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Optimal Tactics for Close Support Operations: Part IV, Perfect Intelligence and Communications

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Abstract. A new generation of C^3 (command, control, and communication) models for military cybernetics is developed. Recursive equations for the solution of the C^3 problem are derived for an amphibious campaign with linear, time-varying dynamics. Air and ground commanders are assumed to have perfect intelligence and perfect communications. Numerical results are given for the optimal decision rules.

Key Words. Command, control, and communications; military cybernetics; optimal decision functions; minimum expected cost.

1. Introduction

Consider a C^3 (command, control, and communication) problem in which there are two subordinate commanders, both striving to coordinate their decisions to attain the tactical objective set down by superordinate headquarters. In an amphibious campaign, the blue naval force lands ground troops and provides close support. The objective is to move inland a certain distance in a specified time. It is desired to obtain optimal decision rules for force commitments in order to attain the objective at minimum expected cost.

Some general concepts of C^3 are discussed in Refs. 1–3. Recursive equations for the solution of the C^3 problem are derived for linear dynamics

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with quadratic costs in Refs. 4–6. In Refs. 4 and 5, the blue naval air and ground commanders are assumed to have perfect intelligence with degraded communications between them. In Ref. 6, it is assumed that the blue commanders have no intelligence and no communications. In this paper, perfect intelligence and perfect communications are assumed. The blue air commander is assumed to have perfect intelligence concerning the red air strength commitment for the coming day and communicates this information to the blue ground commander. Similarly, the blue ground commander is assumed to have perfect intelligence concerning the red ground strength commitment for the coming day and communicates this information to the blue air commander.

The general equations for optimal tactics with perfect intelligence and perfect communications are derived for a perturbation model followed by the recursive equation solution. The equations may be considered to provide either the optimal decisions for the total campaign with main objective s_0 or the optimal decision increments for a perturbed objective s_0. Numerical results are given for campaigns with time-invariant and time-varying dynamics.

2. C^3 Model

The C^3 model to be derived is a perturbation model with linear time-varying dynamics and quadratic costs. It is assumed that the scenario for the main forces has already been planned. Thus, only perturbations about the planned scenario are considered. For example, assume that the main objective is to move inland a distance of 200 miles in 21 days. The perturbed objective might be to move inland an additional 21 miles. The red air and ground strengths p and q are the perturbation strength increments (or decrements) about the main strengths in the 21-day campaign.

Let N be the duration of the campaign, and let the distance to the perturbed objective be s_0. Consider K days remaining with the front line at the perturbed position s. The new perturbed position of the front line is

$$S = S(s, p, q, \alpha, \beta), \tag{1}$$

where S is the new perturbed position increment of the front line about the planned position with $K-1$ days remaining; s is the current perturbed position increment of the front line about the planned position with K days remaining; p, q are red air and ground perturbed strength increments (or decrements) about the main strengths, respectively, with K days remaining; α, β are blue naval air and ground perturbed strength increments (or decrements) about the main strengths, respectively, with K days remaining.

The daily cost increment (or decrement) is given by

$$C = C(s, p, q, \alpha, \beta, K). \tag{2}$$

An additional cost is assessed if the front line at the end of the campaign is at some perturbed position increment s other than s_0. This terminal cost is

$$\phi = \phi(s). \tag{3}$$

The red air and ground commanders make the decision to employ strength increments p and q, respectively, each day. The decision-making of the enemy is simplified by assuming that p and q are random variables with joint probability density function

$$w = w(p, q). \tag{4}$$

Furthermore, assuming that p and q are independent random variables, the joint probability function can be expressed as the product

$$w(p, q) = P(p)Q(q). \tag{5}$$

The minimum expected cost is defined by

$$f_K = f_K(s), \qquad K = 0, 1, 2, \dots, N, \qquad \text{all } s, \tag{6}$$

where $f_K(s)$ is the expected cost increment (or decrement) of a campaign beginning with the front line at s, of duration K, and employing an optimal sequence of decisions. Using Bellman's principle of optimality (Ref. 7), the functions $f_{K+1}(s)$ and $f_K(s)$ are related by the recurrence equation

$$f_{K+1}(s) = \min_{\alpha,\beta} \iint \{C(s, p, q, \alpha, \beta, K) + f_K[S(s, p, q, \alpha, \beta)]\} P(p)Q(q)\, dp\, dq,$$
$$K = 0, 1, 2, \dots, N-1. \tag{7}$$

All integrals on p and q are from $-\infty$ to $+\infty$. When no time remains,

$$f_0(s) = \phi(s). \tag{8}$$

The conditions for obtaining the minimum are

$$0 = (\partial/\partial\alpha) B_{K+1}(s, p, q, \alpha, \beta), \tag{9}$$

$$0 = (\partial/\partial\beta) B_{K+1}(s, p, q, \alpha, \beta), \tag{10}$$

where

$$B_{K+1}(s, p, q, \alpha, \beta) = C(s, p, q, \alpha, \beta, K) + f_K[S(s, p, q, \alpha, \beta)]. \tag{11}$$

For linear, time-varying dynamics with quadratic costs, the new perturbed position of the front line is assumed to be a function of the old perturbed position plus a linear combination of the strengths

$$S = s + C_1\alpha + C_2\beta - C_3p - C_4q, \tag{12}$$

where

$$C_1 = C_1(K), \quad C_2 = C_2(K), \quad C_3 = C_3(K), \quad C_4 = C_4(K).$$

The perturbed daily cost is assumed to be proportional to the losses, which in turn are proportional to the strengths utilized. The ground losses are reduced by the air strength for close support missions. Thus, the daily cost is assumed to be

$$C = C_5\alpha + (C_6 - C_7\alpha)\beta + \tfrac{1}{2}C_8\alpha^2 + \tfrac{1}{2}C_9\beta^2, \tag{13}$$

where

$$C_5 = C_5(K), \quad C_6 = C_6(K), \quad C_7 = C_7(K),$$
$$C_8 = C_8(K), \quad C_9 = C_9(K).$$

The terms in α^2 and β^2 in the above cost expression serve to limit the force commitments made each day. To assue convexity, we assume that

$$C_8 > 0 \quad \text{and} \quad C_7^2 < C_8C_9.$$

The terminal cost is assumed to be

$$\phi(s) = \lambda(s - s_0)^2. \tag{14}$$

From general control theoretical considerations, the minimum expected cost has the form

$$f_K(s) = \zeta_K + \eta_K s + \theta_K s^2, \tag{15}$$

where the coefficients, ζ_K, η_K, θ_K, are computed for K stages remaining. Substituting Eqs. (12), (13), (15) into Eq. (7), the minimum expected cost is

$$\begin{aligned} f_{K+1}(s) = \min_{\alpha,\beta} \iint \{ & C_5\alpha + (C_6 - C_7\alpha)\beta + \tfrac{1}{2}C_8\alpha^2 + \tfrac{1}{2}C_9\beta^2 + \zeta_K \\ & + \eta_K(s + C_1\alpha + C_2\beta - C_3p - C_4q) \\ & + \theta_K(s + C_1\alpha + C_2\beta - C_3p - C_4q)^2\} P(p)Q(q)\, dp\, dq. \end{aligned} \tag{16}$$

Differentiating the integrand with respect to α and β yields

$$0 = C_5 - C_7\beta + C_8\alpha + \eta_K C_1 + 2\theta_K C_1(s + C_1\alpha + C_2\beta - C_3p - C_4q), \tag{17}$$

$$0 = C_6 - C_7\alpha + C_9\beta + \eta_K C_2 + 2\theta_K C_2(s + C_1\alpha + C_2\beta - C_3p - C_4q), \tag{18}$$

which are a set of linear algebraic equations for α and β. Clearly, α and β are linear in s.

3. Recursive Equations

The recursive equations for the solution of Eqs. (6) and (7) are derived as follows. Rearranging Eqs. (17)–(18),

$$\begin{aligned} -[C_5 + \eta_K C_1 + 2\theta_K C_1(s - C_3p - C_4q)] \\ = (2\theta_K C_1^2 + C_8)\alpha + (2\theta_K C_1C_2 - C_7)\beta, \end{aligned} \tag{19}$$

$$\begin{aligned} -[C_6 + \eta_K C_2 + 2\theta_K C_2(s - C_3p - C_4q)] \\ = (2\theta_K C_1C_2 - C_7)\alpha + (2\theta_K C_2^2 + C_9)\beta. \end{aligned} \tag{20}$$

Solving simultaneous Eqs. (19)–(20) for α and β, it can be shown that

$$\alpha(K+1, s, p, q) = u_K + v_K s + w_K p + r_K q, \tag{21}$$

$$\beta(K+1, s, p, q) = x_K + y_K s + z_K p + t_K q, \tag{22}$$

where

$$u_K = [-(2\theta_K C_2^2 + C_9)C_5 + (2\theta_K C_1C_2 - C_7)C_6 - \eta_K(C_1C_9 + C_2C_7)]/D, \tag{23}$$

$$v_K = -B_1, \tag{24}$$

$$w_K = B_1C_3, \tag{25}$$

$$r_K = B_1C_4, \tag{26}$$

$$x_K = [-(2\theta_K C_1^2 + C_8)C_6 + (2\theta_K C_1C_2 - C_7)C_5 - \eta_K(C_2C_8 + C_1C_7)]/D, \tag{27}$$

$$y_K = -B_2, \tag{28}$$

$$z_K = B_2C_3, \tag{29}$$

$$t_K = B_2C_4, \tag{30}$$

$$B_1 = 2\theta_K(C_2C_7 + C_1C_9)/D, \tag{31}$$

$$B_2 = 2\theta_K(C_2C_8 + C_1C_7)/D, \tag{32}$$

$$D = 4\theta_K C_1C_2C_7 - C_7^2 + 2\theta_K(C_2^2C_8 + C_1^2C_9) + C_8C_9. \tag{33}$$

Substituting the optimal α and β into Eq. (12), the new position of the front line becomes

$$\begin{aligned} S &= s + C_1(u_K + v_K s + w_K p + r_K q) + C_2(x_K + y_K s + z_K p + t_K q) - C_3p - C_4q \\ &= C_1u_K + C_2x_K + (1 + C_1v_K + C_2y_K)s + (C_1w_K + C_2z_K - C_3)p \\ &\quad + (C_1r_K + C_2t_K - C_4)q, \end{aligned}$$

that is,

$$S = A_1 + A_2 s + A_3 p + A_4 q, \tag{34}$$

where

$$A_1 = C_1 u_K + C_2 x_K, \tag{35}$$

$$A_2 = 1 + C_1 v_K + C_2 y_K, \tag{36}$$

$$A_3 = C_1 w_K + C_2 z_K - C_3, \tag{37}$$

$$A_4 = C_1 r_K + C_2 t_K - C_4. \tag{38}$$

The minimum expected cost $f_{K+1}(s)$ is obtained by substituting Eqs. (21), (22), (34) into Eq. (16):

$$\begin{aligned} f_{K+1}(s) = \iint \{ & C_5(u_K + v_K s + w_K p + r_K q) + [C_6 - C_7(u_K + v_K s + w_K p + r_K q)] \\ & \times (x_K + y_K s + z_K p + t_K q) + \tfrac{1}{2} C_8 (u_K + v_K s + w_K p + r_K q)^2 \\ & + \tfrac{1}{2} C_9 (x_K + y_K s + z_K p + t_K q)^2 + \zeta_K \\ & + \eta_K (A_1 + A_2 s + A_3 p + A_4 q) \\ & + \theta_K (A_1 + A_2 s + A_3 p + A_4 q)^2 \} P(p) Q(q)\, dp\, dq. \end{aligned} \tag{39}$$

Making use of the equations

$$\int P(p)\, dp = 1, \qquad \int Q(q)\, dq = 1, \tag{40}$$

$$\iint p P(p) Q(q)\, dp\, dq = \bar{p}, \tag{41}$$

$$\iint q P(p) Q(q)\, dp\, dq = \bar{q}, \tag{42}$$

integrating, and setting the coefficients of the constant terms, s, and s^2 equal, it can be shown that

$$\begin{aligned} \zeta_{K+1} = {} & C_5 A_6 + C_6 A_5 - C_7[u_K A_5 + w_K (x_K \bar{p} + z_K \overline{p^2} + t_K \bar{q}\bar{p}) \\ & + r_K (x_K \bar{q} + z_K \bar{p}\bar{q} + t_K \overline{q^2})] \\ & + \tfrac{1}{2} C_8 [u_K^2 + 2(u_K w_K \bar{p} + u_K r_K \bar{q} + w_K r_K \bar{p}\bar{q}) + w_K^2 \overline{p^2} + r_K^2 \overline{q^2}] \\ & + \tfrac{1}{2} C_9 [x_K^2 + 2(x_K z_K \bar{p} + x_K t_K \bar{q} + z_K t_K \bar{p}\bar{q}) + z_K^2 \overline{p^2} + t_K^2 \overline{q^2}] + \zeta_K \\ & + \eta_K (A_1 + A_3 \bar{p} + A_4 \bar{q}) \\ & + \theta_K [A_1^2 + 2(A_1 A_3 \bar{p} + A_1 A_4 \bar{q} + A_3 A_4 \bar{p}\bar{q}) + A_3^2 \overline{p^2} + A_4^2 \overline{q^2}], \end{aligned} \tag{43}$$

$$\begin{aligned} \eta_{K+1} = {} & C_5 v_K + C_6 y_K - C_7 A_5 v_K - C_7 A_6 y_K + C_8 (u_K + w_K \bar{p} + r_K \bar{q}) v_K \\ & + C_9 (x_K + z_K \bar{p} + t_K \bar{q}) y_K + \eta_K A_2 + 2\theta_K (A_1 + A_3 \bar{p} + A_4 \bar{q}) A_2, \end{aligned} \tag{44}$$

$$\theta_{K+1} = -C_7 v_K y_K + \tfrac{1}{2} C_8 v_K^2 + \tfrac{1}{2} C_9 y_K^2 + \theta_K A_2^2, \tag{45}$$

where

$$A_5 = x_K + z_K \bar{p} + t_K \bar{q}, \tag{46}$$

$$A_6 = u_K + w_K \bar{p} + r_K \bar{q}. \tag{47}$$

The second moments of p and q are

$$\overline{p^2} = \bar{p}^2 + \sigma_p^2, \qquad \overline{q^2} = \bar{q}^2 + \sigma_q^2, \tag{48}$$

where $\bar{p}$ and $\bar{q}$ are the average values and σ_p and σ_q are the standard deviations of p and q.

The recursive relations are given by Eqs. (43)–(45) supplemented by Eqs. (23)–(30) and (31)–(33). These equations may be considered to provide either (a) the optimal daily decisions α and β for the total campaign with the main objective s_0, or (b) the optimal daily decision increments α and β for the perturbed objective s_0.

4. Numerical Results

Numerical results were obtained using the recursive equations derived in the previous paragraphs. At each stage, the coefficients in the equations for the optimal blue air strength α and the optimal blue ground strength β are computed. The new position of the front line is computed in Eq. (34) using the optimal decisions.

Consider a campaign with the following duration N and additional distance to be covered s_0:

$$N = 21 \text{ days}, \qquad s_0 = 21 \text{ miles}.$$

The average red strengths and standard deviations are assumed to be

$$\bar{p} = \bar{q} = 1, \qquad \sigma_p = \sigma_q = 0.5.$$

Assume that the coefficients in Eqs. (12)–(14) are constants, with the exception of C_5:

$$\begin{aligned} &C_1 = 0.1, && C_4 = 1.0, && C_8 = 0.02, \\ &C_2 = 1.0, && C_6 = 0.1, && C_9 = 0.02, \\ &C_3 = 0.1, && C_7 = 0.01, && \lambda = 1.0. \end{aligned}$$

The coefficient C_5 in the cost equation is proportional to the red anti-air strength and may vary during the course of the campaign as the red anti-air fortifications are destroyed. Then, for various choices of $C_5(K)$, the optimal air and ground decisions α and β for $K=21$, 11, 1 stages to go are as follows:

Case I, Constant C_5. $C_5 = 0.005$.

$$K=21, \quad \alpha = 1.330 - 2.573\times 10^{-2}(s-0.1p-q),$$
$$K=21, \quad \beta = 1.912 - 4.503\times 10^{-2}(s-0.1p-q);$$
$$K=11, \quad \alpha = 1.794 - 4.911\times 10^{-2}(s-0.1p-q),$$
$$K=11, \quad \beta = 2.723 - 8.594\times 10^{-2}(s-0.1p-q);$$
$$K=1, \quad \alpha = 11.47 - 0.5369(s-0.1p-q),$$
$$K=1, \quad \beta = 19.66 - 0.9396(s-0.1p-q).$$

Case II, Time-Varying C_5. $C_5(K) = 0.005 + 0.0005K$.

$$K=21, \quad \alpha = 0.7493 - 2.573\times 10^{-2}(s-0.1p-q),$$
$$K=21, \quad \beta = 1.770 - 4.503\times 10^{-2}(s-0.1p-q);$$
$$K=11, \quad \alpha = 1.493 - 4.911\times 10^{-2}(s-0.1p-q),$$
$$K=11, \quad \beta = 2.654 - 8.594\times 10^{-2}(s-0.1p-q);$$
$$K=1, \quad \alpha = 11.45 - 0.5369(s-0.1p-q),$$
$$K=1, \quad \beta = 19.67 - 0.9396(s-0.1p-q).$$

In Case I, the coefficient C_5 is constant. Considering the campaign as a whole with 21 days and 21 miles to go, and starting with the front line at $s=0$, if the red air and ground strengths are constant and equal to their average values of one, i.e.,

$$p = \bar{p} = 1 \quad \text{and} \quad q = \bar{q} = 1,$$

then the optimal blue air and ground strengths are also constant and given by

$$\alpha = 1.359 \quad \text{and} \quad \beta = 1.961.$$

The daily cost given by Eq. (15) is

$$C = C_5\alpha + (C_6 - C_7\alpha)\beta + \tfrac{1}{2}C_8\alpha^2 + \tfrac{1}{2}C_9\beta^2$$
$$= 0.00679 + 0.169 + 0.018 + 0.038 = 0.233.$$

The front line will move forward at the increment at approximately one mile per day.

In Case II, $C_5(K)$ decreases as the campaign progresses. If the red air and ground strengths are equal to their average values, then

$$\alpha = 0.7776 \quad \text{and} \quad \beta = 1.819$$

at the beginning of the campaign and gradually increase to

$$\alpha = 1.444 \quad \text{and} \quad \beta = 2.153$$

at the end of the campaign. The front line moves forward at the increment of 0.8 miles per day at the beginning of the campaign and gradually increases to 1.2 miles per day at the end of the campaign. The front line moves forward at a lower rate at the beginning of the campaign where the coefficients of the daily cost are higher, and moves forward at a higher rate near the end of the campaign where the coefficients are lower.

It should be noted that, in the examples given, the optimal decisions are the perturbed strength increments about the main strengths and the positions are the perturbed position increments of the front line about the planned position. Furthermore, α, β, and the minimum expected cost increments are all positve.

The optimal decisions correspond closely to those obtained for perfect intelligence with no communications in Ref. 5 and for no intelligence and no communications in Ref. 6. The optimal decisions for perfect intelligence and perfect communications in this paper have the general form

$$\alpha = u_K + v_K s + w_K p + r_K q,$$
$$\beta = x_K + y_K s + z_K p + t_K q.$$

For no communications between the blue air and ground commanders, the coefficients r_K and z_K are

$$r_K = z_K = 0.$$

For no intelligence and no communications,

$$w_K = r_K = z_K = t_K = 0.$$

The coefficients of s are numerically the same in all three cases of intelligence and communications. If the red air and ground strengths are equal to their average values, then the optimal decisions for perfect intelligence and perfect communications are identical to those with perfect intelligence and no communications, and no intelligence and no communications, assuming that the C_i coefficients given above are the same. The optimal decisions for no intelligence and no communications do not depend directly on p and q. The minimum expected cost is highest for the latter case and lowest for perfect intelligence and perfect communications. The minimum expected cost increases as the standard deviations σ_p and σ_q are increased. The effect of increasing the standard deviations is to increase ζ_{K+1} in Eq. (43).

In this paper, the air and ground commanders were assumed to have perfect intelligence and perfect communications. Future papers will consider detailed comparisons of the effects of intelligence and communications on tactics and costs.

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TECHNICAL NOTE

On the Discrete-Time Regulator Problem in Infinite-Dimensional Spaces[1]

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Abstract. The linear, discrete-time regulator problem is considered in infinite-dimensional spaces without posing in advance any positivity conditions on quadratic criterion. The convergence of the finite-time optimum solution is studied, when time increases to infinity with a stable, stabilizable, and detectable system.

Key Words. Linear optimal control, Hilbert spaces, infinite-dimensional spaces, discrete-time systems.

1. Introduction

The linear discrete-time quadratic optimum control problem in infinite-dimensional Hilbert spaces is considered almost on minimum conditions, i.e., without posing in advance any positivity conditions on Q and R in the quadratic criterion. The problem is formulated in spaces of control and state sequences in which the existence and uniqueness of the optimum solution reduces to a positivity condition of an operator, whose properties are based on system parameters. The optimum solution, when it exists, proves to be a linear and bounded function of the initial state. Then, the discrete Riccati operator is defined and the optimal control law may be derived.

Similar results are deduced in infinite-time set with a stable system. Then, the convergence of the finite-time solution is studied and, under a strict positiveness hypothesis, it turns out to be uniform in the initial state

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