the total amount of funds possessed by the agent, $x_n - d_n$. If the agent decides to save an amount of z_n of non-legitimate funds, in addition to legitimate funds $s_n - d_n$, then her wealth at the beginning of the next period,

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[11] For simplicity, we do not consider the possibility of over-reporting cash flows. Such flexibility, however, would not affect the results, as the agent does not have an incentive to over-report cash flows in equilibrium.

[12] Others in the literature who have distinguished between consumption and investment of diverted funds are Bolton and Scharfstein (1990) and Hart and Moore (1998). In the former, investors can enforce a covenant restricting further investment by the firm (e.g., following low reported cash flows), preventing investment of diverted funds, while in the latter the entrepreneur is free to invest diverted funds in future projects.

w_{n+1}, includes legitimate funds $s_n - d_n$ in addition to (net) diverted funds saved $(1 - \rho)z_n$, i.e.,

$$w_{n+1} = s_n - d_n + (1 - \rho)z_n.$$

2.3. The financial contract

The financial contract specifies investments θ_n, payments to the investor d_n, and auditing by the investor a_n, for all periods $n \in \{1, N\}$.[13] Due to the ability of the agent to approach competitive financial markets prior to each investment and the inalienability of investment, the investor breaks even on her investment in each period. In particular, suppose, by contradiction, that in the last period N, the expected payment to the investor (net of auditing costs), $E(d_N - \delta a_N)$, is higher than the amount of funds the investor contributes for investment in the last period, $\theta_N - w_N$. Then, by approaching a different investor (or renegotiating the contract), the agent can finance the same investment $\theta_N - w_N$, realize cash flows x_n, but pay the investor less (with the same auditing schedule a_N). Thus, in the last period the investor will break even in equilibrium, i.e., $E(d_N - \delta a_N) = \theta_N - w_N$. By backwards induction, it can be shown to be true for all periods, i.e., $E(d_n - \delta a_n) = \theta_n - w_n$ for all $n \in \{1, N\}$. As payments to the investor in any period n do not depend on past payments, we analyze the optimal sequence of one-period contracts.[14]

The financial contract in period n, $\sigma_n = (\theta_n, a_n, d_n)$ cannot directly specify payments to the investor as a function of realized cash-flows; however, payments can depend on reported cash flows t_n and the signal received by the investor s_n, i.e., $d_n = d_n(t_n, s_n)$. The difference $d_n(t_n, s_n) - d_n(s_n, s_n)$ can be interpreted as the *fine* imposed on an agent when lying is detected.[15] Similarly, the degree of auditing can depend on reported cash flows, i.e., $a_n = a_n(t_n)$. Due to limited liability of the agent, a contract is feasible only if the investor's payoff, d_n, can be covered by publicly observed funds, s_n, i.e., $d_n(t_n, s_n) \leqslant s_n$.

2.4. Time line

At the beginning of period n the agent and the investor enter a financial contract to raise $\theta_n - w_n$ for a total investment of θ_n in the project. After cash flows x_n are realized and privately observed by the agent, she reports cash flows t_n to the investor (in equilibrium, the agent will report truthfully). The investor then audits with degree a_n, while the agent either falsifies the state of the world or not (falsification will not take place in equilibrium). The signal s_n is then realized according to (1), and the investor is paid d_n. At the end of the period, the agent can then consume or save her funds.

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[13] It is common in the security design literature to treat auditing efforts by the investor as contractible. Notable examples include Townsend (1979), Gale and Hellwig (1985), Diamond (1984), and Mookherjee and Png (1989). See Krasa and Villamil (2000), however, for an analysis of the case in which the principal cannot commit to an auditing schedule.

[14] See discussion in Section 5.3 regarding long-term contracts.

[15] In Proposition 1, it is shown that under the optimal contract the maximal fine is imposed on the agent for lying, i.e., $d_n(t_n, s_n) = s_n$. In order to simplify the proof of Proposition 1, however, Assumption A1, in the appendix, states that $d_n(t_n, s_n) - d_n(s_n, s_n) \geqslant 0$; i.e., contracts cannot reward the agent for lying (see also Melumad and Mookherjee, 1989 for a similar assumption).

3. The contracting problem

The agent designs a contract that maximizes her expected consumption subject to the investor's participation constraint, limited liability, and her incentive compatibility constraint. Let $V_n(w)$ denote the indirect utility function (i.e., expected future consumption) of an agent with funds w at the beginning of period $n \leqslant N + 1$. Following the last investment period the agent can only consume funds, and $V_{N+1}(w) = w$.

We will eventually show that the value function $V_n(w)$ is linear in w (this is due to the constant returns to scale investment technology). Meanwhile, however, we rely on the induction assumption **(C1)** below, which trivially holds for the last investment period $n = N$ and is eventually verified to hold in equilibrium for all periods. We say that condition **(C1)** holds for period n if

$$\text{for all } k \in \{n + 1, N + 1\}, \quad V_k(w) = wR_k \text{ for some } R_k \geqslant 1. \tag{C1}$$

Consider an arbitrary period n and suppose that condition **(C1)** holds for that period. We start the analysis at the end of the period (i.e., the agent's saving/consumption decision) and go backwards. Consider the contract σ_n, cash-flow x_n, report t_n, and signal s_n. At this point, the agent has funds of $x_n - d_n$, which can either be consumed or saved for next period. From these funds of the agent, $s_n - d_n$ are legitimate (the public is aware of these funds) while $x_n - s_n$ are diverted funds, i.e., are non-legitimate. It is clearly optimal to save *legitimate* funds and use them to finance future investments. However, the agent faces a trade-off with regard to *diverted* funds. It is optimal for the agent to save diverted funds when future investments are sufficiently profitable. Formally, let $\hat{V}_n(x_n, t_n, s_n \mid \sigma_n)$ denote the indirect utility function just before the agent's saving/consumption decision (see Fig. 1). Thus,

$$\hat{V}_n(x_n, t_n, s_n \mid \sigma_n) = \max_{z_n \in [0, x_n - s_n]} \underbrace{x_n - s_n - z_n}_{\text{consumption}} + \underbrace{\left(s_n - d_n + (1 - \rho)z_n\right)}_{\text{savings}} R_{n+1}. \tag{2}$$

It is optimal for the agent to either save or consume *all* diverted funds, if such exist. In particular, only when the net return on an investment of diverted funds is sufficiently high, i.e. $(1 - \rho)R_{n+1} > 1$, is it optimal for the agent to save diverted funds (Lemma 1 in the appendix). In particular,

$$\hat{V}_n(x_n, t_n, s_n \mid \sigma_n) = (s_n - d_n)R_{n+1} + (x_n - s_n)\max\left(R_{n+1}(1 - \rho), 1\right).$$

Next, consider the agent's incentives to falsify the signal s_n, following realized cash flow x_n, and under a contract that implies an expected degree of auditing $E(a_n \mid t_n, \sigma_n)$.[16] Formally, let
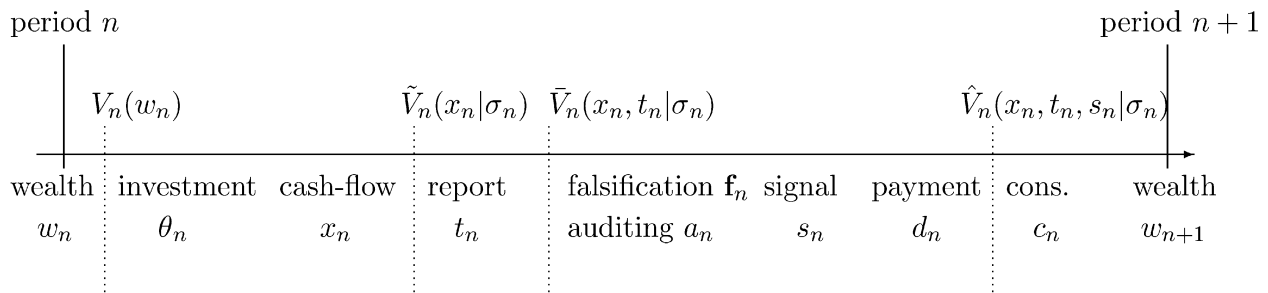
period n period $n + 1$

	$V_n(w_n)$		$\tilde{V}_n(x_n\mid\sigma_n)$	$\bar{V}_n(x_n, t_n\mid\sigma_n)$			$\hat{V}_n(x_n, t_n, s_n\mid\sigma_n)$	
wealth	investment	cash-flow	report	falsification $\mathbf{f}_n$	signal	payment	cons.	wealth
w_n	θ_n	x_n	t_n	auditing a_n	s_n	d_n	c_n	w_{n+1}

Fig. 1. Time line.

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[16] The expectation $E(a_n \mid t_n, \sigma_n)$ accounts for potential stochastic auditing schedules. We will soon show, however, that deterministic auditing is optimal.

the agent's indirect utility function before the falsification decision, but after the report t_n, be $\bar{V}_n(x_n, t_n \mid \sigma_n)$.

$$\bar{V}_n(x_n, t_n \mid \sigma_n) = \max\Big(\underbrace{\hat{V}_n(x_n, t_n, t_n \mid \sigma_n) - E(a_n \mid t_n, \sigma_n)}_{\text{falsification } \mathbf{f}_n = 1}, \underbrace{\hat{V}_n(x_n, t_n, x_n \mid \sigma_n)}_{\text{non-falsification } \mathbf{f}_n = 0}\Big). \tag{3}$$

Thus, it is optimal for the agent to falsify the state of the world (i.e., $s_n = t_n$) if the expected degree of auditing $E(a_n \mid t_n, \sigma_n)$ is sufficiently low or the benefits from falsification are sufficiently large (Lemma 2 in the appendix). Next, denote by $\tilde{V}_n(x_n \mid \sigma_n)$ the agent's indirect utility function just before the report t_n is made. Thus,

$$\tilde{V}_n(x_n \mid \sigma_n) = \max_{t_n \in [0, x_n]} \bar{V}_n(x_n, t_n \mid \sigma_n). \tag{4}$$

The optimal contract must satisfy two conditions. First, the expected payment to the investor net of auditing costs must exceed the loan amount of $\theta_n - w_n$ (Individual Rationality), i.e., $E(d_n - a_n\delta) \geqslant \theta_n - w_n$. Second, the maximal payment to the investor is limited by the amount of publicly observed funds available to the agent (Limited Liability), i.e., $d_n(t_n, s_n) \leqslant s_n$.

In Proposition 1 (see the appendix) we show that the Revelation Principle applies in this setting, and it is therefore sufficient to consider truth-telling equilibria. Further, it is shown that it is optimal to specify the highest possible payment to the investor when a report is discovered to be not truthful, i.e., $d_n = s_n$ whenever $s_n > t_n$. Finally, as the agent's optimal falsification strategy, defined in (3), depends on the expected degree of auditing, it is shown that we can restrict attention to deterministic auditing schedules. Proposition 1 allows us to define the set of feasible contracts, $\mathcal{F}$, to include contracts that satisfy the limited liability constraint, impose the maximal payment on the agent if lying is detected, and specify a deterministic auditing schedule.[17]

An incentive compatible contract σ_n satisfies $\bar{V}_n(x_n, x_n \mid \sigma_n) \geqslant \bar{V}_n(x_n, t_n \mid \sigma_n)$, for all $t_n \leqslant x_n \in \mathcal{X}$. It follows from Proposition 1, however, that lying, i.e., $t_n = \tilde{t} < x_n$, and not falsifying, i.e., $\mathbf{f}_n = 0$, is dominated by truth-telling. Intuitively, by lying and not falsifying, the agent's false report is discovered, and maximal payment is imposed, i.e., $d_n(\tilde{t}, x_n) = x_n$.[18] Consider, therefore, a deviation where $t_n = \tilde{t} < x_n$, and $\mathbf{f}_n = 1$. As lying is not detected, i.e., $s_n = \tilde{t}$ (see (1)), the agent's payoff is $\bar{V}_n(x_n, \tilde{t} \mid \sigma_n) = (\tilde{t} - d_n(\tilde{t}, \tilde{t}))R_{n+1} + (x_n - \tilde{t})\max(R_{n+1}(1 - \rho), 1) - a_n(\tilde{t})$ (see (2) and (3)). This should be compared to the agent's payoff from reporting truthfully, $\bar{V}_n(x_n, x_n \mid \sigma_n) = R_{n+1}(x_n - d_n(x_n, x_n))$. The incentive compatibility constraint (**IC**, below) ensures that truth telling is preferred by the agent under the optimal contract.

We are now ready to formalize the agent's problem in (5) below,

$$V_n(w) \equiv \max_{\sigma_n \in \mathcal{F}} R_{n+1} E\big[x_n - d_n(x_n, x_n)\big] \tag{5}$$

$$\text{s.t.} \quad \textbf{IR} \quad E\big(d_n(x_n, x_n) - a_n(x_n)\delta\big) \geqslant \theta_n - w_n,$$

$$\textbf{IC} \quad R_{n+1}\big(d_n(x, x) - d_n(t, t)\big) \leqslant a_n(t) + R_{n+1}(x - t)\min\Big(\rho, \frac{R_{n+1} - 1}{R_{n+1}}\Big),$$

$$\text{for all } t \leqslant x \in \mathcal{X},$$

$$\textbf{LL} \quad d_n(x, x) \leqslant x.$$

Through the **IC** constraint one can see how future investment opportunities modify the agent's incentives to pay investors. In particular, the potential gain from reporting non-truthfully, i.e.,

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[17] Formally the set $\mathcal{F}$ is given by $\mathcal{F} \equiv \{\sigma \mid a : \mathcal{X} \to [0, \infty), \ d(t, t) \leqslant t, \text{ and } d(t, s) = s \text{ for all } t < s \in \mathcal{X}\}$; we also restrict attention to continuous payment schedules.

[18] It suffices that $d_n(t_n, s_n) \geqslant d_n(s_n, s_n)$; see Lemma 3 in the appendix.

reporting $t < x_n$ and paying the investor $d_n(t, t)$ instead of $d_n(x_n, x_n)$, is balanced by the cost of falsification, determined by $a_n(t)$, and by the cost due to the reduction of $x_n - t$ in legitimate funds. Incentives, thus, change in a dynamic setting. First, future investment opportunities increase the gains from diverting funds, $R_{n+1}(d_n(x_n, x_n) - d_n(t, t))$. Second, however, they increase the opportunity cost of reporting low cash flows and discipline the agent to pay investors, $R_{n+1}(x_n - t) \min(\rho, \frac{R_{n+1}-1}{R_{n+1}})$. The net effect of these forces depends on the optimal contract.

3.1. Optimal contract

The optimal contract is derived using backward induction. To simplify the analysis and provide intuition, we first focus on the case of a fixed scale of investment (Proposition 2) and then proceed to present the optimal contract with variable investment (Theorem 1).

Before formally analyzing the optimal contract for the case of fixed investment, we would like to provide intuition for the optimality of external equity financing. Consider a contract that pays out a fixed fraction $\gamma \in [0, 1]$ of cash flows to outside investors and entails no auditing by the investor, regardless of reported cash flows (i.e., $d_n(x, x) = \gamma x$, and $a_n(x) = 0$, for all x). Would the agent report truthfully and pay investors in the absence of auditing? To illustrate why payments by the agent may still be sustainable in equilibrium, consider an agent with realized cash flows x_n who wishes to report t instead of x_n, for some $t < x_n$. As there is no auditing by the investor, falsification is optimal, and $s_n = t$. Such a strategy would imply diverted funds of $x_n - t$ and benefit of $(x_n - t) \max(1, R_{n+1}(1 - \rho))$ to the agent (see (2)). Thus, the marginal benefit from diversion is endogenously given by $\max(1, R_{n+1}(1 - \rho))$. On the other hand, such a strategy would decrease the agent's legitimate funds from the level of $x_n(1 - \gamma)$ to a level of $t(1 - \gamma)$, and the amount of legitimate funds reinvested would be reduced by $(x_n - t)(1 - \gamma)$. Thus the marginal cost of diversion is endogenously given by $R_{n+1}(1 - \gamma)$. If the fraction of cash flows allocated to the investor is below the threshold,

$$\gamma_n \equiv \min\left(\frac{R_{n+1} - 1}{R_{n+1}}, \rho\right),$$

then strategically reporting low cash flows $t < x_n$ is not optimal for the agent. In other words, payments are sustained in equilibrium as long as the fraction of cash flows paid out to the investor is sufficiently low.

With this in mind, and for a fixed level of investment, say $\theta_n = 1$, the agent can raise at most $\gamma_n E(x_n)$ externally (by selling a fraction γ_n of cash flows x_n) without utilizing the auditing technology. Consistently, we show in Proposition 2 that for $w_n \geqslant 1 - \gamma_n E(x_n)$, it is optimal for the agent to finance the project of scale $\theta_n = 1$ with contract $d_n(x, x) = (\frac{1-w_n}{E(x_n)})x$, and bear no auditing costs. If the return on future investment is sufficiently low, however, then external funds $\gamma_n E(x_n)$ together with the agent's wealth w_n need not suffice to cover the desired investment level, $\theta_n = 1$ (in particular for $w_n < 1 - \gamma_n E(x_n)$). In such cases, additional investment funds are required to finance the project, and the auditing technology must be utilized in order to support higher payments to the investor. But we establish in Proposition 2 that instead of increasing the fraction of cash flows allocated to the investor, it is optimal to pay the investor all cash flows up to a threshold and a fraction γ_n of excess cash flows above the threshold. Moreover, auditing will take place when reported cash flows are below this threshold.

Before deriving the optimal financing contract for a fixed level of investment, it is useful to define the threshold cash flow D_n (in (6)), which, as we show in Proposition 2 below, is part of

the optimal contract. Recall from (5) that the investor will participate in the contract as long as

$$E\big(d_n(x_n, x_n) - a_n(x_n)\delta\big) \geq \theta_n - w_n.$$

To define the threshold level D_n, it is useful to consider a payment schedule $d_n^D(x_n, x_n)$ that allocates to the investor all cash flows up to the threshold d, and a fraction γ_n of excess cash flows above this threshold,

$$d_n^D(x_n, x_n) = \min(d, x_n) + \gamma_n \max(x_n - d, 0).$$

In order for the agent to report truthfully cash flow $x_n = x^*$ that is below the threshold d, i.e., $x^* \leq d$, the degree of auditing a_n must satisfy, $x^* - t \leq \frac{a_n(t)}{R_{n+1}} + (x^* - t)\gamma_n$, for all $t \leq x^*$ (see (5)). Therefore, given payment schedule $d_n^D(x_n, x_n)$ truth telling is optimal for the agent, when

$$a_n^D(t) = \begin{cases} R_{n+1}(1 - \gamma_n)(d - t) & \text{for } t \leq d, \\ 0 & \text{for } t > d. \end{cases}$$

We define D_n as the lowest threshold d that is required to satisfy the investor's participation constraint, while ensuring truth telling by the agent. In particular,

$$D_n = \min\big\{d \geq 0\colon E\big(d_n^D(x_n, x_n) - a_n^D(x_n)\delta\big) \geq \theta_n - w_n\big\}.$$

By substituting the above,

$$D_n = \min\big\{d \geq 0\colon E\big[\min(d, x_n) + \gamma_n \max(x_n - d, 0) - \bar{\delta}_n \max(d - x_n, 0)\big] \geq \theta_n - w_n\big\},$$
$$\bar{\delta}_n = \delta R_{n+1}(1 - \gamma_n), \quad \text{for } n \leq N \text{ and } w_n \leq \theta_n. \tag{6}$$

In Proposition 2 below, we show that for a fixed scale of investment θ_n, the optimal payment schedule is given by $d_n^D(x_n, x_n)$, and the optimal auditing schedule is given by $a_n^D(t)$, for $d = D_n$, as given by (6). When considering a fixed level of investment, $\theta_n = 1$, the possibility of auditing does not guarantee that investment takes place in equilibrium. Investment requires two conditions: the investor must break even (i.e., there must exist such D_n), and also, it must be profitable for the agent to invest wealth w_n in the project. Lemma 4 in the appendix derives the minimal wealth level, $\underline{w}_n$, above which both conditions are satisfied. Consistent with the above discussion, Proposition 2 summarizes the optimal financial contract for scale of investment $\theta_n = 1$.

Proposition 2. *Consider a scale of investment $\theta_n = 1$, wealth level $w_n \in [\underline{w}_n, 1)$, and assume that condition (C1) holds for period n. Under the optimal contract, the investor invests $1 - w_n$, receives payment $d_n(x_n, x_n) = \min(D_n, x_n) + [\min(\gamma_n, (1 - w_n)/E(x_n))] \max(x_n - D_n, 0)$, and audits with degree $a_n(x_n) = R_{n+1}(1 - \gamma_n) \max(D_n - x_n, 0)$. Finally, when $w_n \in [0, \underline{w}_n)$, investment does not take place.*

Allowing the agent to optimally choose the scale of the project θ_n substantially affects the optimal financial contract. The optimal scale investment is chosen to balance the benefits from increased investments and the expected auditing costs that are required to support payments in equilibrium. This trade-off results in a level of investment that is larger than the maximum investment that can be financed without auditing, $\theta_n > w_n + \gamma_n E(x_n)$. Therefore, under the optimal contract with variable investment, the firm issues both debt and equity. Moreover, the optimal mix of debt and equity no longer depends on the agent's wealth w_n when investment is chosen optimally.

In order to address the optimal level of investment, let $\pi_n(w|\theta)$ denote the expected profit of the agent in period n when the scale of investment equals θ and $w_n = w$. According to Proposition 2, $\pi_n(w_n|1) = (1 - \gamma_n)E[\max(x_n - D_n, 0)]$, for $w_n \in [\underline{w}_n, 1 - \gamma_n E(x_n)$ (where D_n is given by (6) for $\theta_n = 1$). It is trivial then to define the expected gross *return* to the agent during period n on her investment of w_n (while $\theta_n = 1$) as $\frac{\pi_n(w_n|1)}{w_n}$. When the agent finds it optimal to invest (see Proposition 2), the expected profit to the agent exceeds her investment in the firm, i.e., $\frac{\pi_n(w_n|1)}{w_n} > 1$ for $w_n > \underline{w}_n$. Moreover, expected profit is increasing in the agent's investment w_n, as higher wealth levels lead to lower expected auditing costs. In Lemma 5, in the appendix, it is shown that there exists a wealth level, w_n^*, for which the return on the agent's investment in firm equity, $\frac{\pi_n(w_n|1)}{w_n}$, is maximized. The optimal level of investment, then, is set such that the agent internally finances a fraction w_n^* of total investment θ_n to maximize her expected profits. In Theorem 1, below, the optimal level of investment is derived, and the agent's continuation function is shown to be linear in wealth (i.e., condition (**C1**) is verified).[19]

Theorem 1. *Assume that condition* (**C1**) *holds for period n and consider* $\underline{w}_n > 0$. *In period n the agent invests* w_n *and the investor* $\theta_n - w_n$, *where* $\theta_n = \frac{w_n}{w_n^*}$. *The investor is entitled to* $\min(D_n, x_n) + \gamma_n \max(x_n - D_n, 0)$ *of cash flows while the agent receives the rest. The degree of auditing is* $a_n(x_n) = R_{n+1}(1 - \gamma_n)\max(D_n - x_n, 0)$ *and finally,* $V_n(w) = R_n \times w$, *where* $R_n = R_{n+1} \times \frac{\pi_n(w_n^*|1)}{w_n^*}$ *(i.e., condition* (**C1**) *holds for period n − 1).*

Theorem 1 recursively defines the sequence of optimal contracts. In any period n the value of future investment opportunities R_{n+1} affects the optimal contract through the division of cash flows, the scale of investment, and the degree of auditing. Before formally analyzing these implications, we discuss the implementation of the optimal contract through the combination of debt and leveraged equity.

3.2. Implementation

In this section we show that the optimal contract derived in Theorem 1 can be implemented using simple financial securities. Guided by the division of cash flows under the optimal contract, the claim $\min(x_n, D_n)$ is interpreted as debt with face value D_n, and the claim $\max(x_n - D_n, 0)$ is interpreted as leveraged equity. Thus, we define ($n \in \{1, N\}$),

$$V_n^D = E[\min(x_n, D_n) - \delta a_n] \quad \text{(value of debt, period } n\text{)},$$
$$V_n^E = E[\max(x_n - D_n, 0)] \quad \text{(value of equity, period } n\text{)}.$$

The investor's claim on cash flows, $d_n(x_n, x_n)$, is interpreted as a combination of debt with face value D_n, and fraction γ_n of firm equity. At the beginning of each period, new one-period debt is issued, and investment θ_n takes place. If the firm defaults on the debt payment, i.e., $x_n < D_n$, then the investor audits with degree $a_n(x_n) = R_{n+1}(1 - \gamma_n)(D_n - x_n)$, and the firm no longer invests in the future. If, however, the firm repays its debt, i.e., $x_n \geqslant D_n$, then profits at the end of the period, $x_n - D_n$, are either paid out to equity holders or reinvested in future projects. When past profits

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together with new debt issued suffices to cover investment, i.e., $\theta_{n+1} \leqslant x_n - D_n + V_{n+1}^D$, the firm pays out any excess cash to equity holders (e.g., as dividends). Alternatively, when past profits together with new debt issued do not suffice to cover investment, i.e., $\theta_{n+1} > x_n - D_n + V_{n+1}^D$, the firm raises the additional required funds by issuing equity. Thus, ($n \in \{1, N - 1\}$)

$$\max(\theta_{n+1} - [x_n - D_n + V_{n+1}^D], 0) \quad \text{(equity issued, period } n + 1),$$
$$\max(x_n - D_n + V_{n+1}^D - \theta_{n+1}, 0) \quad \text{(dividends paid, period } n + 1).$$

4. Empirical implications

4.1. Optimal financing and firm growth

Several studies point to a relation between firms' growth potential and their financing choices (e.g., Goyal et al., 2002, and Helwege and Liang, 1996). In the model, the value of the firm's future growth potential R_{n+1} depends on the properties of the underlying investment technologies in future periods (e.g., the distribution functions F_n) through the financial contract (Theorem 1). In this section we seek to understand the link between the value of future investment opportunities and the optimal financial contract. In particular, we study the equilibrium relation between R_{n+1} and the size of investment, the portion of investment that is financed with equity, and firm profitability in period n. In subsequent analysis we study the evolution of the optimal financial contract over the firm's growth cycle.

Periodical investment is funded with funds allocated by the agent and by the investor. In particular, investment θ_n is financed internally with the agent's wealth, w_n, and externally with debt, V_n^D, and equity, $\gamma_n V_n^E$. The fraction of external financing that is financed with equity (debt) is predicted to increase (decrease) in the value of the firm's future investment opportunities.

Proposition 3. *The portion of external financing $\theta_n - w_n$, that is financed with equity $\gamma_n V_n^E$ (debt V_n^D) is increasing (decreasing) in the value of future growth opportunities R_{n+1}.*

Intuitively, high-growth firms (i.e., high R_{n+1}) rely more on equity financing relative to low-growth firms for two reasons. First, in the region $R_{n+1}(1 - \rho) < 1$ (or equivalently $\frac{R_{n+1}-1}{R_{n+1}} < \rho$) the fraction of equity held by outside investors $\gamma_n \equiv \min(\frac{R_{n+1}-1}{R_{n+1}}, \rho) = \frac{R_{n+1}-1}{R_{n+1}}$ is increasing in R_{n+1}. The reason for this is that it is optimal for the agent to consume diverted funds in this region, and therefore, the opportunity cost of diverting funds (i.e., forgoing future investments) is increasing in R_{n+1}. This, in turn, disciplines the agent to honor payments to equity holders and increases the fraction of external equity under the optimal contract. Second, in the region $R_{n+1}(1 - \rho) \geqslant 1$ (i.e., $\gamma_n = \rho$), higher degrees of auditing are required to keep the agent honest. In particular, higher values of R_{n+1} tighten the agents incentive compatibility constraint (see (5)). This is because it is optimal for the agent to invest diverted funds in this region, and therefore, the benefits from diverting funds, in default states (i.e., when $D_n > x_n$), are increasing in R_{n+1}. This, in turn, increases the need for auditing and reduces the amount of capital raised via debt financing.

The level of investment is endogenous in the model, and it depends on the wealth accumulated by the agent, w_n, and the agent's incentives to repay investors. While the former is determined by past performance, the latter changes with the severity of the agency problem and depends on the future prospects of the firm. As one would expect, the level of investment is increasing in the level

of funds accumulated by the agent. However, investment is predicted to be non-monotonic in the value of future investment opportunities. Namely, an intermediate value of future investment opportunities R_{n+1} maximizes periodical investment θ_n.

Proposition 4. *Investment θ_n is increasing in the level of funds w_n. Moreover, the portion of investment that is financed with internal funds w_n, is decreasing in R_{n+1} when the value of future growth opportunities is low, $R_{n+1}(1 - \rho) < 1$, and is increasing in R_{n+1} otherwise.*

The above non-monotonicity has an intuitive reasoning. When the value of future investment opportunities is relatively small, i.e., $R_{n+1}(1 - \rho) < 1$, then a higher growth potential increases the relative amount of capital raised via external equity and leads to higher levels of investment. However, when $R_{n+1}(1 - \rho) > 1$, a higher growth potential requires more auditing and reduces the relative amount of capital raised via debt, which in turn, leads to lower levels of investment. As a result, investment is largest when the value of future investment opportunities is intermediate.

The above non-monotonicity in investment is reflected in a non-monotonicity in total expected profits $E(\max(x_n - D_n, 0))$ and expected profits to the agent $(1 - \gamma_n)E(\max(x_n - D_n, 0))$. An inspection of the incentive compatibility constraint in (5) reveals that it is tightened (relaxed) by an increase in R_{n+1} when $R_{n+1}(1 - \rho) > 1$ $(R_{n+1}(1 - \rho) < 1)$. As a result, the agency problem is least severe, and expected profits to the agent are the largest, for an intermediate value of future investment opportunities R_{n+1}. This, in turn, together with the definition of γ_n, leads to the non-monotonicity in total expected profits $E(\max(x_n - D_n, 0))$.

Proposition 5. *Expected profits $E(\max(x_n - D_n, 0))$, and expected profits to the agent $(1 - \gamma_n)E(\max(x_n - D_n, 0))$, in period n, are increasing in R_{n+1} when the value of future investment opportunities is low, $R_{n+1}(1 - \rho) < 1$, and are decreasing in R_{n+1} otherwise.*

Both the optimal contract and expected profitability are jointly determined in equilibrium (according to Propositions 3 and 5). In particular, in the region $R_{n+1}(1 - \rho) < 1$, an increase in the value of future investment opportunities reduces the firm's reliance on debt financing (Proposition 3) and also leads to higher expected profits in period n (Proposition 5). Thus, in this region, a negative relation between the firm's reliance on debt financing and firm profitability emerges in equilibrium. In the region $R_{n+1}(1 - \rho) > 1$, this relation is reversed. As we will see next, the aforementioned result implies a negative (positive) relation between profitability and leverage for mature (young) firms.

4.2. Optimal financing over the growth cycle

In this section we analyze the evolution of the optimal financing contract and levels of investment throughout the firm's growth cycle. The dynamics of the optimal contract depend on the specific investment technologies $F_1, \ldots, F_N$ we consider and the way they evolve over time. For any choice of technologies the optimal financial contract and levels of investment are provided by Theorem 1. In order to abstract from any financing patterns that are due to changes in the firm's investment technologies, we conduct the analysis while assuming a fixed investment technology

throughout.[20] Thus, the patterns we identify below result only from the endogenous patterns in the severity of the agency problem throughout the firm's growth cycle.

We find that the growth cycle can be characterized by a critical age, which we denote by n^* and define as $n^* = \max\{n \leqslant N: R_{n+1}(1-\rho) \geqslant 1\}$. Thus, one could classify *young* firms as those for which $n < n^*$ and *mature* firms as those for which $n \geqslant n^*$. An interesting distinction between so called young and mature firms has to do with the role of periodical profits, $\max(x_n - D_n, 0)$. The model predicts that for young firms the level of investment θ_{n+1} exceeds the funds supplied by debt V_{n+1}^D plus the level of profits in the previous period $\max(x_n - D_n, 0)$. Thus, new external equity is raised in every period $n < n^*$. For mature firms, however, this is not the case.

Proposition 6. *Investment in period n exceeds debt issued together with profits, i.e., $\theta_n \geqslant V_n^D + \max(x_{n-1} - D_{n-1}, 0)$, for all $1 < n < n^*$ (i.e., for young firms).*

This result reflects the reduction in the severity of the agency problem over time for young firms. Intuitively, when the firm is young, the profitability of investment increases over time, while the fraction of cash flows allocated to external equity holders remains constant ($\gamma_n = \rho$ in this region). Therefore, the value of their claim on cash flows increases over time. External equity holders (which break even, on average) compensate for this increase in the value of their claims by increasing their investment in the firm each period.

Throughout the firm's growth cycle the value of future investment opportunities decreases over time as the firm matures. Consistent with Propositions 3–5, this has implications for the evolution of the optimal contract as summarized below.

Corollary 1. (i) *The portion of external financing, $\theta_n - w_n$, that is financed with equity $\gamma_n V_n^E$ (debt V_n^D) is decreasing (increasing) in n.*

(ii) *The portion of investment that is financed with internal funds, w_n, is decreasing in n for $n \leqslant n^*$ (i.e., young firms) and is increasing in n for $n > n^*$ (i.e., mature firms).*

(iii) *Expected profits $E(\max(x_n - D_n, 0))$ and expected profits to the agent $(1 - \gamma_n) \times E(\max(x_n - D_n, 0))$ are increasing in n for $n \leqslant n^*$ and are decreasing in n for $n > n^*$.*

5. Extensions

5.1. Monitoring costs and equity financing

The cost of auditing/monitoring in our model is determined by the parameter δ. Recall, the cost to the investor of auditing with degree a is given by δa. Intuitively, the ability to unearth the true realization of cash flows may vary across investors. It may depend on the relation between the investor and the firm (e.g., Petersen and Rajan, 1994), the investor's expertise in lending, such as sector-specialization (e.g., Boot and Thakor, 2000), and the availability and reliability of additional sources of information, such as audited financial statements. In the following we address the natural question whether (and how) a lower cost of monitoring affects the optimal financial contract.

In Theorem 1 we have shown that while equity payments are supported by the existence of future investment opportunities, payments to debt holders require the threat of auditing by the

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[20] Formally, in Proposition 6, we assume that $(x_n^1, F_n, f_n) = (x^1, F, f)$ for $n \leqslant N$.

investor. Thus, one would expect that lower auditing costs would lead to higher firm leverage, investment, and profitability. But the optimal financial contract in period n depends both on the cost of monitoring δ and on the return on future investments R_{n+1}. In particular, an increase in future profitability, due to a reduction in δ, leads to an increase in R_{n+1}. This in turn affects the fraction of firm equity issued to outsiders and the relative amount of funds raised by issuing external equity in earlier periods (Proposition 3 for the case $R_{n+1}(1 - \rho) < 1$). Thus, a change in monitoring costs directly affects the optimal amount of debt issued, through δ, and indirectly affects the amount of funds raised by issuing equity through the profitability in future periods, R_{n+1}. This intuition is reflected in Proposition 7, in which we study the two period case $N = 2$.[21]

Proposition 7. *A more efficient auditing technology, i.e., smaller δ, leads to a larger scale of investment θ_n ($n = 1, 2$), higher values of debt, V_n^D ($n = 1, 2$), and external equity, $\gamma_1 V_1^E$, for a given initial wealth level w_1, provided that $\delta \geqslant \hat{\delta}(\rho)$ and $\rho \in (1 - \frac{1}{E(x_2^1)}, 1]$.*

Proposition 7 emphasizes the role of efficient monitoring (e.g., by banks) in supporting both debt and external equity financing. In particular, access to a lender with an efficient monitoring technology does not only increase the amount of funds borrowed by the firm V_n^D but also increases the amount of funds raised via external equity $\gamma_1 V_1^E$.

5.2. Optimal financing and legal enforcement

In the model there are two ways by which expropriation by insiders may be mitigated: first, through the state contingent auditing schedule specified in the financial contract, and second, through the fraction ρ of diverted funds that can convert to future investment. While the former state contingent auditing schedule is decided upon in equilibrium (Theorem 1), the cost of legitimization, ρ, is taken as a given, as part of the contracting environment.

The opportunities for firms to divert funds at the expense of creditors varies across countries. Namely, the likelihood of investors to be repaid, and the amount recovered, following a breach of contract depends on the amount of funds available for repayment and the extent to which the legal system enforces contracts (e.g., Visaria, 2005; Lerner and Schoar, 2005). As discussed earlier, the fraction ρ represents the cost to the agent of legitimizing investment, financed with funds diverted from investors. Thus, to the extent that weaker enforcement of contracts by the legal system implies lower legitimization costs (e.g., the hiring of more and/or better lawyers, time spent in evidence production/fabrication, and even bribes paid to court officials; Bond, 2007a, 2007b), our model draws a link between the optimal financial contract and the level of legal enforcement.

It follows from Theorem 1 that the optimal contract in a given period depends on the level of ρ relative to the value of future investment opportunities. Intuitively, when the fraction of diverted funds that can convert to investment in future periods is sufficiently low (i.e., ρ is high), the agent will not consider diverting funds for investment, and marginal changes in ρ do not affect the agent's incentives or the optimal contract. Namely, Theorem 1 establishes that the optimal contract in period n is *not* affected by marginal changes in ρ, as long as $\rho > \frac{R_{n+1}-1}{R_{n+1}}$. When the fraction of diverted funds that can convert to investment in future periods is sufficiently

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[21] The two-period case yields the simple expression for the lower bound $\rho > 1 - \frac{1}{E(x_2^1)}$ in Proposition 7.

high (i.e., ρ is low), however, the agent prefers investment over consumption of diverted funds. Further increasing the benefits from investing diverted funds (i.e., decreasing ρ) will exacerbate the agent's incentive problem and reduce investment.

In a two-period version of the model, it can be shown that a marginal increase in ρ leads to larger investment θ_n in both periods ($n = 1, 2$) and higher values of debt issued, V_1^D, and equity issued, V_1^E, for a given initial wealth level w_1, provided that $\rho < \frac{R_2-1}{R_2}$. Note also that even though the optimal contract in the second (last) period is not directly affected by a marginal increase in ρ, the expected wealth at the beginning of the second period is higher (due to the aforementioned increase in investment in the first period), and as a result, expected investment in the second period is higher as well.[22]

5.3. Multi-period contracts

Throughout the paper we have focused our analysis on the sequence of one-period financial contracts. As discussed in Section 2.3, this is optimal, as the agent is free to approach competitive financial markets prior to each investment, and the agent's investment technology is inalienable (Diamond, 1989, 1991a). Otherwise, long-term financial contracts can be beneficial when the investor can affect future cash-flows, e.g., by control of firm assets, or when the agent can be excluded from financial markets. In particular, the agent can be induced to pay investors via the threat of the loss of control of the project (Bolton and Scharfstein, 1990; Diamond, 1991b, 1993; DeMarzo and Fishman, 2007); long-term debt can prevent termination by the investor of good projects too early (Von Thadden, 1995); and loan payments can be induced by the commitment of the investor to deny credit if performance is poor (Albuquerque and Hopenhayn, 2004; Bond and Krishnamurthy, 2004).[23]

In reality, firms enter financial contracts for both short and long durations but do not enter financial contracts to fund *all* future investments. In particular, firms repeatedly approach financial markets to raise debt and issue stock (e.g., Frank and Goyal, 2003). Our assumptions reflect the fact that investors can affect future productivity of the agent, e.g., by having the right to sell firm assets in the case of default, but to a limited extent. This assumption distinguishes our paper from other analyses of long-term contracts, such as Bolton and Scharfstein (1990), where the contract can specify liquidation of the firm's assets after which the agent and the investor receive liquidation payoffs. Namely, in our analysis the investor has no means by which to prevent the agent from raising funds and investing in future periods.

The restriction to one-period contracts in our analysis, however, is not crucial for the optimality of external equity financing. We have shown that a disciplinary role for growth opportunities exists, as the agent has an incentive to enter future periods with more funds in order to finance larger investments. This incentive of the agent to accumulate funds for future periods is the driving force behind the optimality of external equity financing in our model and determines the optimal mix of debt and equity financing. To illustrate why this intuition is more general, suppose that a firm that lives for two periods deploys assets that can be controlled by the investor and are required for production/investment in both periods. Then, the optimal two-period contract could specify payments and control rights that are contingent on performance (e.g., Bolton

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[22] In the more general case, $N > 2$, changes in ρ may also affect the optimal contract in period n indirectly through its effect on the value of R_{n+1}.

[23] See Bolton and Dewatripont (2005) for a thorough overview of the dynamic contracting literature.

and Scharfstein, 1990). Now suppose that this firm is also endowed with a future investment opportunity in the third period, which does not require the aforementioned assets and cannot be controlled by the original investor. If the agent is free to approach any investor to raise funds for the third period investment, then the investor must break even on her investment in the third period, and, consequently, the level of investment depends on the agent's wealth at the beginning of the third period. This, in turn, affects the agent's incentives to accumulate funds, as in our model, and supports truth telling in the first two periods. In particular, for a given return on third period investment R_3 the agent can sell a fraction $\gamma_2 = \max(\frac{R_3-1}{R_3}, \rho)$ of cash flows x_1 and x_2 and raise funds $\gamma_2 E(x_1 + x_2)$, as these payments will be honored in equilibrium. Thus, as long as there exists some time in the future an investment opportunity that is not alienable, i.e., of which control cannot be allocated to the initial investor, the main intuition will still hold, and it will be optimal for the agent to use external equity to finance investment.

6. Concluding remarks

> *The fact that high-growth, high-risk new ventures often obtain angel finance and/or venture capital before they obtain amounts of external debt finance suggests that the moral hazard problem may be particularly acute for these firms.*
> Berger and Udell (1998, p. 624)

In this paper we propose a rationale for the prevalent use of external equity as a mode of financing by both private and public growth firms. Growth firms are eager to fully exploit their growth potential but face information asymmetries leading to under investment. As we have shown, growth opportunities serve as a disciplinary device as long as the diversion of funds implies a reduction in future investment.

Disciplined by growth, insiders are more willing to repay investors and can rely more on external equity and less on monitored finance in the form of debt. This optimality of outside equity is a direct result of the entrepreneur's dynamic considerations. In a one-period version of our model, as in previous static analyses (e.g. Townsend, 1979; Gale and Hellwig, 1985), debt is optimal, and payments to the investor are supported by the threat of auditing. By explicitly modeling the sequence of investments facing growth firms, we demonstrate that future investment opportunities mitigate the need for auditing to support truth telling in equilibrium. In particular, the agent truthfully reports cash flows that exceed the debt payment (implying higher payments to investors) even though auditing takes place only when the debt is not repaid.

Our theory of (risky) debt and outside equity emerges from fundamentals and thus hints of the different underlying economic forces that bring about different securities observed in reality. We use this theory to analyze the optimal financial security over the firm's growth cycle, the relation between profitability and the use of debt financing, and the implications of a better auditing technology and stronger legal enforcement of contracts for the optimal financial contract. We view our analysis as a potential building block for future analyses that seek to explore variations in capital structure, pricing of corporate securities, and incentives of informed managers. Embedding our model into a general equilibrium setting, for example, might shed light on firms' optimal financing methods over the business cycle.

Appendix A

Lemma 1. *Given period $n \leqslant N - 1$, contract σ_n, cash-flows x_n, report t_n and signal s_n the optimal level of consumption in period n satisfies $c_n = 0$ if $(1 - \rho)R_{n+1} > 1$ and $c_n = x_n - s_n$ otherwise. Moreover, $\hat{V}_n(x_n, t_n, s_n \mid \sigma_n) = R_{n+1}(s_n - d_n(t_n, s_n) + (1 - \rho)(x_n - s_n))$ if $(1 - \rho)R_{n+1} > 1$ and $\hat{V}_n(x_n, t_n, s_n \mid \sigma_n) = x_n - s_n + R_{n+1}(s_n - d_n(t_n, s_n))$ otherwise.*

Proof. The corner solution follows from the linearity of $V_{n+1}(w_{n+1})$ specified in condition (C1). □

Lemma 2. *Given period $n \leqslant N$, contract σ_n, cash-flows x_n, report t_n, and expected auditing $E(a_n \mid t_n, \sigma_n)$, the optimal falsification strategy satisfies $\mathbf{f}_n = 0$ if and only if $\hat{V}_n(x_n, t_n, x_n \mid \sigma_n) \geqslant \hat{V}_n(x_n, t_n, t_n \mid \sigma_n) - E(a_n \mid t_n, \sigma_n)$.*

Proof. In deciding on the falsification strategy for x, the agent must compare the expected payoff from choosing $\mathbf{f}_n = 0$, $\hat{V}_n(x_n, t_n, x_n \mid \sigma_n)$, and the expected payoff from choosing $\mathbf{f}_n = 1$, $\hat{V}_n(x_n, t_n, t_n \mid \sigma_n) - E(a_n \mid t_n, \sigma_n)$. □

In order to simplify the proof of Proposition 1 below, we assume that contracts cannot reward the agent when lying is detected, i.e., $s_n > t_n$. This assumption does not bind the optimal contract that implements truth-telling, as it imposes the maximal payment when lying is detected, i.e., $d_n(t, s) = s$, as we will see next in Proposition 1.

Assumption A1. $d_n(t, x) - d_n(x, x) \geqslant 0$, for all $t \leqslant x \in \mathcal{X}$, and $n \leqslant N$.

Lemma 3. *Given period $n \leqslant N$, and contract σ_n, it follows from Assumption A1 that if it is optimal for the agent to lie, then it is also optimal for the agent to falsify. Formally, $t_n(x) < x \Rightarrow \mathbf{f}_n(x) = 1$, or equivalently $s_n = t_n$.*

Proof. The utility of the agent from lying, i.e., reporting $t < x$, and not falsifying, i.e., $\mathbf{f}_n = 0$, is $\hat{V}_n(x, t, x \mid \sigma_n) = R_{n+1}(x - d_n(t, x))$, which is smaller than her utility from reporting truthfully (and not falsifying), $\hat{V}_n(x, x, x \mid \sigma_n) = R_{n+1}(x - d_n(x, x))$, as $d_n(x, x) < d_n(t, x)$ (from Assumption A1). Thus, lying and not falsifying is dominated by truth-telling, i.e., $t_n(x) < x \Rightarrow \mathbf{f}_n(x) = 1$. □

Proposition 1. *Given period $n \leqslant N$, feasible contract $\sigma_n = (\theta_n, a_n, d_n)$ with optimal strategy $\alpha_n = (t_n, \mathbf{f}_n, c_n)$ for the agent, there exists a feasible contract $\sigma'_n = (\theta_n, a'_n, d'_n)$ with corresponding optimal strategy $\alpha'_n = (t'_n, \mathbf{f}'_n, c'_n)$, such that $t'_n(x_n) = x_n$, $\mathbf{f}'_n(x_n) = 0$, and both the investor and*

the agent are weakly better off under σ'_n. Moreover, under the contract σ'_n, the agent consumes only in period $N + 1$, and the auditing schedule a'_n is deterministic.

Proof. Consider period $n \leqslant N$, and contract $\sigma'_n = (\theta_n, a'_n, d'_n)$, where $a'_n(x) = E(a_n \mid t_n(x), \sigma_n)$, $d'_n(t, s) = s$ when $t < s$, and $d'_n(t, t) = d_n(t_n(t), t_n(t)) + (1 - \frac{\max(R_{n+1}(1-\rho),1)}{R_{n+1}})(t - t_n(t))$, otherwise. Note that from $s_n(t) = t_n(t)$ (Lemma 3), it follows that $\hat{V}_n(x, x, x \mid \sigma'_n) = \hat{V}_n(x, t_n(x), t_n(x) \mid \sigma_n)$. We first show that the optimal strategy of the agent under σ'_n is to report truthfully, i.e., $t_n(x_n) = x_n$, and not falsify, i.e., $\mathbf{f}'_n(x_n) = 0$.

Suppose by contradiction that truth telling is *not* optimal under σ'_n, i.e., there exists $x_n = x^*$ such that $t'_n(x^*) = t^* < x^*$ and $\bar{V}_n(x^*, t^* \mid \sigma'_n) > \bar{V}_n(x^*, x^* \mid \sigma'_n)$. This also implies that $\mathbf{f}'_n(x^*) = 1$, as lying is strictly preferred over truth-telling only if it is not detected. Formally, the utility of the agent from lying and not falsifying, $\hat{V}_n(x^*, t^*, x^* \mid \sigma'_n) = R_{n+1}(x^* - d'_n(t^*, x^*)) = 0$, cannot be strictly larger than her utility from reporting truthfully, $\hat{V}_n(x^*, x^*, x^* \mid \sigma'_n) = R_{n+1}(x^* - d'_n(x^*, x^*)) \geqslant 0$. The optimality of the strategy $\langle t'_n(x^*), \mathbf{f}'_n(x^*)\rangle = \langle t^*, 1\rangle$, implies that $\hat{V}_n(x^*, t^*, t^* \mid \sigma'_n) - a'_n(t^*) > \hat{V}_n(x^*, x^*, x^* \mid \sigma'_n)$. We will see next, however, that this contradicts the optimality of the reporting strategy $t_n(x^*) = t^*$ under contract σ_n. In particular, from $t_n(x^*) < x^* \Rightarrow \mathbf{f}_n(x^*) = 1$ (Lemma 3), and (3) it follows that,

$$\bar{V}_n(x^*, t_n(x^*) \mid \sigma_n) = \begin{cases} \hat{V}_n(x^*, t_n(x^*), t_n(x^*) \mid \sigma_n) - E(a_n(t_n(t^*)) \mid \sigma_n), & \text{if } t_n(x^*) < x^*, \\ \hat{V}_n(x^*, t_n(x^*), t_n(x^*) \mid \sigma_n), & \text{if } t_n(x^*) = x^*. \end{cases} \tag{7}$$

Then, from the definition of $d'_n(x, x)$,

$$\begin{aligned} \hat{V}_n(x^*, x^*, x^* \mid \sigma'_n) &= \hat{V}_n(x^*, t_n(x^*), t_n(x^*) \mid \sigma_n) &&(8)\\ &\geqslant \bar{V}_n(x^*, t_n(x^*) \mid \sigma_n) \quad \text{[from (7)]}\\ &\geqslant \bar{V}_n(x^*, t_n(t^*) \mid \sigma_n) \quad \text{[from optimality of } t_n(x^*)]\\ &\geqslant \hat{V}_n(x^*, t_n(t^*), t_n(t^*) \mid \sigma_n) - E(a_n(t_n(t^*)) \mid \sigma_n) \quad \text{[from (3)]} \end{aligned}$$

Moreover,

$$\begin{aligned} R_{n+1}(t^* - d'_n(t^*, t^*)) &= R_{n+1}(t_n(t^*) - d_n(t_n(t^*), t_n(t^*)))\\ &\quad + \max(R_{n+1}(1 - \rho), 1)(t^* - t_n(t^*)), \end{aligned}$$

and, as $a'_n(t^*) = E(a_n(t_n(t^*)) \mid \sigma_n)$, we can write,

$$\begin{aligned} \hat{V}_n(x^*, t^*, t^* \mid \sigma'_n) - a'_n(t^*) &= R_{n+1}(t^* - d'_n(t^*, t^*))\\ &\quad + \max(R_{n+1}(1 - \rho), 1)(x^* - t^*) - a'_n(t^*)\\ &= R_{n+1}(t_n(t^*) - d_n(t_n(t^*), t_n(t^*)))\\ &\quad + \max(R_{n+1}(1 - \rho), 1)(x^* - t_n(t^*)) - E(a_n(t_n(t^*)) \mid \sigma_n)\\ &= \hat{V}_n(x^*, t_n(t^*), t_n(t^*) \mid \sigma_n) - E(a_n(t_n(t^*)) \mid \sigma_n)\\ &\leqslant \hat{V}_n(x^*, x^*, x^* \mid \sigma'_n) \quad \text{[from (8)]}. \end{aligned}$$

Thus, we reach a contradiction, and we conclude that truth-telling (without falsification) is optimal under contract σ'_n.

It remains to verify that both the investor and the agent are weakly better off under σ_n'. Note, for any realization $x_n = x$, $t_n'(x) = x$, $\mathbf{f}_n'(x) = 0$, and $\hat{V}_n(x, x, x \mid \sigma_n') = \hat{V}_n(x, t_n(x), t_n(x) \mid \sigma_n)$. It follows from (1) that $\bar{V}_n(x, t_n(x) \mid \sigma_n) \leqslant \hat{V}_n(x, t_n(x), t_n(x) \mid \sigma_n) = \hat{V}_n(x, x, x \mid \sigma_n') = \bar{V}_n(x, x \mid \sigma_n')$, thus the agent is better off under contract σ_n'. Moreover, as $a_n'(x) = E(a_n \mid t_n(x), \sigma_n)$ and $d_n'(x, x) \geqslant d_n(t_n(x), t_n(x))$, the investor is better off.

Finally, since it may be optimal to consume diverted funds but not legitimate funds in early periods (Lemma 1), and there is no diversion in equilibrium, it follows that consumption will take place only at the last period $n = N + 1$. $\square$

Lemma 4. *There exists $w_n^0 \in [0, 1 - \gamma_n E(x_n))$ such that the debt level D_n in (6) is well defined for $w_n \geqslant w_n^0$, and $\theta_n = 1$. Moreover, there exists $\underline{w}_n \in [w_n^0, 1 - \gamma_n E(x_n))$ for which the aforementioned debt level yields positive expected profit for the agent for $w_n \geqslant \underline{w}_n$, and $\theta_n = 1$.*

Proof. Define $G(d) \equiv E[\min(d, x_n) - \bar{\delta}_n \max(d - x_n, 0) + \gamma_n \max(x_n - d, 0)]$ (for $w_n < 1 - \gamma_n E(x_n)$). D_n is well defined if there exists $d^* \geqslant 0$ such that $G(d^*) = 1 - w_n$. Note that (i) $G(d)$ is continuous, (ii) $G'(d) = 1 - \gamma_n - (1 + \bar{\delta}_n - \gamma_n) F_n(d)$, (iii) $G(0) = \gamma_n E(x_n)$ and (iv) $G(d) \leqslant G(\bar{d})$ where $\bar{d} = F_n^{-1}((1 - \gamma_n)/(1 + \bar{\delta}_n - \gamma_n))$ and F^{-1} is the inverse function of F. It follows from the above that the debt level is well defined if $G(\bar{d}) > 1 - w_n$, i.e. $w_n^0 \equiv \max(1 - (1 + \bar{\delta}_n - \gamma_n) \int_0^{\bar{d}} x f_n(x) \, dx - \gamma_n E(x_n), 0)$. The agent will profit from debt level D_n as long as $E((1 - \gamma_n) \max(x_n - d, 0)) \geqslant w_n$, that is, for $w_n \geqslant \underline{w}_n \equiv \inf\{w \in [w_n^0, 1 - \gamma_n E(x_n)): E((1 - \gamma_n) \max(x_n - d, 0)) \geqslant w\}$. $\square$

Proof of Proposition 2. For ease of notation all subscripts n are omitted during the proof. In particular, let $D = D_n$, $\gamma = \gamma_n$, $R = R_{n+1}$, $x = x_n$, $w = w_n$ and $\underline{w} = \underline{w}_n$. Let $\sigma^* = (1, a^*, d^*)$ denote the contract suggested in Proposition 2. It will be shown that σ^* is optimal, and uniquely so for $w \in (\underline{w}, 1 - \gamma E(x))$.

We start by considering the case $w \in [1 - \gamma E(x), 1)$, where the payment to the investor is $d^*(x, x) = (\frac{1 - w}{E(x)}) x$ and there is no auditing, i.e., $a^*(x) = 0$. Contract σ^* satisfies the **IR** constraint, as $E(d^*(x, x) - a^*(x)\delta) = 1 - w$, and satisfies the **IC** constraint, as $d^*(x, x) - d^*(t, t) = (\frac{1 - w}{E(x)})(x - t) \leqslant \gamma(x - t)$. Moreover, σ^* yields an expected payoff of $w + E(x) - 1$ to the agent, which represents the maximum expected return from investment of scale one, $E(x) - 1$, and therefore σ^* is optimal. In the region $w \in [1 - \gamma E(x), 1)$, the contract σ^* is optimal but not unique.[24]

We now consider the case $w \in (\underline{w}, 1 - \gamma E(x))$ and show that σ^* is unique, i.e., strictly dominates *any* feasible contract $\sigma = (1, a, d) \neq \sigma^*$ that satisfies the **IR** and **IC** constraints. It is useful to define contract $\sigma' = (1, a', d')$ based on contract σ. In particular, $d'(x, x) = \min(x, D') + \gamma \max(x - D', 0)$, and $a'(x) = R(1 - \gamma) \max(D' - x, 0)$, for all $x \in \mathcal{X}$, where

$$D' = \min\{k \geqslant 0: \min(x, k) + \gamma \max(x - k, 0) \geqslant d(x, x) \text{ for all } x \in \mathcal{X}\}.$$

By the definition of σ', payments to the investor are weakly higher under σ' relative σ, i.e., $d'(x, x) \geqslant d(x, x)$ for all $x \in \mathcal{X}$, but there exists at least one cash flow $x_0 \geqslant D'$ for which the two payment schedules coincide, i.e., $d'(x_0, x_0) = D' + \gamma(x_0 - D') = d(x_0, x_0)$. Moreover, the

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[24] For example, consider payment $\tilde{d}(x, x) = \min(\gamma x, \tilde{\gamma} x + z(\gamma - \tilde{\gamma}))$. For $w \in (1 - \gamma E(x), 1)$, it can be shown that for any $0 \leqslant \tilde{\gamma} < \gamma$, there exists $z > 0$, such that $1 - w = E(\tilde{d}(x, x))$, i.e., the **IR** constraint is satisfied. Moreover, the **IC** constraint is satisfied without auditing as $\tilde{\gamma} < \gamma$.

investor audits less under σ' relative σ, i.e., $a'(x) \leqslant a(x)$ for all $x \in \mathcal{X}$, as we show next. Clearly, $a'(x) = 0 \leqslant a(x)$ for all $x \geqslant D'$. To show that $a'(x) \leqslant a(x)$ for all $x \in \mathcal{X}$, recall that the **IC** constraint (which is satisfied for σ) implies that (see (5)),

$$d(x, x) - \gamma x \leqslant d(t, t) - \gamma t + \frac{a(t)}{R}, \quad \text{for all } t \leqslant x \in \mathcal{X}.$$

In particular, for cash flow x_0, and report $t \leqslant D'$, $d(x_0, x_0) - \gamma x_0 \leqslant d(t, t) - \gamma t + \frac{a(t)}{R}$. But $d(x_0, x_0) - \gamma x_0 = D'(1 - \gamma)$, and therefore, for all $t \leqslant D'$, $(1 - \gamma)D' + \gamma t - d(t, t) \leqslant \frac{a(t)}{R}$. By definition of σ', $\frac{a'(t)}{R} = (1 - \gamma)(D' - t)$ for all $t \leqslant D'$, which implies that $a'(t) \leqslant a(t)$ (as $d(t, t) \leqslant t$). Thus, we have established that $a'(x) \leqslant a(x)$ for all $x \in \mathcal{X}$.

Next, we show that $\sigma' = (1, a', d')$ satisfies the **IC**, and **IR** constraints. To establish the **IC** constraint, note that by the definition of σ', $d'(t, t) - \gamma t + \frac{a'(t)}{R} = (1 - \gamma)D'$, for all $t \in \mathcal{X}$. The **IR** constraint follows from $d'(x, x) \geqslant d(x, x)$, and $a'(x) \leqslant a(x)$, for all $x \in \mathcal{X}$, as shown earlier. In particular, $E(d'(x, x) - \delta a'(x)) \geqslant E(d(x, x) - \delta a(x)) \geqslant 1 - w$.

To show that $\sigma^* = (1, a^*, d^*)$ dominates $\sigma = (1, a, d)$, we consider first the case $E(d'(x, x) - \delta a'(x)) > 1 - w$. Then, by the definition of D (given by (6) for $\theta_n = 1$), it follows that $D < D'$. This implies that, $a^*(x) \leqslant a'(x)(\leqslant a(x))$ for $x \in \mathcal{X}$, and $a^*(x) < a'(x)$ for $x < D'$, thus $E(a^*(x)) < E(a'(x))(\leqslant E(a(x)))$. Together with $E(d(x, x) - \delta a(x)) \geqslant 1 - w = E(d^*(x, x) - \delta a^*(x))$, we conclude that $E(d(x, x)) > E(d^*(x, x))$, i.e., contract σ^* strictly dominates contract σ.

Consider now the case $E(d'(x, x) - \delta a'(x)) = 1 - w = E(d^*(x, x) - \delta a^*(x))$ (which also implies that $E(d(x, x) - \delta a(x)) = 1 - w$). Then, by the definition of D, it follows that $D = D'$, and therefore, $a^*(x) = a'(x)(\leqslant a(x))$, and $d^*(x, x) = d'(x, x)$, i.e., $\sigma' = \sigma^*$. As contract σ involves higher payments to the investor and less auditing, relative to contract σ' (i.e., $d'(x, x) \geqslant d(x, x)$, and $a'(x) \leqslant a(x)$), and $E(d'(x, x) - \delta a'(x)) = E(d(x, x) - \delta a(x))$, it follows that, $d'(x, x) = d(x, x)$, and $a'(x) = a(x)$, for all $x \in \mathcal{X}$. Thus, the suggested optimal contract σ^*, the contract σ, and the constructed contract σ' coincide when $E(d'(x, x) - \delta a'(x)) = 1 - w$. This contradicts our assumption that $\sigma = (1, a, d) \neq \sigma^*$; thus, contract σ^* strictly dominates contract σ.

To conclude, σ^* strictly dominates *any* feasible contract $\sigma \neq \sigma^*$, that satisfies the **IR** and **IC** constraints. $\quad\square$

Lemma 5. *The expected end of period wealth of the agent $\pi_n(w|1)$ is strictly concave in the interval $w \in [\underline{w}_n, 1 - \gamma_n E(x_n)]$. Moreover, if $\underline{w}_n > 0$, then there exists a unique wealth level $w_n^* \in (\underline{w}_n, 1 - \gamma_n E(x_n))$ for which the return per dollar wealth invested, $\frac{\pi_n(w|1)}{w}$, is maximized, and $\frac{\pi_n(w_n^*|1)}{w_n^*} > 1$.*

Proof. For simplicity all subscript n's are omitted during the proof. In particular, let $x = x_n^1$ (i.e. for $\theta_n = 1$), $\gamma = \gamma_n$, $\underline{w} = \underline{w}_n$, $\bar{\delta} = \bar{\delta}_n$ and $\pi_n(w|1) = \pi(w|1)$. Let D be given by (6) for $\theta_n = 1$. Note that, $\pi(w|1) = w$ for $w < \underline{w}$, $\pi(w|1) = (1 - \gamma)E(\max(x - D, 0))$ for $w \in (\underline{w}, 1 - \gamma E(x))$ and $\pi(w|1) = E(x) - 1 + w$, otherwise. Thus, $\pi_n(w|1)$ is continuous for all $w \geqslant 0$ excluding the point $\underline{w}$ where there might exist an upward jump in the function. This implies that $\pi'(w|1) = \frac{(1-\gamma)(1-F(D))}{(1-\gamma)(1-F(D))-\bar{\delta}F(D)} > 1$ in the interval $w \in (\underline{w}, 1 - \gamma E(x))$. Notice that $\pi'(w|1)$ is continuous on the interval $w > \underline{w}$. Finally, $\pi(w|1)$ is concave on the interval $w > \underline{w}$ as $\pi''(w|1) = -\frac{\bar{\delta}f(D)(1-\gamma)}{[(1-\gamma)(1-F(D))-\bar{\delta}F(D)]^3} < 0$ for $w \in (\underline{w}, 1 - \gamma E(x))$ and $\pi''(w|1) = 0$ for $w > 1 - \gamma E(x)$.

Now we show that w^* is unique if $\underline{w} > 0$. By the definition, $w^* \in \arg\max_{w \geq \underline{w}} \frac{\pi(w|1)}{w}$; thus, the first-order condition for optimality is $\pi(w^*|1) = w^*\pi'(w^*|1)$. The second derivative of the objective function at w^* is proportional to $\pi''(w^*) < 0$; thus, a unique maximum exists (if the solution is interior). It remains to verify that the optimum is not a corner solution. It must be that $w^* < 1$ as $\pi'(1|1) = 1$ but $\pi(1|1) > 1$. The solution w^* must also satisfy $w^* > \underline{w}$. If $\pi(w|1)$ is continuous at the point $\underline{w}$ then this is clear, and if not then $\pi'(w|1)$ approaches infinity as $w \to \underline{w}$ from above, and thus $\underline{w}$ could not be the solution. Finally, it follows from the interior unique solution that $\frac{\pi(w^*|1)}{w^*} > \frac{\pi(\underline{w}|1)}{\underline{w}} \geq 1$. $\quad\square$

Lemma 6. *Consider scale of investment θ'_n and wealth level $w'_n \geq \theta'_n \underline{w}_n$. (i) The optimal contract to implement investment of scale θ'_n in period n satisfies, $d_n(x, x) = \min(x_n, D_n) + [\min(\gamma_n, (1 - w'_n)/E(x_n))]\max(x_n - D_n, 0)$ and $a_n(x) = R_{n+1}(1 - \gamma_n)\max(D_n - x_n, 0)$, where D_n is given by (6) for scale of investment $\theta_n = \theta'_n$. (ii) The debt level satisfies $D_n = \theta'_n D^1_n$ where D^1_n is defined by (6) for $\theta_n = 1$ and $w_n = w'_n/\theta'_n$. (iii) The expected end of period profit satisfies $\pi_n(w'_n|\theta'_n) = \theta'_n \pi_n(\frac{w'_n}{\theta'_n}|1)$.*

Proof. The optimal contract to implement scale of investment θ'_n follows from the proof of Proposition 2. First suppose that, $w'_n \in (\theta'_n(1 - \gamma_n E(x^1_n)), \theta'_n)$. Then, it follows from the definition of D_n in (6), and $x_n = \theta'_n x^1_n$, that for both pairs $\langle w'_n, \theta'_n \rangle$, and $\langle \frac{w'_n}{\theta'_n}, 1 \rangle$, $D_n = 0$, thus (ii) holds. Also, in the case $w'_n \in (\theta'_n(1 - \gamma_n E(x^1_n)), \theta'_n)$, the agent receives payment $(1 - \frac{\theta'_n - w'_n}{E(x_n)})x_n$ which yields expected profit $\pi_n(w'_n|\theta'_n) = E(x_n) + w'_n - \theta'_n = \theta'_n[E(x^1_n) + \frac{w'_n}{\theta'_n} - 1] = \theta'_n \pi_n(\frac{w'_n}{\theta'_n}|1)$, thus (iii) holds.

Now suppose that $w'_n \in [\theta'_n \underline{w}_n, \theta'_n(1 - \gamma_n E(x^1_n)))$. According to (6),

$$\begin{aligned}
D_n &= \min\{d \geq 0: \ E[\min(d, x_n)] \geq \theta'_n - w_n \\
&\quad + E[\bar{\delta}_n \max(d - x_n, 0) - \gamma_n \max(x_n - d, 0)]\} \\
\ldots &= \min\{d \geq 0: \ E[\min(d, \theta'_n x^1_n)] \geq \theta'_n - w_n \\
&\quad + E[\bar{\delta}_n \max(d - \theta'_n x^1_n, 0) - \gamma_n \max(\theta'_n x^1_n - d, 0)]\} \\
\ldots &= \min\{d \geq 0: \ E[\min(\tfrac{d}{\theta'_n}, x^1_n)] \geq 1 - \tfrac{w_n}{\theta'_n} \\
&\quad + E[\bar{\delta}_n \max(\tfrac{d}{\theta'_n} - x^1_n, 0) - \gamma_n \max(x^1_n - \tfrac{d}{\theta'_n}, 0)]\} \\
\ldots &= \theta'_n D^1_n.
\end{aligned}$$

Finally, this implies that $E(\min(x_n, D_n)) = \theta'_n E(\min(x^1_n, D^1_n))$ and $E[\max(D_n - x_n, 0)] = \theta'_n E[\max(D^1_n - x^1_n, 0)]$. Thus, $\pi_n(w'_n|\theta'_n) = \theta'_n \pi_n(\frac{w'_n}{\theta'_n}|1)$. $\quad\square$

Proof of Theorem 1. It follows from Lemmas 5, and Lemma 6 that the optimal level of investment is $\frac{w_n}{w^*_n}$. In particular,

$$\max_\theta \pi_n(w_n|\theta) = \max_\theta \theta \pi_n\left(\frac{w_n}{\theta} \bigg| 1\right) = w_n \max_\theta \frac{\pi_n(\frac{w_n}{\theta}|1)}{\frac{w_n}{\theta}} = w_n\left(\frac{\pi_n(w^*_n|1)}{w^*_n}\right).$$

Now, from $w^*_n \in (\underline{w}_n, 1 - \gamma_n E(x_n))$ (Lemma 5), and Lemma 6, it is optimal to have, $d_n(x, x) = \min(x_n, D_n) + \gamma_n \max(x_n - D_n, 0)$ and $a_n(x) = R_{n+1}(1 - \gamma_n)\max(D_n - x_n, 0)$,

where D_n is given by (6) for scale of investment $\theta_n = \frac{w_n}{w_n^*}$. Finally, as expected payoff to the agent in period n is linear in wealth w_n, the value function is linear, i.e., $V_n(w_n) = E(w_{n+1}|w_n)R_{n+1} = w_n(\frac{\pi_n(w_n^*|1)}{w_n^*})R_{n+1}$, and we define $R_n = (\frac{\pi_n(w_n^*|1)}{w_n^*})R_{n+1}$. Notice also that $R_n > R_{n+1}$ as $\frac{\pi_n(w_n^*|1)}{w_n^*} > 1$ (according to Lemma 5). $\quad\square$

Lemma 7. *Let D_n^1 be given by (6) for $\theta_n = 1$. The derivatives of D_n^1 with respect to $\bar{\delta}_n$ and γ_n satisfy:* $\frac{dD_n^1}{d\bar{\delta}_n} < 0$ *and* $\frac{dD_n^1}{d\gamma_n} = 0$. *Moreover,* $\frac{\partial w_n^*}{\partial \bar{\delta}_n} > 0$ *and* $\frac{\partial w_n^*}{\partial \gamma_n} < 0$.

Proof of Lemma 7. For simplicity all subscript n's are omitted during the proof. In particular, let $x = x_n^1$ (i.e., for $\theta_n = 1$), $\gamma = \gamma_n$, $\underline{w} = \underline{w}_n$, $\bar{\delta} = \bar{\delta}_n$, $w^* = w_n^*$ and $\pi(w|1) = \pi_n(w|1)$, and $D^1 = D_n^1$ (defined above). In the following we establish the derivatives $\frac{\partial w^*}{\partial \gamma}$ and $\frac{\partial w^*}{\partial \bar{\delta}}$. Consider the function G that defines the debt level D^1 when $G = 0$ according to (6),

$$G = (1 + \bar{\delta} - \gamma) \int_0^D x f(x)\, dx + D\big((1 - \gamma)(1 - F(D)) - \bar{\delta}F(D)\big) - 1 + w + \gamma E(x).$$

As $w^* \in (\underline{w}, 1 - \gamma E(x))$, the end of period expected payoff for scale of investment 1 is given by $\pi(w|1) = (1 - \gamma)\int_{D^1}^\infty (x - D^1)f(x)\, dx$ (Lemma 5). Moreover, $\pi(w|1)$ is a function of wealth w only through the solution D^1. It follows from Theorem 1 that the optimal scale of investment is determined by w^* implicitly given by the solution to the first order condition $Z = 0$ where,

$$Z = w\frac{\partial \pi(w|1)}{\partial w} - \pi(w|1).$$

We start by analyzing the function G,

$$\frac{\partial G}{\partial \gamma} = -\int_0^D x f(x)\, dx - D + DF(D) + E(x) = \int_D^\infty (x - D)f(x)\, dx > 0,$$

$$\frac{\partial G}{\partial \bar{\delta}} = -\int_0^D (D - x)f(x)\, dx,$$

$$\frac{\partial G}{\partial D} = (1 - \gamma)(1 - F(D)) - \bar{\delta}F(D) > 0 \quad \big(\text{at the point } D = D^1\big),$$

$$\frac{\partial G}{\partial w} = 1.$$

It follows from the Implicit Function Theorem and $G \equiv 0$ that,

$$\frac{\partial D^1}{\partial \bar{\delta}} = -\frac{\partial G/\partial \bar{\delta}}{\partial G/\partial D^1} = \frac{\int_0^{D^1}(D^1 - x)f(x)\, dx}{(1 - \gamma)(1 - F(D^1)) - \bar{\delta}F(D^1)} > 0,$$

$$\frac{\partial D^1}{\partial \gamma} = -\frac{\partial G/\partial \gamma}{\partial G/\partial D^1} = -\frac{\int_{D^1}^\infty (x - D^1)f(x)\, dx}{(1 - \gamma)(1 - F(D^1)) - \bar{\delta}F(D^1)} < 0,$$

$$\frac{\partial D^1}{\partial w} = -\frac{\partial G/\partial w}{\partial G/\partial D^1} = -\frac{1}{(1 - \gamma)(1 - F(D^1)) - \bar{\delta}F(D^1)} < 0.$$

It is convenient to rewrite $\frac{\partial D^1}{\partial \gamma}$ using the equilibrium relation $\pi(w^*|1) = w(\frac{\partial \pi(w|1)}{\partial w}|_{w=w^*})$. In particular at the optimum,

$$\frac{\partial D^1}{\partial \gamma} = -\frac{\pi(w|1)/(1-\gamma)}{(1-\gamma)(1-F(D^1)) - \bar{\delta}F(D^1)} = -\frac{w(\frac{\partial \pi(w|1)}{\partial w})/(1-\gamma)}{(1-\gamma)(1-F(D^1)) - \bar{\delta}F(D^1)},$$

where,

$$\frac{\partial \pi(w|1)}{\partial w} = \frac{\partial \pi(w|1)}{\partial D^1} \times \frac{\partial D^1}{\partial w} = \frac{(1-\gamma)(1-F(D^1))}{(1-\gamma)(1-F(D^1)) - \bar{\delta}F(D^1)} > 1.$$

Before analyzing the derivatives of the function Z, it is necessary to establish the following derivatives of the function $\pi(w|1)$:

$$\frac{d\pi(w|1)}{d\gamma} = \frac{\partial \pi(w|1)}{\partial \gamma} + \frac{\partial \pi(w|1)}{\partial D^1} \times \frac{\partial D^1}{\partial \gamma} = \frac{F(D^1)\bar{\delta}\int_{D^1}^{\infty}(x-D^1)f(x)\,dx}{(1-\gamma)(1-F(D^1)) - \bar{\delta}F(D^1)}$$

$$= \frac{F(D^1)\bar{\delta}\pi(w|1)}{(1-\gamma)((1-\gamma)(1-F(D^1)) - \bar{\delta}F(D^1))}$$

$$= \frac{w\bar{\delta}F(D^1)(1-F(D^1))}{[(1-\gamma)(1-F(D^1)) - \bar{\delta}F(D^1)]^2},$$

$$\frac{\partial \pi(w|1)}{\partial \bar{\delta}} = \frac{\partial \pi(w|1)}{\partial D^1} \times \frac{\partial D^1}{\partial \bar{\delta}} = -(1-\gamma)(1-F(D^1)) \times \frac{\partial D^1}{\partial \bar{\delta}}.$$

The above leads to the following second order derivative,

$$\frac{\partial^2 \pi(w|1)}{\partial w^2} = \frac{\partial^2 \pi(w|1)}{\partial w \partial D^1} \times \frac{\partial D^1}{\partial w} = -\frac{\bar{\delta}f(D^1)(1-\gamma)}{[(1-\gamma)(1-F(D^1)) - \bar{\delta}F(D^1)]^3} < 0.$$

And cross-derivatives,

$$\frac{\partial^2 \pi(w|1)}{\partial w \partial D^1} = \frac{\bar{\delta}f(D^1)(1-\gamma)}{[(1-\gamma)(1-F(D^1)) - \bar{\delta}F(D^1)]^2} > 0,$$

$$\frac{d(\frac{\partial \pi(w|1)}{\partial w})}{d\gamma} = \frac{\partial^2 \pi(w|1)}{\partial w \partial \gamma} + \frac{\partial^2 \pi(w|1)}{\partial w \partial D^1} \times \frac{\partial D^1}{\partial \gamma}$$

$$= \frac{(1-F(D^1))F(D^1)\bar{\delta} + f(D^1)\bar{\delta}(1-\gamma) \times \frac{\partial D^1}{\partial \gamma}}{[(1-\gamma)(1-F(D^1)) - \bar{\delta}F(D^1)]^2}$$

$$= \frac{\bar{\delta}(1-F(D^1))}{[(1-\gamma)(1-F(D^1)) - \bar{\delta}F(D^1)]^2}$$

$$\times \left(F(D^1) - \frac{wf(D^1)(1-\gamma)}{[(1-\gamma)(1-F(D^1)) - \bar{\delta}F(D^1)]^2} \right),$$

$$\frac{d(\frac{\partial \pi(w|1)}{\partial w})}{d\bar{\delta}} = \frac{\partial^2 \pi(w|1)}{\partial w \partial \bar{\delta}} + \frac{\partial^2 \pi(w|1)}{\partial w \partial D^1} \times \frac{\partial D^1}{\partial \bar{\delta}} = \frac{f(D^1)(1-\gamma)(1-F(D^1) + \bar{\delta}\frac{\partial D^1}{\partial \bar{\delta}})}{[(1-\gamma)(1-F(D^1)) - \bar{\delta}F(D^1)]^2}.$$

Now we can proceed and analyze the function Z (defined above) at the optimum $w = w^*$.

$$\frac{dZ}{d\gamma} = w\frac{d(\frac{\partial \pi(w|1)}{\partial w})}{d\gamma} - \frac{d\pi(w|1)}{d\gamma}$$

$$= \frac{w\bar{\delta}(1 - F(D^1))}{[(1-\gamma)(1-F(D^1)) - \bar{\delta}F(D^1)]^2}\left(F(D^1) - \frac{wf(D^1)(1-\gamma)}{[(1-\gamma)(1-F(D^1)) - \bar{\delta}F(D^1)]^2}\right)$$

$$- \frac{w\bar{\delta}F(D^1)(1 - F(D^1))}{[(1-\gamma)(1-F(D^1)) - \bar{\delta}F(D^1)]^2}$$

$$= -\frac{w\bar{\delta}(1 - F(D^1))}{[(1-\gamma)(1-F(D^1)) - \bar{\delta}F(D^1)]^2} \times \frac{wf(D^1)(1-\gamma)}{[(1-\gamma)(1-F(D^1)) - \bar{\delta}F(D^1)]^2} < 0.$$

Also,

$$\frac{dZ}{d\bar{\delta}} = w\frac{d(\frac{\partial \pi(w|1)}{\partial w})}{d\bar{\delta}} - \frac{\partial \pi(w|1)}{\partial \bar{\delta}}$$

$$= \frac{w(1-\gamma)(F(D^1)(1 - F(D^1)) + \bar{\delta}f(D^1)\frac{\partial D^1}{\partial \bar{\delta}})}{[(1-\gamma)(1-F(D^1)) - \bar{\delta}F(D^1)]^2} + (1-\gamma)\big(1 - F(D^1)\big)\frac{\partial D^1}{\partial \bar{\delta}} > 0,$$

$$\frac{dZ}{dw} = w\frac{\partial^2 \pi(w|1)}{\partial w^2} = -w\frac{\bar{\delta}f(D^1)(1-\gamma)}{[(1-\gamma)(1-F(D^1)) - \bar{\delta}F(D^1)]^3} < 0.$$

Using the Implicit Function Theorem again for $Z \equiv 0$, we calculate the derivatives $\frac{\partial w^*}{\partial \gamma}$ and $\frac{\partial w^*}{\partial \bar{\delta}}$. In particular,

$$\frac{\partial w^*}{\partial \gamma} = -\frac{\partial Z/\partial \gamma}{\partial Z/\partial w} = -\frac{w(1 - F(D^1))}{(1-\gamma)(1-F(D^1)) - \bar{\delta}F(D^1)} < 0,$$

and

$$\frac{\partial w^*}{\partial \bar{\delta}} = -\frac{\partial Z/\partial \bar{\delta}}{\partial Z/\partial w} = \frac{(1-\gamma)(1-F(D^1)) - \bar{\delta}F(D^1)}{\bar{\delta}f(D^1)}$$

$$\times \left(\big(1 - F(D^1)\big)F(D^1) + \bar{\delta}f(D^1)\frac{\partial D^1}{\partial \bar{\delta}}\right)$$

$$+ \frac{(1 - F(D^1))\frac{\partial D^1}{\partial \bar{\delta}}[(1-\gamma)(1-F(D^1)) - \bar{\delta}F(D^1)]^3}{wf(D^1)\bar{\delta}} > 0.$$

Next, we calculate the derivatives of D^1 with respect to γ and $\bar{\delta}$ at the optimum $w = w^*$. In particular,

$$\frac{dD^1}{d\gamma} = \frac{\partial D^1}{\partial w} \times \frac{\partial w^*}{\partial \gamma} + \frac{\partial D^1}{\partial \gamma}$$

$$= \frac{1}{(1-\gamma)(1-F(D^1)) - \bar{\delta}F(D^1)} \times \frac{w(1 - F(D^1))}{(1-\gamma)(1-F(D^1)) - \bar{\delta}F(D^1)}$$

$$- \frac{w(1 - F(D^1))}{[(1-\gamma)(1-F(D^1)) - \bar{\delta}F(D^1)]^2}$$

$$= 0,$$

and

$$\frac{\mathrm{d}D^1}{\mathrm{d}\bar{\delta}} = \frac{\partial D^1}{\partial w} \times \frac{\partial w^*}{\partial \bar{\delta}} + \frac{\partial D^1}{\partial \bar{\delta}}$$

$$= -\frac{(1-F(D^1))F(D^1)}{\bar{\delta} f(D^1)} - \frac{\partial D^1}{\partial \bar{\delta}}$$

$$-\frac{(1-F(D^1))\frac{\partial D^1}{\partial \bar{\delta}}[(1-\gamma)(1-F(D^1)) - \delta F(D^1)]^2}{w f(D^1)\bar{\delta}} + \frac{\partial D^1}{\partial \bar{\delta}} < 0. \qquad \square$$

Proof of Proposition 3. The portion of external financing, $\theta_n - w_n$, financed with debt equals

$$\frac{E[\min(x_n, D_n) - \bar{\delta}_n \max(D_n - x_n, 0)]}{\theta_n - w_n} = \frac{\theta_n - w_n - \gamma_n E[\max(x_n - D_n, 0)]}{\theta_n - w_n}.$$

The above equality holds as $\theta_n - w_n = V_n^D + \gamma_n V_n^E$. This can be written as

$$\frac{E[\min(x_n^1, D_n^1) - \bar{\delta}_n \max(D_n - x_n, 0)]}{1 - w_n^*} = \frac{1 - w_n^* - \gamma_n E[\max(x_n^1 - D_n^1, 0)]}{1 - w_n^*},$$

where D_n^1 is given by (6) for $\theta_n = 1$. Consider an increase in R_{n+1}. If $R_{n+1}(1-\rho) > 1$ then this results in an increase in $\bar{\delta}_n = \delta R_{n+1}(1-\rho)$ but no change to $\gamma_n = \rho$. This, then, leads to an increase in w_n^* and decrease in D_n^1 (Lemma 7). An examination of the right-hand side of the above equation yields that the fraction of external financing that is financed with debt is decreasing in R_{n+1}. If, on the other hand, $R_{n+1}(1-\rho) < 1$ then this results in no change in $\bar{\delta}_n = \delta$, an increase in $\gamma_n = \frac{R_{n+1}-1}{R_{n+1}}$, and a decrease in w_n^* but no change to D_n^1 (Lemma 7). An examination of the left-hand side of the above equation yields that the fraction of external financing that is financed with debt is decreasing in R_{n+1}. The results with respect to equity follow from $\theta_n - w_n = V_n^D + \gamma_n V_n^E$. $\square$

Proof of Proposition 4. Investment is given by $\theta_n = \frac{w_n}{w_n^*}$. Thus, investment increases in w_n. In addition, investment increases in R_{n+1} when $R_{n+1}(1-\rho) < 1$ (via the increase in γ_n), and decreases in R_{n+1} otherwise (via the increase in $\bar{\delta}_n$), as follows from Lemma 7. $\square$

Proof of Proposition 5. It follows from Theorem 1, and (5) that the expected profits of the agent $(1-\gamma_n)E(\max(x_n - D_n, 0))$ decrease in R_{n+1} when $R_{n+1}(1-\rho) > 1$ and increase in R_{n+1} when $R_{n+1}(1-\rho) < 1$. In particular, an increase in R_{n+1} tightens (relaxes) the **IC** constraint in (5) when $R_{n+1}(1-\rho) > 1$ ($R_{n+1}(1-\rho) < 1$), as $d_n(x,x) - d_n(t,t) \leqslant x - t$ for $t \leqslant x \in \mathcal{X}$ (Theorem 1). Finally, the definition of γ_n, together with the above, imply the non-monotonicity of $E(\max(x_n - D_n, 0))$. $\square$

Proof of Proposition 6. First suppose that $x_{n-1} > D_{n-1}$. Investment in period $1 < n \leqslant n^*$ satisfies $\theta_n = w_n + V_n^D + \gamma_n V_n^E$, where $w_n = (1-\gamma_{n-1})(x_{n-1} - D_{n-1})$. But the agent earns strictly positive profits on her investment in period n, i.e., $w_n < (1-\gamma_n)V_n^E$. This, together with $\gamma_{n-1} = \gamma_n = \rho$ (as $n \leqslant n^*$), imply that $x_{n-1} - D_{n-1} < V_n^E$. Finally, $\theta_n = w_n + V_n^D + \gamma_n V_n^E > x_{n-1} - D_{n-1} + V_n^D$, i.e., investment exceeds past profits plus new debt issued. In the case $x_{n-1} \leqslant D_{n-1}$ there are zero profits, and $\theta_n = 0$. $\square$

Proof of Corollary 1. Follows directly from the monotonicity of R_n in n and Propositions 3–5. $\square$

Proof of Proposition 7. In the second (and last) period there are no future investment opportunities, and the scale of investment, and debt level D_2^1, depend on δ, and the distribution function F_2. In particular, the debt level D_2^1, and return R_2, are decreasing in δ, while w_2^* is increasing in δ (Lemma 7). For any given wealth level w_2, the scale of investment $\theta_2 = \frac{w_2}{w_2^*}$ is decreasing in δ, and the value of debt $V_2^D = \theta_2 - w_2$ is also decreasing in δ. Moreover, it can be shown that there exists $\hat{\delta}(\rho)$ such that $R_2(1 - \rho) > 1$ for $\delta < \hat{\delta}(\rho)$, and $R_2(1 - \rho) \leqslant 1$ for $\delta \geqslant \hat{\delta}(\rho)$, provided that $E(x_2^1)(1 - \rho) < 1$ (from the continuity and monotonicity of R_2 in δ, and $R_2 > E(x_2^1)$).

In the first period, for $\delta \geqslant \hat{\delta}(\rho)$ (i.e., $R_2(1 - \rho) \leqslant 1$), $\gamma_1 = \frac{R_2 - 1}{R_2}$ is decreasing in δ (as R_2 is decreasing in δ), and $\bar{\delta}_1 = \delta$ is increasing in δ. This implies that w_1^* is increasing in δ (from Lemma 7), and the scale of investment $\theta_1 = \frac{w_1}{w_1^*}$ is decreasing in δ. Expected investment in the second period, $E(\theta_2) = \frac{E(w_2)}{w_2^*}$ is decreasing in δ for two reasons. First, w_2^* is increasing in δ, and second, $E(w_2) = (1 - \gamma_1)E(\max(x_1 - D_1, 0)) = (1 - \gamma_1)V_1^E$ is decreasing in δ. The latter follows from Theorem 1 and (5), as an increase in δ tightens the **IR** constraint, and a decrease in R_2 tightens the **IC** constraint for $R_2(1 - \rho) \leqslant 1$. This implies that V_1^E and $\gamma_1 V_1^E$ are also decreasing in δ (as γ_1 and $(1 - \gamma_1)V_1^E$ are decreasing in δ). Finally, the value of debt V_1^D is decreasing in δ (via the decrease in γ_1 and the increase in $\bar{\delta}_1$, see proof of Proposition 3). $\quad\square$

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