

Stochastic Volatility and Asset Pricing Puzzles

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Abstract

This paper builds a real-options, term structure model of the firm to shed new light on the value premium, financial distress, momentum, and credit spread puzzles. The model incorporates stochastic volatility in the firm productivity process and a negative market price of volatility risk. Since the equity of growth firms and financially distressed firms have embedded options, such securities hedge against volatility risk in the market and thus command lower volatility risk premia than the equities of value or financially healthy firms. Abnormal risk-adjusted momentum profits are concentrated among low credit-rating firms for similar reasons. Conversely, since increases in volatility generally reduce the value of debt, corporate debt will tend to command large volatility risk premia, allowing the model to generate higher credit spreads than existing structural models. The paper illustrates that allowing for endogenous default by equityholders is necessary for the model to account for the credit spreads of both investment grade and junk debt. The model is extended to include rare disasters and multiple time scales in volatility dynamics to better account for the expected default frequencies and credit spreads of short maturity debt. Finally, the paper uses a methodology based on asymptotic expansions to solve the model.

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1 Introduction

An enormous body of work in financial econometrics has documented that volatility in market returns is time-varying and has attempted to model its dynamics.¹ Chacko and Viceira (2005) and Liu (2007) point out that this time-varying volatility implies long-term investors will value assets for their ability to hedge against market volatility risk in addition to their expected returns and other hedging properties. In market equilibrium, then, assets which hedge against persistent volatility risk should require lower average returns, all else equal.² Campbell, Giglio, Polk, and Turley (2012) formally model this point, constructing an intertemporal capital asset pricing model (ICAPM) incorporating stochastic volatility and documenting the existence of low-frequency movements in market volatility.

This paper builds on these insights by constructing a real options, term-structure model of the firm that includes persistent, negatively priced shocks to volatility. In contrast to Campbell et al. (2012), the primary focus is to understand how differences in financing, productivity, and investment options across firms and their relationship with volatility generate cross-sectional variability in asset prices. The paper makes two key contributions. I first demonstrate that the model’s qualitative asset pricing predictions offer new insights on the value premium, financial distress, and momentum puzzles in cross-sectional equity pricing and generate new testable predictions.³ I then calibrate key parameters and analyze the model’s quantitative implications for debt pricing. The model, when calibrated to match historical leverage ratios and recovery rates, can generate empirically observed levels of credit spreads and default probabilities across ratings categories *if and only if* equityholders are allowed to optimally decide when to default. Thus, the model quantitatively delivers a resolution of the credit spread puzzle documented by Huang and Huang (2003) that existing structural models with only a single sourced price of risk significantly underpredict credit spreads, especially for investment grade debt, when matching historical recovery rates, leverage ratios, and empirical default frequencies.

To fix ideas and elucidate the key intuitions at work in the model, consider a firm with

¹Much of the literature provides models similar to the ARCH/GARCH models of Engle (1982) and Bollerslev (1986) in which volatility is function of past return shocks and its own lags. More recent literature has used high-frequency data to directly estimate the stochastic volatility process. Papers include Barndorff-Nielsen and Shephard (2002), Bollerslev and Zhou (2002), and Andersen et al. (2003).

²This will be the case if, in particular, asset prices are priced according to the first-order conditions of a long-term, Epstein-Zin representative investor with a coefficient of relative risk aversion greater than one.

³The value premium puzzle, due to Basu (1977,1983) and Fama and French (1993), refers to the greater risk-adjusted return of high book-to-market stocks over low book-to-market stocks. The momentum puzzle, attributed to Jagadeesh and Titman (1993), is the finding that a portfolio long winners and short losers generates positive CAPM alpha. The financial distress puzzle, uncovered by Dichev (1998) and Campbell, Hilscher, and Szilagyi (2008), refers to the positive CAPM alpha on a portfolio long financially healthy firms and short firms close to default.

debt and suppose that the equityholders can optimally choose to either default on the firm's obligations or to continue financing the firm by issuing new equity. Then part of the value of the firm is the equityholders' default option. By the standard logic of option theory, an increase in volatility should raise the value of this embedded option. For example, if uncertainty about demand for the firm's product increases, the potential downside to equityholders is capped by limited liability, while there is substantial upside potential.⁴

However, an increase in volatility has an offsetting effect. Since the firm maintains a stationary capital structure, it is exposed to rollover risk. At every point in time, the firm retires a given quantity of principal and issues new debt with principal exactly equal to this amount. Given that new debt must be issued at market value, the firm will then either face a cash shortfall or windfall as part of its flow dividend depending on the current price. Increases in volatility amplify this risk and lower the expected present value of future dividends accruing to equityholders.⁵ Therefore, an increase in volatility affects equityholders negatively as well as positively.

If the firm is currently in financial distress, i.e. close to default, then most of the equity value is comprised of the default option. At these high levels of default probability, the increased option value from higher volatility dominates the exacerbated rollover risk and financially distressed firms serve as a hedge against volatility in the market, rising in market value when volatility increases. Conversely, the default option only constitutes a small fraction of the equity value of a healthy firm, such that the effect on rollover risk dominates and increases in volatility lower the equity value. According to the logic of the ICAPM, this implies that healthy firms should have higher variance risk premia than financially distressed firms. Empirically, this mechanism would manifest itself as a positive CAPM alpha in a portfolio which is long healthy firms and short financially distressed firms, hence resolving the financial distress puzzle.

In a similar fashion, the model also predicts a positive CAPM alpha of a portfolio which is long recent winners and short recent losers, consistent with the momentum anomaly. More specifically, the model predicts that abnormal risk-adjusted momentum profits should be concentrated among firms with low credit ratings, which has been empirically confirmed by Avramov, Chordia, Jostova, and Philipov (2007). Intuitively, if a firm in financial distress experiences a string of positive returns, then the variance beta of the equity falls as the firm's health improves and the importance of the equityholders' default option decreases. On the other hand, the variance beta of a firm with low credit rating experiencing a string of negative

⁴Limited liability has been studied in the corporate finance literature in papers such as Hellwig (1981), Innes (1990), Laux (2001), and Biais et al. (2012).

⁵Increases in volatility decrease the expected present value of future dividends since the value of newly issued debt, and hence the flow dividend of the firm, are concave in the firm's productivity level.

returns should increase as it is likely the firm is becoming more financially distressed. Since shocks to volatility are negatively priced, the variance risk premium and hence risk-adjusted returns of low-credit-rating winners are higher than those of low-credit-rating losers.

The model's cross-sectional predictions for firms which differ in their book-to-market ratios are more nuanced. In the model, firms can have different book-to-market ratios in a variety of ways. I focus on two.⁶ First, consider two firms and suppose their book leverage ratios and ratios of growth options to book value are the same. Then, by scale invariance, the firm with lower productivity will have the higher book-to-market ratio. If this firm is also financially distressed, then the firm with the *lower* book-to-market ratio, i.e. higher productivity, should earn greater risk-adjusted returns since its variance beta would be lower.

However, if both firms are healthy and the average maturity of their debt is short, then it is the firm with lower productivity and *higher* book-to-market ratio that earns greater risk-adjusted returns. To see this, note that among healthy firms, exposure to volatility is largely due to rollover risk, which becomes more important as a component of equity valuation as the average maturity of the firm's debt falls. But since the market value of debt is more sensitive to volatility at lower productivities, increases in volatility exacerbate rollover risk more at lower productivities. Therefore, among two healthy businesses, the firm with lower productivity and higher book-to-market ratio has greater negative exposure to volatility and thus commands higher risk premia. In this fashion, the model can generate both a value premium *and* a growth premium.

Second, firms within the model having the same book leverage ratios and productivity can still differ in their book-to-market ratios if one, the growth firm, has a higher ratio of investment options to book value. These embedded options operate in a similar manner for growth firms as does the default option for financially distressed firms. Since the values of the growth options increase with volatility, all else equal, a firm with a higher ratio of growth options to book value will serve as a better hedge against volatility risk in the market and should earn a lower variance risk premium. This channel therefore generates a value premium.

Turning to fixed income, note that while equity is long the option to default, debt is short. An increase in volatility raises the probability of default, which decreases the value of debt. As a result, the required return on debt should be higher in a model with stochastic volatility than a model in which volatility remains constant. This effect lowers prices and increases credit spreads. However, while this intuition is strong and important, quantitative

⁶In addition to the mechanisms discussed, book-to-market ratios can also differ if firms have different levels of asset volatility or different book leverage ratios.

calibration of the model illustrates that it is itself insufficient to match empirical credit spreads across ratings categories. If the default barrier is set exogenously, then the model simply creates a credit spread puzzle in the other direction. It is able to better explain the credit spreads on investment grade debt, but ends up substantially *overpredicting* the credit spreads on junk debt. This result is a reflection of the fact that existing structural models with constant volatility actually do a reasonably good job predicting the credit spreads of junk debt.

The paper demonstrates that it is exactly the interaction of stochastic volatility *and* endogenous default which allows the model to quantitatively match credit spreads across credit ratings and therefore resolve the credit spread puzzle. The reasons are twofold. First, as has been discussed, when volatility increases, it raises the value of the equityholders' default option. If equityholders optimally decide when to default, they respond to an increase in volatility by postponing default. This channel ameliorates the adverse effects of higher volatility on debt value and, in fact, can reverse the sign if the firm is sufficiently financially distressed. That is, the debt of a firm extremely close to default actually benefits from an increase in volatility and thus hedges volatility risk.⁷ In a precise sense to be made clear, the price of junk debt reflects these hedging properties such that the model generates lower credit spreads than a model with exogenous default.

Second, the simple fact that volatility is stochastic raises the value of the default option such that at all volatility levels, the endogenous default barrier is lower than in a model with constant volatility. This once again tends to raise debt values, but the effect of the shift is strongest for junk debt since it is closest to default and the future is discounted. With these two effects present, the calibrated model is able to quantitatively match well the target credit spreads and historical default probabilities across all ratings categories for intermediate and long maturity debt, holding *fixed* the market price of variance risk.

The model is less successful at shorter maturities. While still providing a substantial improvement over a model with constant volatility, it accounts for less of the empirically observed credit spreads than at longer maturities. The calibration further demonstrates, though, that some of this underprediction is due to the model significantly understating credit risk at these maturities. In two extensions, I consider additions to the model which allows it to generate higher credit risk at short maturities, while only marginally affecting credit risk at longer maturities. In the first, I include a fast-moving volatility time scale in addition to the slow one. In the second, I include rare disasters in the form of low-frequency jumps in the firm productivity process.

⁷This result implies that the value of the firm increases with volatility close to the default boundary, since the equity of financially distressed firms also increases with volatility.

That stochastic volatility has not received much attention in structural credit modeling is likely due to the considerable technical difficulties involved, a fact pointed out by Huang and Huang (2003). To overcome the technical hurdles, I employ novel perturbation techniques from mathematical finance and physics to construct accurate, approximate asymptotic series expansions of contingent claim valuations. The fundamental assumption which makes this methodology operative in the primary model is that volatility is slowly-moving and persistent. While the econometric literature has uncovered multiple time scales in volatility dynamics, note that it is exactly these persistent fluctuations which long-run investors should care about and should therefore be significantly priced in equilibrium. The key advantages of this approach are twofold. First, it provides analytic tractability by transforming the solution of difficult partial differential equations problems with an unknown boundary to recursively solving a standard S problem, and then a straightforward ordinary differential equations problem involving key comparative statics of the model in which volatility remains constant. Second, an application of the Feynman-Kac formula provides a useful probabilistic interpretation of the first-order correction terms which allows one to cleanly see the various mechanisms at work in the model.

The paper proceeds as follows. Section 2 briefly reviews the relevant literature. Section 3 introduces the baseline model and provides characterizations of contingent claims valuations as solutions to appropriately defined partial differential equations problems. Section 4 discusses the perturbation methodology used to solve the problems. Section 5 considers the qualitative implications of the model for the equity puzzles and provides testable predictions. Section 6 calibrates the model and demonstrates how it quantitatively resolves the credit spread puzzle. Section 7 considers extensions to the baseline model. Finally, Section 8 concludes.

2 Literature Review

In addition to the ICAPM of Campbell et al. (2012), this paper is connected to a broader literature recognizing the asset pricing implications of stochastic volatility. The long-run risks model of Bansal and Yaron (2004) incorporates time-varying consumption volatility into a consumption-based asset pricing framework. Later calibrations of the model by Beeler and Campbell (2012) and Bansal, Kiku, and Yaron (2012) emphasize the importance of this feature in delivering empirically reasonable results. Coval and Shumway (2001), Ang, Hodrick, Xing, and Zhang (2006), Adrian and Rosenberg (2008), and Carr and Wu (2009) provide empirical evidence that innovations in market volatility are priced risk factors in the cross-section of stock returns. This paper differs from those above by explicitly constructing

a structural model of the firm to understand how differences across firms can explain cross-sectional asset pricing patterns observed in the data for both equity and debt.

Previous papers have offered non-behavioral explanations for the equity puzzles considered in this paper. Gomes, Kogan, and Zhang (2003) construct a model in which book-to-market serves as a proxy for the systematic risk of assets in place. Zhang (2005) demonstrates how a model with both idiosyncratic productivity and convex capital adjustment costs can generate a value premium. Cooper (2006) is similar but includes non-convex adjustment costs. Carlson, Fisher, and Giammarino (2004) generate book-to-market effects in a model with operating leverage. Garlappi and Yan (2011) argue that the financial distress puzzle can be accounted for by a model of partial shareholder recovery.

One consistent feature in all of this work is that there is a single source of priced risk. Therefore, in explaining the value premium puzzle, for instance, the models generate higher conditional market betas of value firms than growth firms, contradicting empirical evidence. This paper differs by including a second source of priced risk, such that abnormal risk-adjusted returns can be explained by variance betas rather than market betas. In this fashion, the study is similar to Papanikolaou (2011), which includes investment shocks as a second source of priced risk to explain the value premium puzzle.

The paper is part of a large literature on structural credit modeling. Important contributions are Merton (1974), Black and Cox (1976), Leland (1994a,1994b), Longstaff and Schwartz (1995), Anderson, Sundaresan and Tychon (1996), Leland and Toft (1996), Leland (1998), Duffie and Lando (2001), and Collin-Dufresne and Goldstein (2001). Features of these models include stochastic interest rates, endogenous default, shareholder recovery, incomplete accounting information, and mean-reverting leverage ratios. Yet, as Huang and Huang (2003) show, these models cannot jointly produce historical default probabilities and realistic credit spreads. My work is most similar to Hackbarth, Miao, Morellec (2006), Chen, Collin-Dufresne (2009), and Chen (2010) in analyzing how business cycle variation in macroeconomic conditions impacts credit spreads. My work differs from these in allowing for both independent diffusive movements in volatility as well as endogenous default and, moreover, analyzing the credit spreads on junk debt in addition to the Aaa-Baa spread.

3 Structural Model of the Firm

In this section, I develop a continuous-time, real options model of the firm incorporating both stochastic volatility of the firm productivity process and strategic default by equityholders. The model will allow for an analysis of equity pricing as well as the full term structure of credit spreads. In later sections, I enrich this base framework by including multiple time

scales in the volatility dynamics and rare disasters in the productivity process. Table 1 defines the model's key variables for convenient reference.

3.1 Firm Dynamics - Physical Measure

Set a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let \mathbf{W}_t be a Wiener process or standard Brownian motion in two dimensions under \mathbb{P} , which I will call the physical measure. Associated with this Brownian motion is a filtration \mathcal{F}_t satisfying the usual properties.⁸ Firms are value-maximizing and operate in a perfectly competitive environment. The productivity of a representative firm's capital is assumed to follow a stochastic process given by

$$dX_t/X_t = \mu dt + \sqrt{Y_t} dW_t^{(1)}, \quad (1)$$

where $W_t^{(1)}$ is a standard Brownian motion and μ is the expected growth rate of productivity under the physical measure \mathbb{P} .^{9,10} The variance of this process Y_t is itself stochastic and follows a mean-reverting Cox-Ingersoll-Ross (CIR) process under the physical measure given by¹¹:

$$dY_t = \kappa_Y (\theta_Y - Y_t) + \nu_Y \sqrt{Y_t} dW_t^{(2)}. \quad (2)$$

Here, κ_Y is the rate of mean reversion, θ_Y is the long-run mean of variance, and ν_Y controls the volatility of variance. The process $W_t^{(2)}$ is a Brownian motion which has correlation ρ_Y with the process $W_t^{(1)}$. In particular, I have that

$$\begin{pmatrix} W_t^{(1)} \\ W_t^{(2)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \rho_Y & \sqrt{1 - \rho_Y^2} \end{pmatrix} \mathbf{W}_t, \quad (3)$$

where \mathbf{W}_t is the Wiener process defined above.

⁸In particular, for $0 \leq s < t$, $\mathcal{F}_s \subseteq \mathcal{F}_t$. For each $t \geq 0$, the process \mathbf{W}_t is \mathcal{F}_t -measurable. Finally, for $0 \leq t < \tau$, the increment $\mathbf{W}_\tau - \mathbf{W}_t$ is independent of \mathcal{F}_t . See definition 3.3.3. of Shreve (2004).

⁹The assumption of stochastic cash flows differs from Leland (1994) and many other models of corporate debt which directly specify a stochastic process for the value of the unlevered firm. As discussed by Goldstein et al. (2001) this assumption has several advantages with regards to the modeling of tax shields and calibration of the risk neutral drift. It is also easier to incorporate growth options into such a framework.

¹⁰For technical reasons, I actually require the diffusion for cash flow to be given by:

$$dX_t/X_t = \mu dt + f(Y_t) dW_t^{(1)},$$

where $f(\cdot)$ is a smooth, positive function which is bounded and bounded away from zero. Set $f(Y_t) = \sqrt{Y_t}$ over a sufficiently large compact interval and use bump functions (also known as mollifiers) to guarantee smoothness at the boundaries.

¹¹I could have alternatively considered a mean-reverting Ornstein-Uhlenbeck process in which the volatility of variance does not scale with the current level. I choose a CIR process to be consistent with much of the empirical literature on index options.

Table 1: Variables in Structural Model of the Firm

Variable	Description
State Variables	
X_t	Asset productivity
Y_t	Asset variance
Contingent Claims	
U^a, U	Value of assets in place
E^a, E	Equity values
\tilde{d}^a, \tilde{d}	Value of newly issued debt
D^a, D	Total debt values
V^a, V	Total debt value plus equity value
u^a, u	Cumulative survival probabilities
Parameters	
r	Riskfree rate
μ, g	Expected productivity growth rates
κ_Y, κ_Z	CIR rates of mean-reversion
θ_Y, θ_Z	Long-run variances
ν_Y, ν_Z	Volatility of variance
ρ_Y, ρ_Z	Correlations between productivity and variance
ρ_{YZ}	Correlation between variance shocks
K, K^a	Capital stocks of mature, young firms
λ	Arrival rate of rare disasters
I	Cost of investment
ϕ	Corporate tax rate
ξ	Bankruptcy costs
P	Total principal
C	Total coupon
m	Rollover rate of debt
$\pi_X(Y_t)$	Asset risk premium
$\Gamma(Y_t)$	Market price of variance risk

Note: This table defines the key variables, parameters, and notation used throughout the paper. A subscript a refers to young firms. The subscript Y refers to the slow-moving volatility factor and the subscript Z refers to the fast-moving volatility factor. The parametric specification for the market price of volatility risk is given by $\Gamma(Y_t) = \Gamma_0 \sqrt{Y_t}$.

The firm has initial capital $K^a < 1$ and can irreversibly expand its productive capacity to $K = 1$ at the discretion of the equityholders. Exercising the growth option costs $I > 0$.¹² I define firms with capital K^a to be “young” or “growth” firms and those with capital K as “mature” firms. Finally, firms pay taxes on their income at the corporate rate ϕ so that the flow of after-tax profits of the unlevered firm at time t is given by

$$(1 - \phi) X_t K_t. \quad (4)$$

As is standard, this profit function reflects optimal choices by the firm in all other variable inputs such as labor and raw materials.

3.2 Firm Dynamics - Risk Neutral Measure

I assume the existence of an equivalent martingale measure \mathbb{P}^* under which contingent claims will be priced. In particular, I suppose there exist processes $\gamma_1(Y_s), \gamma_2(Y_s) \in \mathcal{L}^2$ such that the process

$$\mathbf{W}_t^* = \mathbf{W}_t + \int_0^t \begin{pmatrix} \gamma_1(Y_s) \\ \gamma_2(Y_s) \end{pmatrix} ds \quad (5)$$

is a martingale under \mathbb{P}^* . I define the *market price of risk* as $\gamma_1(Y_t) > 0$ and the *market price of variance risk* as

$$\Gamma(Y_t) = \rho_Y \gamma_1(Y_t) + \sqrt{1 - \rho_Y^2} \gamma_2(Y_t). \quad (6)$$

These definitions are motivated by the fact that due to Girsanov’s theorem, under the risk-neutral measure, the dynamics of productivity and variance are given by

$$dX_t = \left(\mu - \gamma_1(Y_t) \sqrt{Y_t} \right) X_t dt + \sqrt{Y_t} X_t dW_t^{(1)*} \quad (7)$$

$$dY_t = \left(\kappa_Y (\theta_Y - Y_t) - \Gamma(Y_t) \nu_Y \sqrt{Y_t} \right) dt + \nu_Y \sqrt{Y_t} dW_t^{(2)*}, \quad (8)$$

where $W_t^{(i)*}$ is defined as in (3) replacing \mathbf{W} with \mathbf{W}^* .

Since I am considering the setting of random, non-tradeable volatility, I am not in a complete markets framework and therefore the equivalent martingale measure is not unique. In fact, there exists a family of pricing measures parameterized by the market price of variance risk. Note that the market price of variance risk will not be zero if there is correlation between movements in variance and movements in productivity. That is, the first term in (6) will

¹²The choice of $K = 1$ is simply a normalization reflecting the scale invariance of the model. Investment occurs discretely, which allows me to solve for the contingent claims of mature and young firms recursively.

be nonzero and have the same sign as ρ_Y . However, theory suggests that independent fluctuations in variance may also be priced. As discussed in the introduction, if long-term investors see increases in volatility as deteriorations in the investment opportunity set, then the price $\gamma_2(Y_t)$ will be negative.

I define the *asset risk premium* to be $\pi_X(Y_t) = \gamma_1(Y_t)\sqrt{Y_t}$ and assume it to be independent of current variance Y_t . This allows me to define $g = \mu - \pi_X$ as the expected growth rate of productivity under the risk-neutral measure. Finally, I will refer to the expression $\nu_Y\sqrt{Y_t}\Gamma(Y_t)$ as the *variance risk premium*.¹³ As this object will be of central importance in the paper, it therefore warrants further discussion. The variance risk premium is the expected excess return over the riskfree rate on any asset with zero market beta and a variance beta equal to one, where these betas are respectively defined as¹⁴

$$\beta_X = x \frac{\partial \log(\cdot)}{\partial x} \tag{9}$$

$$\beta_Y = \frac{\partial \log(\cdot)}{\partial y}. \tag{10}$$

For instance, it gives the expected excess return on a portfolio of delta-neutral straddles or a portfolio of variance swaps constructed to have a unit variance beta. Coval and Shumway (2001) estimate the expected excess returns on delta-neutral straddles and find them to be negative. Carr and Wu (2009) document a similar fact for variance swaps. These empirical findings are supportive of a negative variance risk premium as predicted by the ICAPM.

For my calibration and quantitative analysis, I will adopt the following parametric specification for the market price of variance risk:

$$\Gamma(Y_t) = \Gamma_0\sqrt{Y_t}, \tag{11}$$

which leads to a variance risk premium that is affine in the current variance Y_t . Beginning with Heston (1993), this has been the standard assumption in both the option pricing and financial econometrics literature. Its appealing feature is it implies the variance dynamics follow a Cox-Ingersoll-Ross (CIR) process under both the risk-neutral and physical measures. The only difference is the rate of mean-reversion. Note, however, that the theory which follows will not rely at all on this parameterization and holds for any arbitrary specification $\Gamma(Y_t)$.

The value of the assets in place is given by the discounted present value under the risk-

¹³The distinction between prices of risk and risk premia are confused in the literature. I take care here to be explicit about their definitions.

¹⁴These definitions reflect the use of Ito's lemma in calculating the required excess return of an asset under the physical measure.

neutral measure of future after-tax profits generated by installed capital. Denoting this value by U_t for mature firms and letting expectations under \mathbb{P}^* conditional on \mathcal{F}_t be denoted by $\mathbb{E}_t^*(\cdot)$, I have

$$U_t(x) = \mathbb{E}_t^* \left[\int_t^\infty e^{-r(s-t)} (1 - \phi) X_s ds \mid X_t = x \right] = \frac{(1 - \phi)x}{r - g}, \quad (12)$$

which is simply the Gordon growth formula given that after-tax profits grow at the rate g under the risk-neutral measure. From this I can see that

$$dU_t/U_t = gdt + \sqrt{Y_t}dW_t^{(1)*}, \quad (13)$$

so that the value of assets in place and productivity share the same dynamics under \mathbb{P}^* . Finally, one can show that

$$\frac{dU_t + (1 - \phi) X_t}{U_t} = rdt + \sqrt{Y_t}dW_t^{(1)*}, \quad (14)$$

which implies that the expected risk-neutral total return on the assets in place, including the dividend payment, is equal to the riskless rate as required by the existence of an equivalent martingale measure. The value of assets in place for young firms, denoted by U_t^a , is calculated similarly and is provided in Appendix B.

3.3 Capital Structure

Firms are financed with both debt and equity issues. I employ the exponential model of Leland (1994a) and Leland (1998) to describe the capital structure of the firm. This modeling device will allow me to analyze the full term structure of credit spreads, while at the same time will maintain tractability.¹⁵ The firm adopts a stationary debt structure with total principal P and total coupon C . The firm continuously rolls over debt at the fractional rate m . That is, at every point in time, debt with principal equal to mP matures and is replaced with new debt of equal coupon, principal, and seniority to maintain stationarity. Let p denote the principal on newly issued debt and c the coupon. Moreover, let $p(s, t)$ and $c(s, t)$ be the principal and coupon outstanding at time t for debt issued at date $s \leq t$. Since the firm retires the principal of all vintages at fractional rate m , the principal and

¹⁵An alternative term structure model is given by Leland and Toft (1996). However, such a model would require me to solve partial differential equations for the debt and equity values. A significant advantage of the exponential model is that, as will be shown, approximate equity and debt values can be derived by solving only ordinary differential equations.

coupon of each vintage declines exponentially with time:

$$p(s, t) = e^{-m(t-s)}p \quad (15)$$

$$c(s, t) = e^{-m(t-s)}c. \quad (16)$$

Integrating over all vintages at time t gives the total principal and coupon of the firm:

$$P = \int_{-\infty}^t p(s, t) ds = p \int_{-\infty}^t e^{-m(t-s)} ds = p/m \quad (17)$$

$$C = \int_{-\infty}^t c(s, t) ds = c \int_{-\infty}^t e^{-m(t-s)} ds = c/m. \quad (18)$$

Thus, the principal and coupon of newly issued debt is always equal to a fraction m of the total principal and coupon in the capital structure of the firm.

Now, note that for a time s vintage, the fraction of currently outstanding debt principal retired at time $t > s$ is given by $me^{-m(t-s)}$. This implies that the average maturity of debt M is given by

$$M = \int_s^{\infty} t (me^{-m(t-s)}) dt = 1/m. \quad (19)$$

In other words, the inverse of the fractional rollover rate is a measure of the average maturity of the firm's debt. Note that the limiting case $m = 0$ corresponds to the Leland (1994b) model of consol debt.

Of course, the price of newly issued debt will reflect current state variables in the market. This exposes the equityholders of the firm to *rollover risk*. The firm will face either a cash windfall or shortfall depending on whether debt is currently priced above or below par. Additional equity must be issued if there is a cash shortfall. Since interest payments provide a tax shield, the total flow payment to equityholders of mature firms is therefore given by

$$(1 - \phi) [X_t - C] + \tilde{d}(X_t, Y_t) - p, \quad (20)$$

where $\tilde{d}(X_t, Y_t)$ is the value of newly issued debt. The first term reflects the flow operating profits generated by the the firm as well as the corporate tax rate and the tax shield, while the final two terms reflect the rollover risk faced by the equityholders.

Equityholders optimally decide when to default on their debt obligations and may issue additional equity to finance coupon payments if current cash flow is insufficient to meet their obligations. In the event of default, equityholders receive nothing and the value of their claims is zero. Debtholders receive the assets of the firm according to their vintage; however,

a fraction ζ of the total value of assets in place is lost due to bankruptcy costs.¹⁶

Finally, I will assume for simplicity that investment spending is financed entirely with equity. I now turn to the valuation of the firm's contingent claims.

3.4 Equity Valuation

Contingent claims are priced according to the risk-neutral measure. Given equation (20), the value of equity for a mature firm is given by

$$E(x, y) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_t^* \left[\int_t^\tau e^{-r(s-t)} \left\{ (1 - \phi)(X_s - C) + \tilde{d}(X_s, Y_s) - p \right\} ds \right], \quad (21)$$

where \mathcal{T} is the set of $\{\mathcal{F}_t\}$ -stopping times. I denote the optimal stopping time, i.e. time of default, by τ^B . Intuitively, the equity value of a mature firm is simply the risk-neutral expected discounted present value of all future dividends accruing to equityholders given that the point of default is chosen optimally.

Note that simple Monte-Carlo computation of this value is not feasible due to the unknown optimal stopping time. Additionally, one cannot implement the recursive least-squares Monte-Carlo (LSM) procedure of Longstaff and Schwartz (2001) since there is no terminal date.¹⁷ Instead, I will seek a partial differential equations characterization of the equity valuation. Specifically, I show that the equity value of the mature firm (21) can be given as the solution to a Dirichlet/Poisson free boundary problem.

¹⁶Bankruptcy costs include the direct costs of the legal proceedings, but also indirect costs such as losses of specialized knowledge and experience, reductions in trade credit, and customer dissatisfaction. In practice, the indirect costs of bankruptcy may be of an order of magnitude larger than the direct costs.

¹⁷Essentially, the methodology uses recursive Monte-Carlo simulation backwards in time along with least-squares regression to compute conditional expectations and compares the expected continuation value against the intrinsic value at each discrete time step to approximate the exercise boundary.

Theorem 1 *The equity value of a mature firm $E(x, y)$ is the solution to*

$$(1 - \phi)(x - C) + \tilde{d}(x, y) - p + \mathcal{L}_{X,Y}E = rE \quad \text{for } x > x_B(y) \quad (22a)$$

$$(1 - \phi)(x - C) + \tilde{d}(x, y) - p + \mathcal{L}_{X,Y}E \leq rE \quad \text{for } x, y > 0 \quad (22b)$$

$$E(x, y) = 0 \quad \text{for } x < x_B(y) \quad (22c)$$

$$E(x, y) \geq 0 \quad \text{for } x, y > 0 \quad (22d)$$

$$E(x_B(y), y) = 0 \quad (22e)$$

$$\lim_{x \rightarrow \infty} E(x, y) = U(x) - \frac{C + mP}{r + m} + \frac{\phi C}{r} \quad (22f)$$

$$\frac{\partial E}{\partial x} \Big|_{x=x_B(y)} = 0 \quad (22g)$$

$$\frac{\partial E}{\partial y} \Big|_{x=x_B(y)} = 0, \quad (22h)$$

where $\mathcal{L}_{X,Y}$ is the linear differential operator given by:

$$\mathcal{L}_{X,Y} = gx \frac{\partial}{\partial x} + \frac{1}{2} y x^2 \frac{\partial^2}{\partial x^2} + (\kappa_Y (\theta_Y - y) - \Gamma(y) \nu_Y \sqrt{y}) \frac{\partial}{\partial y} + \frac{1}{2} \nu_Y^2 y \frac{\partial^2}{\partial y^2} + \rho_Y \nu_Y y x \frac{\partial^2}{\partial x \partial y}, \quad (23)$$

and $x_B(y)$ is a free boundary to be determined.

Proof. See Appendix A. ■

Here $x_B(y)$ is the point at which equityholders optimally default on their debt. That is, the optimal stopping time $\tau^B = \min \{t : (X_t, Y_t) = (x_B(Y_t), Y_t)\}$, which is clearly adapted to the filtration \mathcal{F}_t and is thus well-defined. Crucially, note that the boundary at which default occurs can depend on the current volatility.

The other conditions in the above system are intuitive. By the multidimensional Ito's formula, I can write equation (22a) as

$$\frac{(1 - \phi)(x - C) + \tilde{d}(x, y) - p}{E} + \frac{E^* [dE]}{E} = r, \quad (24)$$

which simply says that the expected return on equity, given by the dividend yield plus the expected capital gain, must be equal to the risk-free rate under the risk-neutral measure if the firm is not in default. Equation (22b) implies that in the stop region the expected return from continuing operations must be less than or equal to the riskless rate. Otherwise, stopping would not be optimal. Equation (22e) is the usual value-matching equation which states that at the point of default the value of equity must be equal to zero. If the value, for instance, were positive then the continuation value of equity would be larger than the value of equity under default, and thus default should be optimally postponed. As $X_t \rightarrow \infty$, the

probability of default in finite time approaches zero, and therefore the equity value is simply given by the present value of the assets in place U_t , minus the present value of future debt obligations $(C + mP)/(r + m)$, plus the present value of the tax shield $\phi C/r$. This logic yields the limiting condition (22f).

Finally, equations (22g) and (22h) provide the smooth-pasting conditions for the problem. These are standard for optimal stopping problems in which the stochastic process follows a regular diffusion. To see why this should be the case, note that if the equity value were not smooth across the free boundary in both variables, there would be a kink in the valuation at the point of default. The nature of diffusion is such that this kink would imply defaulting slightly earlier or later would be optimal. For instance, suppose that the kink were convex. Loosely speaking, within a short period of time the diffusion would be equally likely to be on either side of the kink and since the slope into the continuation region is positive, there is positive expected value to waiting. The payoff from stopping immediately is zero and thus it would be optimal to postpone default.

Similarly, letting $E^a(x, y)$ denote the equity value of young firms, it solves the optimal stopping problem:

$$E^a(x, y) = \sup_{\tau', \tau'' \in \mathcal{T}} \mathbb{E}_t^* \left[\begin{aligned} & \left\{ \int_t^{\tau' \wedge \tau''} e^{-r(s-t)} (1 - \phi) (K_a X_s - C) \tilde{d}(x, y) - p \right\} ds \\ & + 1[\tau' < \tau''] (E(X_{\tau'}, Y_{\tau'}) - I) \end{aligned} \right], \quad (25)$$

where $\tau' \wedge \tau'' = \min(\tau', \tau'')$. Here τ' is a stopping time which indicates exercising the growth option and τ'' is a stopping time which indicates default. If the firm exercises its growth option prior to default, then the value of the equity becomes $E(X_{\tau'}, Y_{\tau'}) - I$, which gives the second term in the expression above. The equity value of a young firm can be described as the solution to an appropriate free boundary problem just as with mature firms. Basically, the limiting condition (22f) is removed and there are value-matching and smooth-pasting conditions at the boundary for exercise of the growth option. It is characterized explicitly in Appendix B.

3.5 Debt Valuation

As is evident from the equations above, solving the equityholders' problem requires the value of newly issued debt. This is due to the assumptions on the capital structure of the firm and the rollover risk inherent in the flow dividend to the equityholders. Let $d(t)$ be the value at date t of debt issued at time 0 for a mature firm.¹⁸ Then according to risk-neutral pricing,

¹⁸Note that we could have considered any vintage of debt. Analyzing the date 0 vintage is without loss of generality.

this valuation is given by

$$d(t) = \mathbb{E}_t^* \left[\begin{array}{l} \int_t^{\tau^B} \{e^{-r(s-t)} e^{-ms} (c + mp)\} ds \\ + e^{-r(\tau^B-t)} \left(e^{-m\tau^B} p/P \right) (1 - \xi) U_{\tau^B} \end{array} \right]. \quad (26)$$

Note that the payments to the debtholders are declining exponentially and are therefore time dependent. Furthermore, the debtholders' claim on the assets of the firm will also depend on the point in time at which bankruptcy occurs. In general, this time-dependency would indicate that a partial differential equation involving a time derivative would need to be solved to find $d(t)$. Instead multiply both sides of the equation above by e^{mt} . Noting that $p/P = m$ yields

$$e^{mt} d(t) = \mathbb{E}_t^* \left[\begin{array}{l} \int_t^{\tau^B} \{e^{-(r+m)(s-t)} (c + mp)\} ds \\ + e^{-(r+m)(\tau^B-t)} m (1 - \xi) U_{\tau^B} \end{array} \right]. \quad (27)$$

Applying Feynman-Kac, I then have the following result:

Theorem 2 *The value of the date 0 debt vintage at time t is given by $d(t) = e^{-mt} \tilde{d}(X_t, Y_t)$ where $\tilde{d}(X_t, Y_t)$ is the value of the newly issued debt and satisfies*

$$c + mp + \mathcal{L}_{X,Y} \tilde{d} = (r + m) \tilde{d} \quad \text{for } x > x_B(y) \quad (28a)$$

$$\tilde{d}(x_B(y), y) = m(1 - \xi) U(x_B(y)) \quad (28b)$$

$$\lim_{x \rightarrow \infty} \tilde{d}(x, y) = \frac{c + mp}{r + m}. \quad (28c)$$

The value of newly issued debt is therefore the solution to a partial differential equation that does not involve a time derivative. Rather, the exponential decline in the flow payments to the debtholders is reflected in a higher implied interest rate $r + m$. This is the key advantage of employing the Leland (1994a) model of capital structure. To find the total value of debt at any point in time, simply integrate over the value of all outstanding debt vintages:

$$D(t) = \int_{-\infty}^t e^{-m(t-s)} \tilde{d}(X_t, Y_t) ds = \frac{\tilde{d}(X_t, Y_t)}{m}. \quad (29)$$

Thus, the total value of the firm's aggregate debt is a function only of the current productivity and volatility and does not depend on time, which is consistent with the stationarity of the overall debt structure.

The debt of young firms is characterized similarly and is discussed in Appendix B.

4 Methodology

The analysis up until this point has provided time-stationary partial differential equation (PDE) characterizations of the debt and equity values. In particular, the equity value is the solution to a two-dimensional free boundary problem. It is the higher dimensional counterpart to the usual sS problem well known in economics and similar to the obstacle problem and the Stefan problem in physics.¹⁹ While it is often possible to obtain closed form solutions to sS problems with only one state variable, this is usually not feasible when the free boundary is of a higher dimension. I could attempt to directly solve the problem numerically using an appropriate discretization scheme and projected successive over-relaxation (PSOR) methods. Such techniques have been successfully applied to the solution of American option problems and involve a variant of the Gauss-Seidel method for the solution of systems of linear equations in which the relevant inequality constraint is enforced at each iteration. However, these numerical procedures can be computationally burdensome and, more importantly, would not allow me to analytically grasp the intuition and economics underlying the results.

Instead, I utilize a semi-analytic methodology based on asymptotic expansions which allows me to generate accurate approximations to the values of the contingent claims. The method relies on making assumptions on the rate of mean-reversion of the stochastic volatility process and then using either regular or singular perturbations around the appropriate parameter in the partial differential equations to recursively solve for the formal power series expansions of the debt and equity values, as well as the default boundary. I will confine myself to first-order expansions, although higher-order terms can be calculated in a straightforward manner.

The basic principles of this method have been developed recently in the mathematical finance literature for the pricing of options under stochastic volatility.²⁰ Lee (2001) develops an approach for the pricing of European options under slow-variation asymptotics of the stochastic volatility process.²¹ In a sequence of papers, authors Fouque, Papanicolaou, Sircar, Sølna have considered fast-variation asymptotics in a variety of option pricing settings (European, barrier, Asian, etc.) and have developed perturbation procedures for stochastic volatility processes with both slow and fast variation components.²² My baseline setting will

¹⁹The classic Stefan problem studies phase changes in homogenous mediums and the resulting temperature distributions. The obstacle problem considers elastic membranes and solves for their equilibrium position given a fixed boundary and the constraint that they lie above a certain geometric obstacle.

²⁰More broadly, these methods are based on the use of perturbation theory to solve differential equations, which has a rich history in both mathematics and physics. Perturbation methods, for instance, are widely used in the modern study of quantum mechanical models.

²¹Precisely, Lee (2001) considers the cases of both slow-variation and small-variation stochastic volatility.

²²Contributions by these authors include Fouque et al. (2003a, 2003c, 2004a, 2004b, 2006, 2011). In

use an adaptation of the method proposed by Lee (2001) for small variation asymptotics to a free boundary problem. I will be able further extend this approach in a novel manner to allow for rare disasters in the firm productivity process. Finally, I will adapt the multiscale methods of Fouque, Papanicolaou, Sircar, Sølna, which are significantly more complicated, to consider a setting with both slowly-varying and fast-varying components of volatility.

To the best of my knowledge, the use of this methodology is novel in the economics literature outside of mathematical finance, and as I will argue below, is particularly well-suited for use by economists in a variety of economic settings. I now turn to the discussion of the existence of volatility time scales, which is important for justifying the asymptotic expansions considered, and how they are to be modeled.

4.1 Time Scales in Volatility Modeling

A substantial number of empirical studies have shown that market volatility appears to evolve on multiple time scales and to exhibit forms of long-run dependencies, often termed long memory. This has led to a number of important econometric developments that attempt to move beyond the standard one-component ARCH/GARCH/EGARCH specifications. Andersen and Bollerslev (1997) and Baillie et al. (1996) introduce the Fractionally Integrated ARCH (FIGARCH) model in which autocorrelations have hyperbolic decline rather than geometric decline. Beginning with Engle and Lee (1999), a body of papers has argued that two-dimensional volatility models with both a short-run component and a long-run component show significantly better performance in matching the data.²³ Building on this work further, Calvet and Fisher (2001, 2004) have developed the Markov-switching multifractal (MSM) stochastic volatility model, allowing for an arbitrary number of time scales on which volatility can evolve. Here, volatility is the product of a large but finite number of factors, each of which are first-order Markov and have identical marginal distributions but differ in their switching probabilities. The authors find that a specification with ten time scales fits the volatility of exchange rates well, with the highest frequency component on the order of a day and the lowest component on the order of 10 years.

I adopt this perspective in my modeling of asset variance. In my primary specification, though, I will only model low frequency movements. One reason for this is that high fre-

particular, Fouque et al. (2006) shows how defaultable bond prices can be calculated using asymptotic expansions in a Black-Cox first passage model with multiscale stochastic volatility. This paper differs by discussing the economic intuitions, including endogenous default and a stationary capital structure with rollover risk, analyzing the implications of stochastic volatility for both equity and debt, and examining the credit spread puzzle.

²³Other papers include Engle and Rosenberg (2000), Alizadeh, Brandt, and Diebold (2002), Bollerslev and Zhou (2002), Fouque et al. (2003), Chernov et al. (2003), and Adrian and Rosenberg (2008).

quency fluctuations are likely to be less important in the actual dynamics of asset volatility than market volatility, since changes in asset volatility should be driven largely by fundamentals rather than market sentiment and other transient factors. More importantly, though, it is exactly those persistent, low frequency movements in volatility which long-run investors should care about and which should be priced in equilibrium.

Rather than directly calibrating the parameter κ_Y to be small, I instead slightly modify the process for variance under the physical measure as

$$dY_t = \delta \kappa_Y (\theta_Y - Y_t) dt + \nu_Y \sqrt{\delta Y_t} dW_t^{(2)}, \quad (30)$$

where $\delta > 0$ is a small parameter. Note that this is again a CIR process exactly the same as in equation (2) except that the drift term is now multiplied by δ and the diffusion term is multiplied by $\sqrt{\delta}$. This parameter directly controls the rate of mean-reversion of the process for volatility. Since I am assuming it is small, I will say that variance is *slowly mean-reverting*. Specifically, δ scales the spectral gap of the process for Y_t , or the distance between the zero eigenvalue and the first negative eigenvalue. Using eigenfunction expansions, it is possible to show that it is exactly this spectral gap which determines the rate of mean-reversion for the process. However, denoting the invariant or long-run distribution of Y_t by Δ_Y , one can show that

$$\Delta_Y \sim \text{Gamma} \left(\frac{2\kappa_Y \theta_Y}{\nu_Y^2}, \frac{\nu_Y^2}{2\kappa_Y} \right),$$

which is in fact independent of δ . This indicates that in the long run, the level of variability in the volatility of asset productivity does not depend on the parameter δ , even though its square root multiplies the diffusion term in (30). It is in this sense that movements in volatility are indeed *slow*, but not necessarily *small*.²⁴ Appendix C provides further technical details on this modeling of volatility time scales.

In an extension of the model, I will construct a model in the spirit of Calvet and Fisher (2001, 2004) in which volatility is the product of both a low frequency component and a high frequency component.

²⁴To model the implications of small variation in asset volatility, I would specify:

$$dY_t = \delta \kappa (\theta - Y_t) dt + \delta \nu \sqrt{Y_t} dW_t^{(2)}.$$

Lee (2001) considers the implications of such a stochastic volatility model for the pricing of European options.

4.2 Asymptotic Approximation

Given this model of volatility dynamics, I can now write the partial differential equations in (22a) and (28a) as:

$$(1 - \phi)(x - C) + \tilde{d}_\delta(x, y) - p + \left(\mathcal{L}_r^y + \sqrt{\delta}\mathcal{M}_1^y + \delta\mathcal{M}_2^y\right) E_\delta = 0 \quad (31)$$

$$c + m + \left(\mathcal{L}_{r+m}^y + \sqrt{\delta}\mathcal{M}_1^y + \delta\mathcal{M}_2^y\right) \tilde{d}_\delta = 0, \quad (32)$$

where E_δ and \tilde{d}_δ are the values of equity and newly issued debt respectively given the choice of δ and the operators $\mathcal{L}_r^y, \mathcal{M}_1^y, \mathcal{M}_2^y$ are given by

$$\mathcal{L}_r^y = gx \frac{\partial}{\partial x} + \frac{1}{2}yx^2 \frac{\partial^2}{\partial x^2} - r(\cdot) \quad (33)$$

$$\mathcal{M}_1^y = \rho_Y \nu_Y yx \frac{\partial^2}{\partial x \partial y} - \Gamma(y) \nu_Y \sqrt{y} \frac{\partial}{\partial y} \quad (34)$$

$$\mathcal{M}_2^y = \kappa_Y (\theta_Y - y) \frac{\partial}{\partial y} + \frac{1}{2} \nu_Y^2 y \frac{\partial^2}{\partial y^2}. \quad (35)$$

Here, \mathcal{L}_r^y is the time-invariant Black-Scholes operator with volatility y and riskfree rate r , \mathcal{M}_2^y is the infinitesimal generator of the CIR process, and \mathcal{M}_1^y is an operator which accounts for correlation between the processes for asset productivity and asset volatility, as well as the Girsanov transformation between the physical and risk-neutral measures.²⁵

To solve for the contingent claims of mature firms by regular perturbation, I expand the equity value, the value of newly issued debt, and the free default boundary in powers of $\sqrt{\delta}$:

$$E_\delta(x, y) = E_0^y(x) + \sqrt{\delta}E_1^y(x) + \delta E_2^y(x) + \dots \quad (36)$$

$$\tilde{d}_\delta(x, y) = \tilde{d}_0^y(x) + \sqrt{\delta}\tilde{d}_1^y(x) + \delta\tilde{d}_2^y(x) + \dots \quad (37)$$

$$x_B(y) = x_{B,0}^y + \sqrt{\delta}x_{B,1}^y + \delta x_{B,2}^y + \dots \quad (38)$$

I then plug these asymptotic expansions into equations (31) and (32), as well as equations (22e)-(22h) and (28b)-(28c). Taylor expansions centered around $x_{B,0}(y)$ are used to appropriately expand value-matching and smooth-pasting conditions. The system is solved by collecting terms in the powers of δ and using the method of undetermined coefficients, where here I understand the coefficients to be functions.

²⁵Note that the dynamics of volatility under the risk-neutral measure are given by:

$$dY_t = \left(\delta\kappa(\theta - Y_t) - \Gamma(Y_t)\nu\sqrt{\delta Y_t}\right) dt + \nu\sqrt{\delta Y_t}dW_t^{(2)*}$$

for a given choice of parameter δ .

4.2.1 Principal Order Terms

I begin by computing the principal order terms in the asymptotic expansions. Collecting the order one terms for newly issued debt yields the problem:

$$c + mp + \mathcal{L}_{r+m}^y \tilde{d}_0^y = 0 \quad \text{for } x > x_{B,0}^y \quad (39a)$$

$$\tilde{d}_0^y(x_{B,0}^y) = m(1 - \xi)U(x_{B,0}^y) \quad (39b)$$

$$\lim_{x \rightarrow \infty} \tilde{d}_0^y(x) = \frac{c + mp}{r + m}. \quad (39c)$$

Under closer inspection, I see that this principal order term is simply the value of newly issued debt under constant return variance y in the firm productivity process, given the fixed boundary $x_{B,0}^y$. As such, it can be solved using standard single variable ODE techniques. I obtain

$$\tilde{d}_0^y(x) = \frac{c + mp}{r + m} + \left[m(1 - \xi)U(x_{B,0}^y) - \frac{c + mp}{r + m} \right] \left\{ \frac{x}{x_{B,0}^y} \right\}^{\gamma_1}, \quad (40)$$

where γ_1 is the *negative* root of the following quadratic equation:

$$g\gamma_1 + \frac{1}{2}y\gamma_1(\gamma_1 - 1) - (r + m) = 0. \quad (41)$$

This expression is consistent with that derived in Leland (1994a). The first term in the valuation reflects the present value of future coupon and principal payments assuming no default. The latter term takes into account the risk of default. The term $(x/x_{B,0}^y)^{-\gamma_1}$ is akin to a probability of default although this is not exactly correct. If default occurs, debtholders receive a fraction of the value of the assets of the firm but lose any future coupon and principal payments remaining. More precisely, the second component of the valuation is a perpetual digital option which pays off the term in brackets the first time the process X_t crosses the boundary $x_{B,0}$.

Intuitively, the principal order term reflects the value of newly issued debt in the limiting case $\delta = 0$. Of course, this is simply a model in which volatility is fixed at its current value. This intuition then carries over to computing the principal order terms for both the equity value and the default boundary. That is, I simply need to compute the value of equity and the default boundary in the case where volatility is fixed in time at its current level \sqrt{y} .

These are given by

$$\begin{aligned}
E_0^y(x) &= U(x) - \frac{C + mP}{r + m} + \frac{\phi C}{r} \\
&\quad + \left[\frac{C + mP}{r + m} - (1 - \xi) U(x_{B,0}^y) \right] \left\{ \frac{x}{x_{B,0}^y} \right\}^{\gamma_1} \\
&\quad - \left[\frac{\phi C}{r} + \xi U(x_{B,0}^y) \right] \left\{ \frac{x}{x_{B,0}^y} \right\}^{\gamma_2}
\end{aligned} \tag{42}$$

$$x_{B,0}^y = -\frac{(C + mP)\gamma_1 / (r + m) - \phi C \gamma_2 / r}{1 - (1 - \xi)\gamma_1 - \xi\gamma_2} \frac{r - g}{1 - \phi}, \tag{43}$$

where γ_2 is the negative root to the quadratic equation:

$$g\gamma_2 + \frac{1}{2}y\gamma_2(\gamma_2 - 1) - r = 0. \tag{44}$$

These expressions are once again consistent with Leland (1994a) and are derived in Appendix A. The equity value incorporates the value of the assets in place, the present value of future debt payments, the present value tax shield, and two perpetual digital options which account for the equityholders' default option.

4.2.2 First-Order Correction Terms

Continuing with the recursive approach, collecting terms of order $\sqrt{\delta}$ now yields the problems to solve for the first-order correction terms. For the value of newly issued debt, this is given by

$$\begin{aligned}
\mathcal{L}_{r+m}^y \sqrt{\delta} \tilde{d}_1^y &= -\sqrt{\delta} \mathcal{M}_1^y \tilde{d}_0^y \\
&= \left(A^\delta y \frac{\partial \tilde{d}_0^y}{\partial y} - B^\delta y x \frac{\partial^2 \tilde{d}_0^y}{\partial x \partial y} \right) \quad \text{for } x > x_{B,0}^y
\end{aligned} \tag{45a}$$

$$\sqrt{\delta} \tilde{d}_1^y(x_{B,0}^y) = \sqrt{\delta} x_{B,1}^y \left[\frac{m(1 - \xi)(1 - \phi)}{r - g} - \frac{\partial \tilde{d}_0^y}{\partial x}(x_{D,0}^y) \right] \tag{45b}$$

$$\lim_{x \rightarrow \infty} \sqrt{\delta} \tilde{d}_1^y(x) = 0, \tag{45c}$$

where the constants A^δ, B^δ are defined by

$$A^\delta = \sqrt{\delta} \nu_Y \Gamma_0 \tag{46}$$

$$B^\delta = \sqrt{\delta} \nu_Y \rho_Y. \tag{47}$$

Note that this is a one-dimensional inhomogeneous second-order boundary value problem in which y operates as a parameter. The source term is a function of comparative statics or Greeks of the debt principal order term. In particular, it is a function of the vega and the vanna of the debt value when volatility is held constant at the level \sqrt{y} .²⁶ Note also that the default boundary correction term $x_{B,1}^y$ appears in the boundary condition of the problem above, given by equation (45b). The problem for the first-order equity correction term takes a similar form and is given by

$$\begin{aligned}\mathcal{L}_r^y \sqrt{\delta} E_1^y &= -\sqrt{\delta} \left(\mathcal{M}_1^y E_0^y + \tilde{d}_1^y \right) \\ &= \left(A^\delta y \frac{\partial E_0^y}{\partial y} - B^\delta y x \frac{\partial^2 E_0^y}{\partial x \partial y} - \sqrt{\delta} \tilde{d}_1^y \right) \quad \text{for } x > x_{B,0}^y\end{aligned}\quad (48a)$$

$$\sqrt{\delta} E_1^y (x_{B,0}^y) = 0 \quad (48b)$$

$$\lim_{x \rightarrow \infty} \sqrt{\delta} E_1^y (x) = 0. \quad (48c)$$

Once again, this is a one-dimensional second-order boundary value problem with a source term. Different from the problem for the debt value correction above, the Black-Scholes operator here has for its riskfree rate r instead of $r + m$ as in equation (48a). Furthermore, the source term for the problem involves the debt correction term, due to the rollover risk in the original problem, in addition to the vega and vanna of the equity value in the constant volatility case.

Finally, the Taylor expansion of the equity value smooth-pasting condition in x , equation (22g), gives the correction to the default boundary as:

$$\sqrt{\delta} x_{B,1}^y \frac{\partial^2 E_0^y}{\partial x^2} (x_{B,0}^y) = -\sqrt{\delta} \frac{\partial E_1^y}{\partial x} (x_{B,0}^y). \quad (49)$$

Equations (45a)-(45c), (48a)-(48c), and (49) form a system of equations which can be jointly solved to determine the first-order corrections for the debt value, the equity value, and the default boundary. While I am unable to give closed-form solutions to these corrections, it is now very straightforward to solve for them numerically since the differential equations are one-dimensional and the problems have fixed rather than free boundaries. The MATLAB function *bvp4c*, which implements a finite elements scheme, is used to solve the boundary value problems while searching over $\sqrt{\delta} x_{B,1}^y$ to satisfy equation (49). Note that I have not used condition (22h) at all; however, it is easy to show that the approximation $E_0(x, y) + \tilde{E}_1^\delta(x, y)$ is smooth across the boundary $x_{B,0}(y)$ as required.

To summarize, by making an assumption on the rate of mean-reversion of the process for

²⁶Here, I define vega to be the comparative static with respect to variance y .

volatility, this method reduces the solution of two-dimensional partial differential equation problems, in which one has a free boundary, to a recursive sequence of one-dimensional problems. The principal order terms simply reflect the contingent claims valuations and default boundary in the limiting case where volatility is fixed at its current level. The first-order correction terms are solved as a system of equations in which the contingent claims corrections are the solutions to one-dimensional, fixed boundary problems and the source terms are functions of the comparative statics or Greeks of the principal order terms. This method is tractable and will provide intuitive expressions demonstrating the effects of stochastic volatility on debt and equity valuation.

Moreover, a significant advantage of the methodology is that it reduces the number of parameters which need to be calibrated to generate quantitative results. Given the parametric assumption on the market price of variance risk, the only further calibration required beyond the baseline constant volatility model is that of two constants: $A^\delta = \sqrt{\delta}\nu_Y\Gamma_0$ and $B^\delta = \sqrt{\delta}\nu_Y\rho_Y$. Essentially, the variance risk premium needs to be calibrated as well as a constant which has the same sign as the correlation between volatility shocks and productivity shocks. The rate of mean-reversion κ_Y and the long-run mean of variance θ_Y do not appear at all.²⁷ This is especially useful in a structural model of credit incorporating stochastic volatility, as there are not good empirical estimates for the structural parameters of the asset volatility process.

Finally, I believe that this methodology is particularly well suited for use in a variety of other economic settings. Indeed, the approach offers a general framework for introducing additional state variables into either deterministic or stochastic dynamic models, including but certainly not limited to volatility. First, one constructs a baseline framework in which this additional state variable is a fixed parameter and then calculates the appropriate comparative statics. If one is able to make limiting assumptions on the dynamics of this additional state variable, such as slow, small, or fast, then approximate solutions to the full model can be derived by calculating correction terms as the solutions to differential equations whose source terms are functions of the comparative statics. For example, in the context of credit modeling, one could use this approach to develop a model which includes both stochastic volatility *and* stochastic interest rates.

4.3 Cumulative Default Probabilities

Given that I would like to study the credit spread puzzle, it is crucial that I am able to calculate cumulative default probabilities as well as valuations. As Huang and Huang

²⁷This is because the operator \mathcal{M}_2 does not appear in the derivation of either the principal order term or the first-order correction term.

(2003) point out, it is this computation in particular which has been one of the primary stumbling blocks in the construction of structural credit models which include stochastic volatility. My methodology, however, provides an effective means of overcoming this hurdle. The key, just as with the contingent claims valuations, is to find a suitable partial differential equations characterization of the probability and then utilize a perturbation to simplify the solution of the problem. Letting $u(l, x, y)$ denote the cumulative *survival* probability within l years, by the backwards Kolmogorov equation:

$$\left(-\frac{\partial}{\partial l} + \mathcal{L}_0 + \sqrt{\delta}\nu_Y\rho yx\frac{\partial^2}{\partial x\partial y} + \delta\mathcal{M}_2^y\right)u = 0 \quad (50a)$$

$$u(l, x_B(y), y) = 0 \quad (50b)$$

$$\lim_{x \rightarrow \infty} u(l, x, y) = 1 \quad (50c)$$

$$u(0, x, y) = 1, \quad (50d)$$

where $x_B(y)$ is the optimal default boundary from the equityholders' problem.²⁸ Importantly, note that the cumulative survival probabilities are calculated with respect to the dynamics of X_t and Y_t under the physical measure rather than the risk-neutral measure. Now expand this survival probability in powers of $\sqrt{\delta}$:

$$u(l, x, y) = u_0^y(l, x) + \sqrt{\delta}u_1^y(l, x) + \delta u_2^y(l, x) + \dots, \quad (51)$$

substitute into equations (50a)-(50d), and expand the boundary conditions using Taylor expansions. Once again, the principal order term reflects the cumulative survival probability in the limiting case where volatility is fixed at the current level. I do not reproduce the expression here, but it can be looked up in any standard textbook treatment on the hitting times of geometric Brownian motion.

The correction term is an inhomogeneous partial differential equation with y as a parameter:

$$\left(-\frac{\partial}{\partial l} + \mathcal{L}_0\right)\sqrt{\delta}u_1^y = -B^\delta yx\frac{\partial^2 u_0^y}{\partial x\partial y} \quad (52a)$$

$$\sqrt{\delta}u_1^y(l, x_{B,0}^y) = -\sqrt{\delta}x_{B,1}^y\frac{\partial u_0^y}{\partial x}(l, x_{B,0}^y) \quad (52b)$$

$$\lim_{x \rightarrow \infty} \sqrt{\delta}u_1^y(l, x, y) = 0 \quad (52c)$$

$$\sqrt{\delta}u_1^y(0, x, y) = 0. \quad (52d)$$

²⁸Recall that \mathcal{L}_0 is the Black-Scholes operator with a riskfree rate set equal to zero. Cumulative default probabilities are simply given by $1 - u(l, x, y)$.

As expected, the source term is a function of a comparative statics of the survival probability in the constant volatility case; however, contrary to the expressions above, the vega of the principal order term and the variance risk premium do not appear in equation (52a). The correction term can be calculated in MATLAB using the function *pdepe*, which implements a finite-difference scheme.

4.4 Debt Valuation under Exogenous Default

Finally, I will want to compare my results to a model in which the default trigger is specified exogenously, rather than determined endogenously. Let this exogenous boundary be given by \bar{x}_B . A perturbation approach can be used in a similar fashion as above to determine the approximate value of debt in the stochastic volatility model. As should be familiar by now, the principal order term reflects the constant volatility case and is simply given by equation (40) with $x_{B,0}^y$ replaced by \bar{x}_B . It turns out that in the case of $\rho_Y = 0$, a relatively simple explicit expression can be derived for the first-order correction term.

Theorem 3 *If the default boundary is set exogenously at \bar{x}_B and the correlation between productivity shocks and volatility shocks is equal to zero, then the first-order correction term in the asymptotic expansion of newly issued debt is given by*

$$\sqrt{\delta} \tilde{d}_1^y(x) = \frac{\ln(x/\bar{x}_B)}{2(g + \gamma_1 y - \frac{1}{2}y)} A^\delta y \left[\frac{\partial \tilde{d}_0^y}{\partial y} + \frac{1}{2(g + \gamma_1 y - \frac{1}{2}y)^2} y x^2 \frac{\partial^2 \tilde{d}_0^y}{\partial y^2} \right], \quad (53)$$

where γ_1 is the negative root of equation (41).

Proof. See Appendix A. ■

Thus, the first-order correction term involves both the vega and gamma of the value of newly issued debt in the constant volatility model. These are provided explicitly in Appendix A. In fact, an explicit expression can be computed for the case of $\rho_Y \neq 0$ as well, but it and its derivation are particularly cumbersome. Moreover, it is not needed in subsequent work and I therefore omit it. Note, however, that the method of derivation is very similar to that described in the proof of the expression above.

While the primary focus of this paper is on debt which allows for endogenous default by equityholders, this result is interesting and useful in its own right from an asset pricing perspective. There are forms of debt in which the exogenous trigger is in fact more appropriate. For instance, the debt may have certain covenants which enforce a zero net worth requirement as discussed in Leland (1994b).²⁹ Alternatively, default may occur once there is

²⁹Leland specifically discusses the example of a contractual arrangement in which the firm has access to a continuously renewable line of credit. Debt is rolled over at each instant at the fixed interest rate if and

insufficient cash flow to meet the debt servicing obligations and new equity cannot be raised to make up the shortfall, a scenario which likely most accurately describes municipal debt. The expression above, along with the principal order term, allow for a closed-form solution to the pricing of such debt in a setting of stochastic volatility.

5 Equity Valuation

I begin my analysis by examining the model's qualitative predictions for equity pricing. Applying Ito's formula, substituting in equation (22a), and taking expectations shows that the required return of a firm's equity at time t under the physical measure is given by

$$\frac{(1 - \phi)(x - C) + \tilde{d}(x, y) - p + \mathbb{E}_t [dE^\delta]}{E^\delta} - r = \pi_X(y) \beta_X + \sqrt{\delta y} \nu_Y \Gamma(y) \beta_Y, \quad (54)$$

where the market and variance betas β_X, β_Y are respectively defined in equations (9) and (10). The excess expected return of equity over the riskfree rate is the sum of a risk premium due to its exposure to productivity risk and a risk premium due to its exposure to volatility risk. The exposures are priced according to the asset risk premium and variance risk premium, respectively. The first term is standard, while the second is novel. Note that the qualitative nature of expected returns will largely be driven by the comparative statics of the principal order terms. The model potentially resolves a number of empirical puzzles which have been documented in cross-sectional equity pricing and generates testable predictions.

5.1 Financial Distress and Momentum

Dichev (1998) and Campbell et al. (2008) find that the equity of financially distressed firms, i.e. those close to default, have lower returns on average than healthy firms, despite having higher market betas. That firms close to default have higher market risk is not surprising; indeed, this is the usual leverage effect. Then, according to a model in which market fluctuations are the only source of priced risk, financially distressed firms should, on average, earn higher returns. Hence, there is a puzzle.

If stochastic volatility is an additional source priced source of risk, though, then a firm's exposure to volatility risk also affects expected returns. An increase in volatility has both a positive and negative impact on the equity value. First, an increase in the volatility raises the value of the equityholders' default option, thereby positively impacting the equity

only if the firm's asset value is sufficient to cover the loan's principal. If not, default occurs.

valuation. Intuitively, since the equityholder’s downside is capped by limited liability, while the upside potential is unlimited, an increase in uncertainty is beneficial.

However, the effect of higher volatility on rollover risk presents an offsetting negative channel. Debt is a concave function of productivity, reflecting the fact that the probability of default is most sensitive to movements in productivity near the default barrier. Since the flow dividend accruing to equityholders depends on the market value of newly issued debt, it too is concave in firm productivity. This concavity implies that an increase in volatility decreases the expected present value of future dividends.

The equity value of a financially distressed firm is largely comprised of its default option. For such firms, the first effect dominates, indicating that the value of equity rises with volatility. That is, a financially distressed firm’s equity hedges against volatility risk in the market, or $\partial (\log E^\delta) / \partial y > 0$. If the variance risk premium is negative, reflecting a view by investors that persistent increases in volatility represent deteriorations in the investment opportunity set, then investors will be willing to accept a lower return on such equities due to their hedge value. On the other hand, the default option is less important in the valuation of healthy firms. For these firms, it is the exacerbating effect of volatility on rollover risk which dominates and equity values fall when volatility increases. Given this exposure to volatility risk, the required return on a sufficiently healthy firm will be higher than in a model with constant volatility.

Panel A of Figure 1 illustrates these points. It shows the variance betas of mature firms with the same book leverage as a function of financial distress and the average maturity of their debt. As expected, averaging across maturities, the variance betas of financially distressed firms are higher than those of healthy firms. Thus, the model offers a mechanism potentially resolving the financial distress puzzle. Furthermore, the figure generates new, testable predictions. First, note that variance betas decline as the average debt maturity of the firm decreases. The shorter the maturity structure, the greater the fraction of total principal which has to be rolled over at any point in time. Consequently, as the maturity structure shortens, rollover risk becomes a greater component of the present value of future dividends accruing to equityholders, which in turn implies that firms with shorter maturity debt are more adversely impacted by increases in volatility. In terms of returns, by equation (54), this pattern of betas indicates that once market risk has been controlled for, firms with shorter maturity debt should have higher average returns than firms with longer maturity debt.

Panel B of Figure 1 also shows that for firms with shorter debt maturity, there is a clear hump-shaped relationship between financial distress and the equity vega.³⁰ That is,

³⁰There is a hump-shaped relationship at longer maturities as well, but it is harder to detect.

a decrease in the probability of default will increase the variance beta (towards zero) if the firm is sufficiently healthy. This is due to the fact that the effect of volatility on rollover risk is ameliorated as the health of the firm continues to increase. To understand this, note that the value of newly issued debt asymptotes to $(c + mp) / (r + m)$ as $x \rightarrow \infty$. In other words, debt becomes approximately flat as a function of productivity when the firm is very healthy. But in the region in which the debt valuation is flat, the firm faces little rollover risk and thus increases in volatility have little impact on the expected present value of future dividends. Equation (54) therefore indicates that variance risk premia should be maximized at intermediate probabilities of default. In fact, Garlappi and Yan (2011) find evidence supportive of this in their empirical work, documenting that average equity returns are hump-shaped as a function of the KMV distance to default.

Finally, consider forming a portfolio of financially distressed firms which is long recent winners and short recent losers. It is likely that, on average, the financial health of the winners has improved while that of the losers has deteriorated further. Then, by Figure 1, the variance betas of the losers should, on average, be higher than that of the winners. Thus, the winners should require higher variance risk premia than the losers, indicating that the portfolio should earn a positive CAPM alpha. This is consistent with the empirical evidence provided by Avramov et al. (2007) that the profits of momentum strategies are highly concentrated among a small subset of firms with low credit ratings. In fact, another empirical prediction of the model is that the momentum relation should reverse among healthy firms with a short debt maturity structure due to the hump-shaped nature of variance betas. Specifically, a portfolio which is long losers and short winners should earn a positive CAPM alpha.

5.2 Book-to-Market Effects

The implications of the model for the value premium puzzle are somewhat more subtle. The first channel I highlight is intimately related to the effects of financial distress discussed above. Consider two financially distressed, mature firms with the same book leverage and maturity structure. Suppose, though, that one firm has a higher productivity, and thus lower book-to-market ratio, than the other. This is similar to Gomes, Kogan, and Zhang (2003) in that differences in the book-to-market ratio can be driven by cross-sectional variation in productivity across firms. Note then, however, that a *growth premium* actually emerges as a consequence of the explanation above for the financial distress puzzle. On the other hand, if the two firms are both healthy and one has a higher productivity, then the figure

above shows that it is the firm with the higher book-to-market ratio that has the greater risk-adjusted return if the average maturity of debt is low. In this fashion, the model is capable of generating both a value premium and a growth premium depending on the level of financial distress.

Alternatively, two firms in the model with the same book leverage, productivity, and maturity structure can differ in their book-to-market ratios if they vary in the ratio of growth options to book value. As with financially distressed firms, much of the value of the young/growth firm in the model is comprised of its option to expand its installed capital. By the same logic as before, since volatility increases raise the value of this embedded option, growth firms should hedge against volatility risk in the market, more so than mature firms of similar default risk which have already exercised their growth options. If variance risk carries a negative price in the market, then investors will demand a higher variance risk premium to hold value stocks than growth stocks, all else equal.

6 Debt Valuation

I now move beyond qualitative considerations and turn to my primary quantitative analysis of the model's implications for debt pricing. I confine myself to analyzing the debt of mature firms. In order to study the debt valuations and default probabilities generated by the model, I must first articulate a suitable calibration of the key parameters.

6.1 Calibration

Given that my interest is in studying the credit spread puzzle, I will use a calibration procedure consistent with those previous studies which have demonstrated the puzzle, with a few notable exceptions. Most importantly, I will not calibrate the asset volatility to force the model to match historical default probabilities, but rather set the asset volatility according to *model-free* empirical estimates and then ask if the model is able to *jointly* generate reasonable credit spreads and default rates by credit rating.³¹ This is in marked contrast to Huang and Huang (2003) who require the models they analyze to match historical default frequencies. The problem with the Huang and Huang approach, as I will demonstrate, is that the implied asset volatilities can be unreasonably high given the empirical estimates, especially at short maturities. For computational reasons, I also do not set the bankruptcy cost ξ to match recovery ratios by rating category, as does Huang and Huang (2003), but instead set this parameter directly and verify ex-post that recovery rates are approximately equal to the

³¹Recall that asset volatility is the same as the volatility of productivity.

Table 2: Calibration of Model Parameters

r	ϕ	g	π_X
.08	.15	.02	.04

Note: This table provides the calibration values for those parameters which do not vary by credit rating: interest rate r , tax rate ϕ , risk-neutral productivity growth g , and asset risk premium π_X .

historical average of 51%.³² This approach is consistent with Leland (2004).

Table 2 summarizes those parameters which do not vary by credit rating. These include the interest rate, the corporate tax rate, the rate of productivity growth under the risk-neutral measure, and the asset risk premium. The riskfree rate is set equal to 8% as in Huang and Huang (2003) and Leland (2004), the historical average of Treasury rates between 1973-1998. The tax rate is equal to 15% as in Leland (2004), reflecting the corporate tax rate offset by the personal tax advantage of equity returns.³³ I set the rate of productivity growth equal to 2%. This indicates that the expected return on the value of assets in place is equal to 2% by equation (13), reflecting a payout rate of 6%. Finally, I set the asset risk premium equal to 4% such that the asset beta is equal to approximately 0.6 for all credit ratings. This is once again consistent with Leland (2004) and is slightly less than the asset risk premia in Huang and Huang (2003). Note that the asset risk premium does not affect the pricing of corporate debt and only impacts cumulative default probabilities.

Table 3 details those parameters of the model which do vary by credit rating, including leverage ratios, bankruptcy costs, average maturity of debt, and finally asset volatility. Target leverage ratios are from Standard & Poors (1999) and are consistent with both Huang and Huang (2003) and Leland (2004). As discussed previously, bankruptcy costs are set such that ex post recovery rates are approximately equal to 51%. Fractional costs of $\xi = 30\%$ works well for all credit ratings except for Caa, which has a slightly higher cost of $\xi = 35\%$. Average maturities and average asset volatilities are from Schaefer and Strebulaev (2008).

Their estimation of the asset volatilities warrants further discussion. Importantly, these estimates are model free, to the extent discussed below, and are therefore not dependent on assuming a particular structural model of credit, which would make them ineligible for use. Specifically, the authors estimate asset volatility of a firm j at time t , denoted as $\sigma_{A_j,t}^2$,

³²Specifically, Huang and Huang (2003) set bankruptcy costs by credit rating so that the models generate recovery rates of exactly 51.31%.

³³Specifically, as shown in Leland (2004), given a corporate tax rate of 35%, a personal tax on bond income of 40%, and a tax rate on stock returns of 20%, the effective tax advantage of debt can be calculated as $1-(1-.35)(1-.20)/(1-.40)=.133$.

Table 3: Calibration of Model Parameters by Credit Rating

Parameter	Credit Rating						
	Aaa	Aa	A	Baa	Ba	B	Caa
Target Leverage	13.1%	21.1%	32.0%	43.3%	53.5%	65.7%	80.0%
Avg. Asset Vol.	22%	22%	22%	22%	23%	28%	28%
Bankruptcy Costs	30%	30%	30%	30%	30%	30%	30%
Avg. Maturity (yr.)	10.16	9.45	10.13	9.14	7.11	7.39	2.48

Note: This table provides calibration values for the target leverage ratio, average asset volatility, fractional bankruptcy costs, and average maturity of debt by credit rating.

according to

$$\widehat{\sigma_{A_{j,t}}^2} = (1 - L_{j,t})^2 \sigma_{E_{j,t}}^2 + L_{j,t}^2 \sigma_{D_{j,t}}^2 + 2L_{j,t}(1 - L_{j,t}) \sigma_{ED_{j,t}}, \quad (55)$$

where $L_{j,t}$ is the market leverage of firm j at time t , $\sigma_{E_{j,t}}$ is the equity volatility at time t , $\sigma_{D_{j,t}}$ is the debt volatility at time t , and $\sigma_{ED_{j,t}}$ is the covariance between debt and equity returns at time t . The volatilities and covariances are calculated directly from the time series of equity and debt returns. While this estimation procedure is indeed ostensibly independent of a specific model, the authors do implicitly assume that movements in the asset value are the only source of fluctuations in the debt and equity values. This is, of course, not the case in a model with stochastic volatility, as movements in the volatility level also constitute a source of fluctuations in debt and equity values. *However*, given my assumption that movements in asset volatility are in fact *slow* in the manner described above, these estimates are accurate to principal order.³⁴

Finally, I initially set $\rho_Y = 0$; that is, productivity shocks and variance shocks are uncorrelated. The variance risk premium, i.e. the constant A^δ , is set to generate the target credit spread on 10-year Baa debt, but then the implied specification for the market price of variance risk is then held constant for all other credit ratings and at other maturities. I set the current productivity level $X_0 = 7.0588$ such that the current value of assets in place is equal to 100. For each credit rating, I set the total principal P equal to the target leverage ratio multiplied by 100 and then solve for the coupon such that newly issued debt, and the total current value of debt, is priced at par. This will imply that the credit spread can be calculated as the coupon rate C/P .

³⁴Given a regime of slowly mean-reverting asset volatility, the estimates of Schaefer and Strebulaev (2008) will slightly overestimate the true average asset volatilities.

6.2 Credit Spreads on Intermediate/Long Maturity Debt

I begin my quantitative analysis by examining the credit spreads of 10-year and 20-year maturity debt. The second column of Table 4 reports the target historical credit spreads for 10-year maturity debt by rating category. The targets for investment grade through speculative grade debt (Aaa-Baa) are from Duffee (1998) while the targets for speculative grade through junk debt (Ba-Caa) are from Caouette, Altman, and Narayanan (1998). The questions I seek to answer are twofold. Is it possible to match historical credit spreads on intermediate to long maturity debt with a reasonable variance risk premium and can a single specification for the market price of variance risk explain credit spreads across credit ratings and maturities?

First, consider the performance of the model for 10-year maturity debt in which volatility is constant. The results are reported in the third and fourth columns of Table 4. As is evident, I recover the credit puzzle in this baseline model, especially for investment grade and speculative debt. For all credit ratings between Aaa-Baa, the model is never able to account for more than 30% of the target credit spread, although the performance is increasing as the rating worsens. On the other hand, the table illustrates that there is significantly less of a credit puzzle for junk debt. The baseline model performs much better at these ratings, explaining approximately 61% of the historical B credit spread and 87% of the historical Caa credit spread. This observation highlights one of the key challenges that a model needs to overcome to fully resolve the credit spread puzzle. In other words, it is important not to create a credit spread puzzle in the other direction, whereby the new model is able to better explain the credit spreads of investment grade debt, but then *overpredicts* the credit spreads of junk debt.

This is essentially what happens in the model incorporating stochastic volatility but in which the default boundary is specified *exogenously*. As can be seen from columns 5 and 6 of the table, the model is now able to explain a substantially higher proportion of the target credit spread for investment grade and speculative debt than the baseline model. However, the model significantly overpredicts the credit spreads on junk debt. The credit spread on the B-rated debt is overpredicted by 30% and the credit spread on Caa-rated debt is substantially overpredicted by 70%. This is not a particularly compelling resolution of the credit spread puzzle.

Conversely, the model incorporating stochastic volatility in which the default boundary is determined *endogenously* performs much better. The model now only slightly overpredicts the spreads on junk debt. B-rated debt is overpredicted by only 8% and Caa-rated debt by only 15%, a substantial improvement. Not only that, but the endogenous default model outperforms the exogenous default model for investment grade debt as well, especially Aaa

Table 4: Credit Spreads on 10-Year Maturity Debt (bps)

	Target	Baseline		Exog. Default		Endog. Default	
		Model	% Explained	Model	% Explained	Model	% Explained
Aaa	47	2	4.3%	14	29.8%	19	40.4%
Aa	69	6	8.7%	37	53.6%	46	66.7%
A	96	19	19.8%	84	87.5%	90	93.8%
Baa	150	43	28.7%	150	100.0%	150	100.0%
Ba	310	93	30.0%	260	83.87%	242	75.6%
B	470	286	60.9%	607	129.2%	508	108.1%
Caa	765	663	86.7%	1307	170.9%	879	114.9%

Note: This table provides target and model-generated 10-year credit spreads by ratings category. Calibration parameters are provided in Tables 2 and 3. There is zero correlation between productivity shocks and variance shocks. The variance risk premium is set to match the 10-year historical Baa credit spread. Historical target credit spreads for Aaa-Baa debt are from Duffee (1998) while those for lower ratings categories are from Caouette, Altman, and Narayanan (1998). The baseline model holds volatility constant. The exogenous default model incorporates stochastic volatility, but sets the default barrier to be equal to that of the baseline model. The endogenous default model is the full stochastic volatility model.

an Aa debt. While the exogenous default model accounts for approximately 30% and 54% of the target credit spreads for Aaa and Aa debt respectively, the endogenous default model can explain 40% and 67%. The only rating category that the exogenous default model wins in is Ba-rated debt. It explains 84% of the target credit spread relative to 76% for the endogenous default model, a slight improvement.

The calibration yields a parameter of $A^\delta = -.2264$, which in turn implies a variance risk premium of -1.1% for an asset volatility of 22% and a premium of -1.77% for an asset volatility of 28%. This result for A^δ is significantly lower than most estimates reported in the empirical literature using market returns and market volatility. This is to be expected and, indeed, it would be worrisome if the calibrated parameter were equal to or higher than empirical estimates. The reasons are twofold. First, since I am operating under the assumption of slowly-moving asset volatility, the volatility of asset variance should be significantly lower than the volatility of market variance. Second, innovations in the asset variance of an individual firm are likely only partially correlated with innovations in aggregate market variance, indicating that only a fraction of the volatility in asset variance can be accounted for by exposure to market variance risk. In other words, the Wiener process $W_t^{(2)}$ is only partially correlated with the process driving movements in market variance, which I denote by $W_t^{(m)}$. Thus, the price of $W_t^{(2)}$ risk should be lower in magnitude than the price of $W_t^{(m)}$ risk. This too will lead to a lower A^δ . Both of these explanations are consistent with

Table 5: Credit Spreads on 20-Year Maturity Debt (bps)

	Target	Baseline		Exog. Default		Endog. Default	
		Model	% Explained	Model	% Explained	Model	% Explained
Aaa	59	3	5.1%	28	47.5%	35	59.3%
Aa	87	8	9.2%	63	72.4%	70	80.4%
A	117	22	18.8%	121	103.4%	131	112.0%
Baa	198	46	23.2%	202	102.0%	194	98.0%
Ba	N/A	91	N/A	326	N/A	282	N/A
B	N/A	258	N/A	692	N/A	513	N/A
Caa	N/A	535	N/A	1256	N/A	716	N/A

Note: This table provides target and model-generated 20-year credit spreads by ratings category. Calibration parameters are provided in Tables 2 and 3. There is zero correlation between productivity shocks and variance shocks. The variance risk premium is set to match the 10-year historical Baa credit spread. Historical target credit spreads for Aaa-Baa debt are from Duffee (1998) while those for lower ratings categories are from Caouette, Altman, and Narayanan (1998). The baseline model holds volatility constant. The exogenous default model incorporates stochastic volatility, but sets the default barrier to be equal to that of the baseline model. The endogenous default model is the full stochastic volatility model.

the results of Carr and Wu (2009) who find that the expected returns on variance swaps of individual equities is well-explained by the stocks' volatility betas.

Using this value of A^δ , I next look at the credit spreads of 20-year debt in Table 5. While I unfortunately do not have target credit spreads for junk debt at this maturity, it is clear that there once again exists a significant credit puzzle for investment grade debt. A model without stochastic volatility is unable to explain more than 25% of the historical credit spreads at any rating. Adding stochastic volatility greatly improves the pricing of investment grade debt, with the endogenous default model once again outperforming the exogenous default model for Aaa and Aa debt by a substantial amount. I also report the credit spreads for junk debt and it is apparent that the exogenous default model yet again produces significantly higher credit spreads at these ratings categories than a model with endogenous default.

Finally, given that I do not force the model to match historical default rates, it is important to see what cumulative default probabilities the model is actually generating. In particular, I need to make sure that I am not delivering higher credit spreads by simply overstating the credit risk. The targets are given by the average cumulative issuer-weighted global default rates from 1970-2007 as reported by Moody's. The model is quite successful at matching long-maturity historical default rates for A, Baa and B rated debt as Table 6 demonstrates. It underpredicts the default probabilities in the Ba category somewhat and

Table 6: Cumulative Default Probabilities - Long Maturity

	10yr		15yr		20yr	
	Target	Model	Target	Model	Target	Model
Aaa	0.53%	0.01%	1.00%	0.08%	1.20%	0.22%
Aa	0.52%	0.15%	1.09%	0.58%	1.88%	1.14%
A	1.31%	1.08%	2.40%	2.54%	4.08%	3.91%
Baa	4.35%	4.16%	7.60%	7.13%	10.51%	9.41%
Ba	18.43%	13.40%	27.53%	18.40%	34.85%	21.73%
B	40.92%	38.33%	50.21%	45.34%	52.38%	49.75%

Note: This table reports historical and model-generated cumulative default probabilities of firms within 10, 15, and 20 years by ratings category. Target expected default frequencies are the average cumulative issuer-weighted global default rates from 1970-2007 as reported by Moody's. Calibration parameters are provided in Tables 2 and 3. There is zero correlation between productivity shocks and variance shocks.

substantially so in the Aaa and Aa categories. Note, though, that the model-generated default probabilities for Aaa and Aa debt are much closer to reported historical rates from 1983-2007.³⁵

6.3 Intuition

These quantitative results naturally lead to two essential questions. How does the inclusion of stochastic volatility increase credit spreads and why is endogenous default important in matching credit spreads across ratings categories? The answer to the first question is intuitive. The negative market price of volatility risk indicates that investors see an increase in volatility as a deterioration in the investment opportunity set. Therefore, investors require a premium in the form of higher expected returns to hold assets which do poorly when volatility increases. In general, debt is such an asset since an increase in volatility raises the probability of default which lowers the value of the claim. Consequently, the discount rates on debt should be higher in a structural credit model with stochastic volatility and a negative market price of volatility risk than in a specification in which volatility is constant. These higher discount rates then lead to lower debt prices and higher credit spreads.

To see this explicitly, I apply Feynman-Kac to the boundary value problem defining the first-order correction for debt to derive a probabilistic interpretation. This gives:

³⁵Moody's reports a global default rate of only 0.19% for Aaa debt between the years 1983-2007 at all maturities of 8 years and above. The rates for Aa-rated debt are also substantially lower in this period than between 1970-2007.

$$\sqrt{\delta}\tilde{d}_1^y(x) = E_t^* \left[\begin{aligned} & \int_t^{\tau^B(y)} e^{-(r+m)(s-t)} \left\{ - \left(A^\delta y \frac{\partial \tilde{d}_0^y}{\partial y} - B^\delta y \frac{\partial^2 \tilde{d}_0^y}{\partial x \partial y} \right) \right\} ds \\ & + e^{-(r+m)(\tau^B(y)-t)} \sqrt{\delta} x_{B,1}^y \left\{ mU'(X_{\tau^B(y)}) - \frac{\partial \tilde{d}_0^y}{\partial x}(X_{\tau^B(y)}) \right\} \end{aligned} \right], \quad (56)$$

where the expectation is taken over the process $dX_s = gX_s + \sqrt{y}X_s dW_s^{(1)*}$ with $X_t = x$ and the stopping time defined by:

$$\tau^B(y) = \min \{s : X_s = x_{B,0}^y\}.$$

In words, the correction term is an average discounted present value of comparative statics in the constant volatility model over all possible sample paths of productivity given an initial value of x , taking into account the stopping time and in which the process for productivity is a geometric Brownian motion with drift g and constant volatility \sqrt{y} . At the stopping time, the payoff is the correction to the debt value at the default boundary. The fact that the averaging holds volatility constant reflects the assumption of a slow-moving variance process. The first term in the expression captures the intuition described above. Recall that $A^\delta y$ is the variance risk premium and $\partial \tilde{d}_0^y / \partial y$ is the debt vega in the constant volatility model. So the correction term takes into account the extent to which debt in the constant volatility model covaries with variance, prices this risk according to the variance risk premium, and averages over all possible sample paths for productivity going forward. Since the vega should in general be negative for debt and the variance risk premium is negative, this constitutes a negative contribution to the correction term, raising credit spreads beyond the constant volatility baseline model.

The second term was not operative in the quantitative results previously since I set $\rho_Y = 0$, which implies $B^\delta = 0$. However, in a model with nonzero correlation, this term would capture the skewness effect of stochastic volatility. A negative correlation between productivity and volatility shocks means that bad times for the firm are more volatile times. This increases the credit risk of debt, i.e. the probability of default, which increases credit spreads. Technically, the cross partial $\partial^2 \tilde{d}_0^y / \partial x \partial y$ is generally positive and with negative correlation the contribution to the debt correction term is negative ($B^\delta < 0$). Note that while the previous effect was one of discount rates, the skewness effect is a statement about expected cash flows.

The mechanisms underlying the improved performance of the endogenous default model are more subtle. There are two driving forces. First, while the debt vega is indeed usually negative in the constant volatility model, it is actually positive when the firm is

close to default if the barrier is chosen endogenously. This does not occur in the exogenous default model, as shown in Figure 2. When volatility increases, the option value of the equityholders increase and they respond, if able to, by postponing default. That is, $x_{B,0}^y$ is a decreasing function of y . Increasing volatility therefore has two effects on the value of debt. The increased riskiness raises the probability of default directly, but the shifting boundary lowers it indirectly. Near the default barrier, the latter effect dominates and the increased volatility actually increases the value of debt. This can be seen mathematically. Differentiating equations (39a)-(39c) gives the problem to solve for the debt vega of the principal order term in the endogenous default model:

$$\mathcal{L}_{r+m}^y \frac{\partial \tilde{d}_0^y}{\partial y} + \frac{1}{2} x^2 \frac{\partial^2 \tilde{d}_0^y}{\partial y^2} = 0 \quad (57a)$$

$$\frac{\partial \tilde{d}_0^y}{\partial y} (x_{B,0}^y) = -\frac{\partial \tilde{d}_0^y}{\partial x} (x_{B,0}^y) \frac{dx_{B,0}^y}{dy} \quad (57b)$$

$$\lim_{x \rightarrow \infty} \frac{\partial \tilde{d}_0^y}{\partial y} (x) = 0 \quad (57c)$$

Since the value of debt is increasing at the default boundary and the boundary is decreasing with volatility, the debt vega at the boundary is positive. The result follows by continuity.

The second effect comes from the fact that in the probabilistic representation of the debt correction term, the payoff at the stopping time is positive. That is,

$$\sqrt{\delta} x_{B,1}^y \left\{ mU' (x_{B,0}^y) - \frac{\partial \tilde{d}_0^y}{\partial x} (x_{B,0}^y) \right\} > 0. \quad (58)$$

Intuitively, even the prospect of future movements in volatility increases the value of the equityholders' default option, leading to a lower default boundary than in the constant volatility case. Debtholders benefit from this because it lowers the probability of default. Technically, the term in brackets in equation (58) is negative. To see this, note that:

$$\begin{aligned} \frac{\partial \tilde{d}_0^y}{\partial x} (x_{B,0}^y) &= \frac{\partial V_0^y}{\partial x} (x_{B,0}^y) - \frac{\partial E_0^y}{\partial x} (x_{B,0}^y) \\ &= \frac{\partial V_0^y}{\partial x} (x_{B,0}^y) \\ &= U' (x_{B,0}^y) - \gamma_2 \left[\xi U (x_{B,0}^y) + \frac{\phi C}{r} \right] \\ &\geq U' (x_{B,0}^y), \end{aligned}$$

where V_0^y is the principal order term of total shareholder value as defined in Appendix A.

Table 7: Credit Spreads on 4-Year Maturity Debt (bps)

	Target	Baseline		Exog. Default		Endog. Default	
		Model	% Explained	Model	% Explained	Model	% Explained
Aaa	46	0.5	1.1%	2	4.4%	2	4.4%
Aa	56	3	5.4%	13	23.2%	14	25.0%
A	87	12	13.8%	38	43.7%	42	48.3%
Baa	149	36	24.2%	91	61.1%	96	64.4%
Ba	310	94	30.3%	189	61.0%	193	62.3%
B	470	344	73.2%	541	115.1%	520	110.6%
Caa	N/A	1072	N/A	1570	N/A	1445	N/A

Note: This table provides target and model-generated 4-year credit spreads by ratings category. Calibration parameters are provided in Tables 2 and 3. There is zero correlation between productivity shocks and variance shocks. The variance risk premium is set to match the 10-year historical Baa credit spread. Historical target credit spreads for Aaa-Baa debt are from Duffee (1998) while those for lower ratings categories are from Caouette, Altman, and Narayanan (1998). The baseline model holds volatility constant. The exogenous default model incorporates stochastic volatility, but sets the default barrier to be equal to that of the baseline model. The endogenous default model is the full stochastic volatility model.

The second inequality follows from smooth-pasting at the default boundary in the constant volatility model, the third inequality follows from differentiating the expression provided in Appendix A, and the final inequality follows from $\gamma_2 < 0$.

Crucially, note that these two effects are more significant in the pricing of junk debt than investment grade debt. When considering investment grade debt, default is far away and so most sample paths from the initial point will not hit the default boundary or the region near it for an extended period of time. As such, these two effects, which only occur near or at the default boundary, are heavily discounted among most sample paths. Thus they do not contribute much to the pricing of the correction term for investment grade debt, leading to similar pricing in both the endogenous and exogenous default models given a particular variance risk premium. On the other hand, it is exactly junk debt that is at risk of moving into the default region in the near future. Consequently, these effects are not discounted heavily in many of the sample paths from the initial point. Since these effects contribute positively to the value of debt, this indicates that the exogenous default model can significantly overpredict credit spreads.

It is now apparent why the endogenous default model is more successful. Since the variance risk premium is calibrated to the speculative grade Baa-rated credit spread, the implied variance risk premium is higher in the endogenous default model than the exogenous default model. However, the effects highlighted above are even stronger for junk debt than

speculative grade debt. Therefore, the endogenous default model generates *lower* credit spreads at these ratings categories. On the other hand, since the effects are quite weak for investment grade debt and the implied variance risk premium is higher in the endogenous default model, it generates *higher* credit spreads at these categories as desired.

7 Short Maturity Debt and Extensions

Having quantitatively analyzed long maturity debt and discussed the mechanisms underlying the results, I now turn to a study of short maturity debt. As Table 7 illustrates, the credit spread puzzle is still present at short maturities and has familiar features. A model with constant volatility is unable to explain more than a third of the historical credit spread for Aaa-Ba debt, but accounts for a significantly greater fraction of junk credit spreads. Moreover, while the model with stochastic volatility and endogenous default certainly does improve on this baseline by a substantial amount, the performance is not as good as for long maturity debt. The model is never able to account for more than two-thirds of the historical credit spreads on investment and speculative grade debt. The performance for Aaa and Aa debt is particularly disappointing, with the model only accounting for 4.4% and 25.0% respectively of observed credit spreads.

One reason for this is that the effects of stochastic volatility are weaker at short maturities. In other words, the first-order correction term for total debt is smaller. This is because the vega of the principal order term is declining with maturity, as shown in Figure 3. Loosely stated, since productivity follows a diffusion, it can only move so far within a short period of time. Thus, increases in volatility do not significantly raise the riskiness of the firm. Since debt is not as sensitive to volatility fluctuations at short maturity, the discount rate correction in the stochastic volatility model is not as large.

This is not the whole story, however. Examining Table 8 indicates that the model is underpredicting the cumulative default probabilities of short-maturity Aaa-Baa debt. That is, the model is not only underpredicting credits spreads, but also credit risk. A key question, therefore, is how well the model could match historical prices if it more accurately reflected empirical default frequencies.

The approach of Huang and Huang (2003) and other studies to this question has been to set the asset volatility to match this moment. As discussed, though, this method is somewhat unsatisfactory since the implied asset volatility is then significantly higher than model-free empirical estimates. A more appropriate approach is to ask whether the model as currently structured is missing some element of realism which, if included, could increase the credit risk of short maturity debt. One possibility would be to include negative correlation

between volatility shocks and productivity shocks, i.e. set $\rho_Y < 0$. This does not work as well as one would like though, since under the assumption of slow-moving volatility, the skewness effects are weak at especially short maturities.

Instead, I consider two extensions to the baseline model. In the first, I add a second, high-frequency factor in the volatility dynamics specification. In the second, I allow for rare disasters in the firm productivity process. In both cases, I will be able to extend the perturbation methodology to solve the model.

7.1 Multiscale Stochastic Volatility

Let \mathbf{W}_t now be a Wiener process or standard Brownian motion in three dimensions under \mathbb{P} . The dynamics of productivity and volatility are given by

$$dX_t/X_t = \mu dt + \sqrt{Y_t Z_t} dW_t^{(1)} \quad (59)$$

$$dY_t = \delta \kappa_Y (\theta_Y - Y_t) dt + \nu_Y \sqrt{\delta Y_t} dW_t^{(2)} \quad (60)$$

$$dZ_t = \frac{1}{\varepsilon} \kappa_Z (\theta_Z - Z_t) dt + \nu_Z \sqrt{\frac{1}{\varepsilon} Z_t} dW_t^{(3)}, \quad (61)$$

where both $\delta > 0$ and $\varepsilon > 0$ are small parameters and

$$\begin{pmatrix} W_t^{(1)} \\ W_t^{(2)} \\ W_t^{(3)} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \rho_Y & \sqrt{1 - \rho_Y^2} & 0 \\ \rho_Z & \rho_{YZ} & \sqrt{1 - \rho_Z^2 - \rho_{YZ}^2} \end{pmatrix} \mathbf{W}_t. \quad (62)$$

As in Calvet and Fisher (2004), the variance of productivity shocks is now a product of multiple factors. The first factor Y_t mean-reverts slowly just as before.³⁶ However, there now is an additional factor Z_t which mean-reverts quickly. The correlation between shocks to productivity and Y_t is given by ρ_Y , while the correlation between shocks to productivity and Z_t is given by ρ_{YZ} . The parameter ρ_{YZ} denotes the correlation between shocks to the high and low frequency components of volatility.

To price contingent claims, the dynamics under the risk neutral measure \mathbb{P}^* need to be specified. The dynamics of X_t and Y_t are given in equations (7) and (8). I assume that the dynamics of Z_t under the risk-neutral measure are the same as under the physical measure. This assumption is equivalent to saying that innovations in the high-frequency component to volatility are not priced. Long-run investors should not view highly transitory, independent

³⁶Note that the parameter values κ_Y, θ_Y , and ν_Y may be different than before. However, under the assumption that $\delta > 0$ is small, the perturbation approach means that they will not need to be calibrated.

Table 8: Cumulative Default Probabilities - Short Maturity

	2yr		4yr		6yr	
	Target	Model	Target	Model	Target	Model
Aaa	0.00%	0.00%	0.03%	0.00%	0.17%	0.00%
Aa	0.02%	0.00%	0.11%	0.00%	0.26%	0.01%
A	0.09%	0.00%	0.34%	0.03%	0.61%	0.21%
Baa	0.48%	0.01%	1.36%	0.43%	2.32%	1.48%
Ba	3.02%	0.53%	7.65%	3.70%	11.77%	7.42%
B	10.20%	8.33%	20.33%	20.39%	28.74%	28.48%

Note: This table reports historical and model-generated cumulative default probabilities of firms within 2, 4, and 6 years by ratings category. Target expected default frequencies are the average cumulative issuer-weighted global default rates from 1970-2007 as reported by Moody's. Calibration parameters are provided in Tables 2 and 3. There is zero correlation between productivity shocks and variance shocks.

innovations in volatility as deteriorations in the investment opportunity set. However, if shocks to high-frequency volatility are partially correlated with either innovations in asset productivity or innovations in low-frequency volatility, then they would carry a price. I ignore this effect since the goal at hand is to see how to increase the credit risk of short maturity debt, which the price of high-frequency volatility shocks would not affect.

The assumptions on capital structure and bankruptcy remain unchanged. The equity value $E(x, y, z)$ can again be characterized as the solution to a free boundary problem and the value of newly issued debt $\tilde{d}(x, y, z)$ as the solution to a PDE boundary value problem. These are provided in the online appendix.³⁷ To use the asymptotic expansion approach, equity values, debt values, and the default boundary are expanded in powers of both $\sqrt{\delta}$ and $\sqrt{\varepsilon}$:

$$E_{\delta, \varepsilon}(x, y, z) = E_0^{y, z}(x) + \sqrt{\delta}E_{1,0}^{y, z}(x) + \sqrt{\varepsilon}E_{0,1}^{y, z}(x) + \sqrt{\delta\varepsilon}E_{1,1}^{y, z}(x) + \dots \quad (63)$$

$$\tilde{d}_{\delta, \varepsilon}(x, y, z) = \tilde{d}_0^{y, z}(x) + \sqrt{\delta}\tilde{d}_{1,0}^{y, z}(x) + \sqrt{\varepsilon}\tilde{d}_{0,1}^{y, z}(x) + \sqrt{\delta\varepsilon}\tilde{d}_{1,1}^{y, z}(x) + \dots \quad (64)$$

$$x_B(y) = x_{B,0}^{y, z} + \sqrt{\delta}x_{B,1,0}^{y, z} + \sqrt{\varepsilon}x_{B,0,1}^{y, z} + \dots \quad (65)$$

A close adaptation of calculations in Fouque et al. (2003) based on both regular and singular perturbations derives the systems of equations which solve for the principal order terms and first-order correction terms. The derivation is quite complicated and can be found by the interested reader in the online appendix. I summarize the results here.

Principal order terms once again reflect the equity/debt valuations and endogenous de-

³⁷The online appendix can be found at <http://www.scholar.harvard.edu/mcquade>

fault boundary in the constant volatility model. However, the volatility plugged into these expressions is no longer \sqrt{y} , the current value of the slow-moving factor, but

$$\bar{\sigma}(y) = \sqrt{y} \int \sqrt{z} d\Lambda_Z(z), \quad (66)$$

where Λ_Z is the invariant distribution of Z_t as defined in Appendix C. That is, the volatility is set equal to the product of the slow-moving factor and the long-run average of the fast-moving factor. Note, therefore, that the principal order terms do not depend on the current level of Z_t .

First-order corrections are again the solutions to ordinary differential equations with a source term reflecting comparative statics of the constant volatility model. The slow-moving equity correction term is given by the same problem as before:

$$\mathcal{L}_r^{\bar{\sigma}^2(y)} \sqrt{\delta} E_{1,0}^{y,z} = \left(A^\delta y \frac{\partial E_0^{y,z}}{\partial y} - B^\delta y x \frac{\partial^2 E_0^{y,z}}{\partial x \partial y} - \sqrt{\delta} \tilde{d}_{1,0}^{y,z} \right) \quad \text{for } x > x_{B,0}^{y,z} \quad (67a)$$

$$\sqrt{\delta} E_{1,0}^{y,z}(x_{B,0}^{y,z}) = 0 \quad (67b)$$

$$\lim_{x \rightarrow \infty} \sqrt{\delta} E_{1,0}^{y,z}(x) = 0, \quad (67c)$$

except that the operator $\mathcal{L}_r^{\bar{\sigma}^2(y)}$ now reflects volatility $\bar{\sigma}(y)$ instead of \sqrt{y} . The fast-moving equity correction term is found according to

$$\mathcal{L}_r^{\bar{\sigma}^2(y)} \sqrt{\varepsilon} E_{0,1}^{y,z} = \left(C^\varepsilon \left[x^3 \frac{\partial^3 E_0^{y,z}}{\partial x^3} + 2x^2 \frac{\partial^2 E_0^{y,z}}{\partial x^2} \right] - \sqrt{\varepsilon} \tilde{d}_{0,1}^{y,z} \right), \text{ for } x > x_{B,0}^{y,z} \quad (68a)$$

$$\sqrt{\varepsilon} E_{0,1}^{y,z}(x_{B,0}^{y,z}) = 0 \quad (68b)$$

$$\lim_{x \rightarrow \infty} \sqrt{\varepsilon} E_{0,1}^{y,z}(x) = 0, \quad (68c)$$

where the constant C^ε is provided in the online appendix and has the opposite sign of ρ_Z .³⁸ Solving for the fast-moving correction term requires the gamma and speed of the constant volatility model. Both correction terms are independent of the current level of the fast-moving factor Z_t . The correction terms for debt are similarly found with Taylor expansions providing the boundary conditions for the differential equations. Finally, to complete the

³⁸If innovations to the high-frequency component of volatility were priced, there would be an additional parameter in the coefficient multiplying gamma.

Table 9: Baa-Rated Credit Spreads with Multiscale Stochastic Volatility

	Target	$C^\varepsilon = 0$	$C^\varepsilon = .0005$	$C^\varepsilon = .0010$	$C^\varepsilon = .0015$	$C^\varepsilon = .0020$
20yr	195	194	200	207	212	218
10yr	150	150	155	161	168	174
4yr	149	95	102	108	115	120
1yr	N/A	28	35	41	46	52
3mo	N/A	3	5	5	7	8

Note: This table provides target and model-generated Baa-rated credit spreads at 20yr, 10yr, 4yr, 1yr, and 3mo maturities in a model with multiscale stochastic volatility. The parameter C^ε has the opposite sign of the correlation between shocks to the high-frequency volatility factor and productivity. Other calibration parameters are provided in Tables 2 and 3. $A^\delta = -.2264$ and $B^\delta = 0$. Historical target credit spreads are from Duffee (1998).

system of equations, the corrections to the default boundary must satisfy

$$\sqrt{\delta} x_{B,1,0}^{y,z} \frac{\partial^2 E_0^{y,z}}{\partial x^2} (x_{B,0}^{y,z}) = -\sqrt{\delta} \frac{\partial E_{1,0}^{y,z}}{\partial x} (x_{B,0}^{y,z}) \quad (69a)$$

$$\sqrt{\varepsilon} x_{B,0,1}^{y,z} \frac{\partial^2 E_0^{y,z}}{\partial x^2} (x_{B,0}^{y,z}) = -\sqrt{\varepsilon} \frac{\partial E_{0,1}^{y,z}}{\partial x} (x_{B,0}^{y,z}). \quad (69b)$$

Like the contingent claims, the endogenous default boundary is independent of the current Z_t . Cumulative survival probabilities can also be approximated using perturbation and are discussed in the online appendix.

7.1.1 Quantitative Results and Discussion

I examine the credit spreads and default probabilities of Baa-rated debt as I vary the parameter C^ε . The choice of this rating category is for illustrative purposes only. The key intuitions and quantitative results carry over to other ratings. All other parameters in the model remain as before. In particular, I set $A^\delta = -.2264$ and $B^\delta = 0$. Table 9 shows that a increasing C^ε from zero to 0.002 raises credit spreads at all maturities. For instance, the spreads on 4-year maturity debt are raised from 95bps to 120bps, or from explaining 64.4% of the target spread to 80.5%.

Credit spreads rise because the model-generated cumulative default probabilities are increasing, as Table 10 illustrates. This is a reflection of the skewness effect described in Section 6. Recall that a positive C^ε indicates negative correlation between productivity shocks and shocks to fast-moving volatility. This means that bad times for the firm are also the most volatile times, which increases the probability of default.

The model achieves the stated goal of proportionally increasing credit spreads and default

Table 10: Baa-Rated Cumulative Default Probabilities with Multiscale Stochastic Volatility

	Target	$C^\varepsilon = 0$	$C^\varepsilon = .0005$	$C^\varepsilon = .0010$	$C^\varepsilon = .0015$	$C^\varepsilon = .0020$
20yr	10.51%	9.41%	10.64%	11.75%	12.86%	14.00%
10yr	4.35%	4.16%	5.14%	6.04%	6.95%	7.88%
4yr	1.36%	0.43%	0.77%	1.11%	1.46%	1.82%
2yr	0.48%	0.01%	0.05%	0.08%	0.12%	0.16%

Note: This table reports historical and model-generated cumulative default probabilities of Baa-rated firms within 20, 10, 4, and 2 years in a model with multiscale stochastic volatility. Target expected default frequencies are the average cumulative issuer-weighted global default rates from 1970-2007 as reported by Moody's. The parameter C^ε has the opposite sign of the correlation between shocks to the high-frequency volatility factor and productivity. Other calibration parameters are provided in Tables 2 and 3. $A^\delta = -.2264$ and $B^\delta = 0$.

probabilities more at short maturities than at long maturities. As C^ε increases from 0 to 0.002, the credit spreads of 20-year and 10-year maturity debt increase by 11.8% and 16.0% respectively. The 4-year credit spread increases by 26.3%. Increases at shorter maturities, for which I do not have target credit spreads, are even more stark. The credit spreads on 1-year and 3-month debt increase by 85.7% and 166.7% respectively. Similarly, while the baseline model is only able to explain 31.6% of the probability of default within four years, setting $C^\varepsilon = .0015$ allows the model to match relatively well this target, while only overpredicting the probability of default within 20 and 10 years by 22.4% and 38.9% respectively.

The intuition behind this result is that periods of increased volatility correlated with negative market movements are highly transitory due the fast mean-reversion of the Z_t factor. Debt with long maturity will be averaging sample paths evolving over many years in which these skewness episodes will be brief and of small measure. Thus, they do not greatly impact the probability of default or valuation. Conversely, such a skewness episode may coincide with the lifetime of especially short maturity debt despite the fast mean-reversion. This accounts for the large proportional increases observed at shorter maturities.

7.2 Rare Disasters

In my second extension to the baseline model, I allow for jumps in the firm productivity process:

$$dX_t/X_t = \mu_t dt + \sqrt{Y_t} dW_t^{(1)} + d \left(\sum_{i=1}^{N(t)} (Q_i - 1) \right) \quad (70)$$

$$dY_t = \delta \kappa_Y (\theta - Y_t) + \nu_Y \sqrt{\delta Y_t} dW_t^{(2)}, \quad (71)$$

where $N(t)$ is a Poisson process with rate $\lambda > 0$ and $\{Q_i\}$ is a sequence of independent, identically distributed random variables which take values between zero and one. Note that a jump in this model always corresponds to a negative event, although this could easily be modified. Jumps have been included in structural modeling previously by Hilberink and Rogers (2002) and Chen and Kou (2009). My contribution is to include jumps, stochastic volatility, and endogenous default in a unified model and to describe a tractable method for solving it.

I will assume that the jump risk premium is zero to maintain my focus on increasing the credit spreads of short maturity debt. Then, the risk-neutral dynamics are given by

$$dX_t/X_t = g dt + \sqrt{Y_t} dW_t^{(1)*} + d \left(\sum_{i=1}^{N(t)} (Q_i - 1) \right) \quad (72)$$

$$dY_t = \left(\kappa_Y (\theta_Y - Y_t) - \Gamma(Y_t) \nu_Y \sqrt{Y_t} \right) dt + \nu_Y \sqrt{Y_t} dW_t^{(2)*}. \quad (73)$$

In the online appendix, I provide the free boundary problem which characterizes the equity valuation. The key innovation is that the equity value must now satisfy an integro-partial differential equation, instead of a partial differential equation, to account for the jumps. The value of newly issued debt also satisfies an integro-partial differential equation.

The key assumption of the model is that the jumps are *rare*, i.e. $\lambda > 0$ is small. This allows me to utilize regular perturbation and expand the contingent claims and default boundary in powers of $\sqrt{\delta}$ and λ :

$$E_{\delta,\lambda}(x, y, z) = E_0^y(x) + \sqrt{\delta} E_{1,0}^y(x) + \lambda E_{0,1}^y(x) + \sqrt{\delta} \lambda E_{1,1}^y(x) + \dots \quad (74)$$

$$\tilde{d}_{\delta,\lambda}(x, y, z) = \tilde{d}_0^y(x) + \sqrt{\delta} \tilde{d}_{1,0}^y(x) + \lambda \tilde{d}_{0,1}^y(x) + \sqrt{\delta} \lambda \tilde{d}_{1,1}^y(x) + \dots \quad (75)$$

$$x_B(y) = x_{B,0}^y + \sqrt{\delta} x_{B,1,0}^y + \lambda x_{B,0,1}^y + \dots \sqrt{\delta} \lambda x_{B,1,1}^y + \dots \quad (76)$$

As should be familiar by now, the principal order terms will reflect the model with constant volatility and no jumps. The first-order correction terms accounting for the slow-moving stochastic volatility will also be the same as before. The equity correction term accounting

Table 11: Baa-Rated Credit Spreads with Multiscale Stochastic Volatility

	Target	$\lambda = 0$	$\lambda = .001$		
			[.50, .75]	[.25, .75]	[.25, .50]
20yr	195	194	216	236	253
10yr	150	150	165	180	196
4yr	149	95	109	124	138
1yr	N/A	28	37	50	66

Note: This table provides target and model-generated Baa-rated credit spreads at 20yr, 10yr, 4yr, 1yr maturities in a model with rare disasters. The parameter λ controls the arrival rate of the disasters and the intervals reflect the uniform distribution of the jump sizes. Other calibration parameters are provided in Tables 2 and 3. $A^\delta = -.2264$ and $B^\delta = 0$. Historical target credit spreads are from Duffee (1998).

for the jumps is:

$$\mathcal{L}_r^y \lambda E_{0,1}^y = - \left(\lambda \int_0^1 [E_0^y(yx) - E_0^y(x)] f_Q(y) dy - \lambda \tilde{d}_{0,1}^{y,z} \right), \text{ for } x > x_{B,0}^{y,z} \quad (77)$$

$$\lambda E_{0,1}^{y,z}(x_{B,0}^y) = 0 \quad (78)$$

$$\lim_{x \rightarrow \infty} \lambda E_{0,1}^y(x) = 0, \quad (79)$$

where $f_Q(y)$ is the density of the jump size distribution. A similar correction term is derived for debt and, together with the corrections to the default boundary, form a system of equations.

7.2.1 Quantitative Results and Discussion

To calibrate the model, I set $\lambda = .001$ such that the probability of a disaster event in any given year is 1%. I assume the jump size distribution to be uniform over an interval $[Q_{\min}, Q_{\max}] \subset [0, 1]$. Note that this implies the firm will lose between Q_{\min} and Q_{\max} percent of its asset value in the event of a jump. Table 11 displays the credit spreads for Baa-rated debt for jump size intervals of [.50, .75], [.25, .75], and [.25, .50]. Clearly, as the magnitudes of the jumps increase, the credit spreads rise as well.

This model also achieves the goal of proportionally increasing short-maturity credit spreads more than long-maturity credit spreads. There is a 30.4% increase in the 20-year credit spread and 30.7% increase in the 10-year credit spread as one moves from a model with no jumps to a model with $\lambda = .001$ and $Q \sim U([.25, 50])$. Credit spreads on 4-year debt increase by 45.3%, while those on 1-year debt increase by 135.7%.

The reason behind this pattern is that credit risk is increased more proportionally at

shorter maturities. The intuition is that the model allows for jumps to default. Since it is difficult for pure diffusions to reach the default barrier within a short period of time, the majority of the default probability for especially short maturity debt comes from this risk. This leads to large proportional increases in credit risk as one moves beyond a pure diffusion model. However, at longer maturities it is possible to both diffuse to default as well as jump to default. Therefore, the inclusion of jump risk does not proportionally increase credit risk as much.

8 Conclusion

This paper has constructed a real-options, term structure model of the firm incorporating stochastic volatility. Shocks to variance carry a negative price of risk in the market to reflect the view of long-term investors that a persistent increase in volatility constitutes a deterioration in the investment opportunity set. The stocks of financially distressed firms are long the default option and hedge against volatility risk in the market, thus requiring lower variance risk premia than healthy firms. Risk-adjusted momentum profits are concentrated among the stocks of low credit-rating firms and results from changing conditional variance betas. The model is capable of generating both a growth premium and a value premium among firms depending on the level of financial distress. A firm with growth options hedges volatility risk and, all else equal, requires smaller variance risk premia than firms without growth options, generating a value premium.

Firm debt is short the option to default such that if volatility is stochastic and shocks are negatively priced, discount rates are higher than in a model with constant volatility. This effect in turn lowers prices and raises credit spreads. However, it is the interaction of stochastic volatility and endogenous default which resolves the credit spread puzzle. Under a setting of endogenous default, extremely distressed junk debt hedges volatility risk in the market and the default threshold is lower at all levels of volatility than in the model with constant volatility. This prevents the model from overpredicting the credit spreads on junk debt and also improves the performance of the model at higher ratings categories.

The paper has demonstrated a novel solution methodology from mathematical finance and physics which should have applicability in other areas of economics. Principal order terms in the asymptotic expansion reflect contingent claims valuations in the constant volatility setting, while correction terms are the solutions to ODEs involving key comparative statics of the constant volatility model. The approach yields tractability, reduces the number of parameters which need to be calibrated, and provides clean mathematical expressions to guide understanding of the mechanisms at work in the model.

Finally, the paper has considered an additional high-frequency volatility time scale and rare disasters as extensions to the primary model. These additional elements allow the model to increase credit risk at short maturities and improve the model's performance at matching empirical default frequencies and credit spreads at short maturities.

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A Derivations and Proofs

A.1 Proof of Theorem 1

I prove Theorem 1 in a sequence of lemmas. Let

$$DIV(x, y) = (1 - \phi)(x - C) + \tilde{d}(x, y) - p \quad (80)$$

denote the flow dividend which accrues to the equityholders and set:

$$E^*(x, y) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_t^* \left[\int_t^\tau e^{-r(s-t)} DIV(X_s, Y_s) ds \right]. \quad (81)$$

The goal is to show that the solution $E(x, y)$ to equations (22a)-(22h) satisfies $E(x, y) = E^*(x, y)$ for all $x, y > 0$. Let $\mathcal{C} = \{(x, y) : x > x_B(y)\}$ denote the continuation set and $\mathcal{D} = \{(x, y) : x \leq x_B(y)\}$ the stopping set. Finally, define the stopping time:

$$\tau^B = \min \{t \geq 0 : (X_t, Y_t)\} \in \mathcal{D}. \quad (82)$$

The first result is as follows:

Lemma 4 *The flow dividend to equityholders must be nonpositive in the stopping region; that is, $DIV(x, y) \leq 0$ for all $(x, y) \in \mathcal{D}$.*

Proof. The proof is very simple. Recalling equation (22b), we must have:

$$DIV(x, y) + \mathcal{L}_{X,Y}E \leq rE$$

for all $(x, y) \in \mathcal{D}$. But equation (22c) says that $E(x, y) = 0$ for all $(x, y) \in \mathcal{D}$. Substituting this into the equation above immediately gives the desired result. ■

This result is intuitive. Since equityholders receive nothing in the event of default, it can never be optimal for them to default when they are still receiving positive dividends. Now define:

$$Z_t = e^{-rt}E(X_t, Y_t) + \int_0^t e^{-rs}DIV(X_s, Y_s) ds. \quad (83)$$

Lemma 5 *The process Z_t satisfies:*

$$\int_0^t e^{-rs}DIV(X_s, Y_s) ds \leq Z_t \leq Z_0 + M_t \quad (84)$$

where M_t is a continuous local martingale. The stopped process $Z_{t \wedge \tau^B}$ satisfies:

$$Z_{t \wedge \tau^B} = Z_0 + M_{t \wedge \tau^B}. \quad (85)$$

Proof. The first inequality in equation (84) follows directly from the fact that $E(x, y) \geq 0$ for all x, y . For the second inequality, note that the Ito-Doebelin formula applies to functions whose second derivatives are discontinuous on measure zero sets as long as the

first-derivatives are everywhere continuous, which itself is a consequence of smooth-pasting. Therefore, under the risk-neutral measure \mathbb{P}^* :

$$dZ_t = e^{-rt} \left[\{-rZ_t + \mathcal{L}_{X,Y}Z_t + DIV_t\} dt + \sqrt{Y_t}X_t \frac{\partial E}{\partial x} dW_t^{(1)*} + \nu_Y \sqrt{Y_t} \frac{\partial E}{\partial y} dW_t^{(2)*} \right]. \quad (86)$$

By equation (22a), this is:

$$dZ_t = e^{-rt} DIV_t \mathbb{I}[X_t \leq x_B(Y_t)] dt + e^{-rt} \left[\sqrt{Y_t}X_t \frac{\partial E}{\partial x} dW_t^{(1)*} + \nu_Y \sqrt{Y_t} \frac{\partial E}{\partial y} dW_t^{(2)*} \right]. \quad (87)$$

However, by the previous lemma I know that $DIV_t \leq 0$ for all $X_t \leq x_B(Y_t)$, which implies that:

$$Z_t \leq Z_0 + M_t \quad (88)$$

where:

$$M_t = \int_0^t e^{-rs} \sqrt{Y_s} X_s \frac{\partial E}{\partial x} dW_s^{(1)*} + \int_0^t e^{-rs} \nu_Y \sqrt{Y_s} \frac{\partial E}{\partial y} dW_s^{(2)*} \quad (89)$$

is a local continuous martingale by property of the Ito integral.

For the stopped process $Z_{t \wedge \tau^B}$, it is the case that $X_t > x_B(Y_t)$ for all $t < \tau^B$, which means that the indicator function in the drift term is equal to zero. Therefore:

$$Z_{t \wedge \tau^B} = Z_0 + M_{t \wedge \tau^B} \quad (90)$$

as desired. ■

To complete the proof, I finally show:

Lemma 6 *The solution $E(x, y)$ to equations (22a)-(22h) satisfies $E(x, y) = E^*(x, y)$*

Proof. Let $X_0 = x$ and $Y_0 = y$. Then for every stopping time τ and $n \in \mathbb{N}$:

$$\int_0^{\tau \wedge n} e^{-rs} DIV(X_s, Y_s) ds \leq E(x, y) + M_{\tau \wedge n}. \quad (91)$$

By the optional sampling theorem, $\mathbb{E}_0^*[M_{\tau \wedge n}] = 0$, so that:

$$\mathbb{E}_0^* \left[\int_0^{\tau \wedge n} e^{-rs} DIV(X_s, Y_s) ds \right] \leq E(x, y). \quad (92)$$

Taking limits as $n \rightarrow \infty$ and applying Fatou's lemma yields:

$$\mathbb{E}_0^* \left[\int_0^{\tau} e^{-rs} DIV(X_s, Y_s) ds \right] \leq E(x, y). \quad (93)$$

Taking the supremum over all stopping times gives $E^*(x, y) \leq E(x, y)$.

For the other direction, by considering the stopped process $Z_{t \wedge \tau^B}$ and once again using the optional sampling theorem, I have that:

$$\mathbb{E}_0^* \left[e^{-r(\tau^B \wedge n)} E(X_{\tau^B \wedge n}, Y_{\tau^B \wedge n}) + \int_0^{\tau^B \wedge n} e^{-rs} \text{DIV}(X_s, Y_s) ds \right] = E(x, y). \quad (94)$$

Taking limits as $n \rightarrow \infty$ and noting that $e^{-r\tau^B} E(X_{\tau^B}, Y_{\tau^B}) = 0$ gives:

$$E(x, y) = \mathbb{E}_0^* \left[\int_0^{\tau^B} e^{-rs} \text{DIV}(X_s, Y_s) ds \right]. \quad (95)$$

This show that τ^B is optimal and that $E(x, y) = E^*(x, y)$ for all $x, y > 0$ as desired. ■

A.2 Derivation of Equation (42)

The problem for the principal order equity term is given by:

$$(1 - \phi)(x - C) + \tilde{d}_0^y(x) - p + \mathcal{L}_r^y E_0^y = 0 \quad \text{for } x > x_{B,0}^y \quad (96a)$$

$$E_0^y(x_{B,0}^y) = 0 \quad (96b)$$

$$\lim_{x \rightarrow \infty} E_0^y(x) = U(x) - \frac{C + mP}{r + m} + \frac{\phi C}{r} \quad (96c)$$

$$\frac{\partial E_0^y}{\partial x} \Big|_{x=x_{B,0}^y} = 0 \quad (96d)$$

To solve this problem I will introduce V_0^y as the solution to:

$$(1 - \phi)x + \phi C + \mathcal{L}_r^y V_0^y = 0 \quad \text{for } x > x_{B,0}^y \quad (97a)$$

$$V_0^y(x_{B,0}^y) = (1 - \xi)U(x_{B,0}^y) \quad (97b)$$

$$\lim_{x \rightarrow \infty} V_0^y(x) = U(x) + \frac{\phi C}{r} \quad (97c)$$

Note that V_0^y is value of debt plus equity in the case of constant volatility. It is not equal to total firm value since the government is a residual claimant on a portion of the firm's cash flows. I now prove the following lemma:

Lemma 7 *The principal equity value term $E_0^y(x) = V_0^y(x) - \tilde{d}_0^y(x)/m$.*

Proof. Recalling that $P = p/m$ and $C = c/m$, it is straightforward to check that:

$$V_0^y(x_{B,0}^y) - \tilde{d}_0^y(x_{B,0}^y)/m = 0 \quad (98)$$

$$\lim_{x \rightarrow \infty} V_0^y(x) - \tilde{d}_0^y(x)/m = U(x) - \frac{C + mP}{r + m} + \frac{\phi C}{r} \quad (99)$$

It therefore remains to check that $V_0^y(x) - \tilde{d}_0^y(x)/m$ satisfies the appropriate differential equation. To this end:

$$\begin{aligned}
& (1 - \phi)(x - C) + \tilde{d}_0^y(x) - p + \mathcal{L}_r^y \left(V_0^y - \tilde{d}_0^y/m \right) \\
&= -C + \tilde{d}_0^y - p - (1/m) \mathcal{L}_r^y \tilde{d}_0^y \\
&= -C + \tilde{d}_0^y(x) - p + (1/m) \left(c + mp - m\tilde{d}_0^y(x) \right) \\
&= 0,
\end{aligned} \tag{100}$$

where here I used the fact that $c + mp + \mathcal{L}_{r+m}^y \tilde{d}_0^y = 0$. This completes the proof. ■

Standard ODE techniques for the Cauchy-Euler equation give:

$$V_0^y(x) = U(x) + \frac{\phi C}{r} - \left[\xi U(x_{B,0}^y) + \frac{\phi C}{r} \right] \left(\frac{x}{x_{B,0}^y} \right)^{\gamma_2}, \tag{101}$$

where γ_2 is the negative root of the following polynomial equation:

$$g\gamma_2 + \frac{1}{2}y\gamma_2(\gamma_2 - 1) - r = 0. \tag{102}$$

The expression for $\tilde{d}_0^y(x)$ is provided in equation (40). Computing $V_0^y(x) - \tilde{d}_0^y(x)/m$ then gives equation (42). Differentiating this equation with respect to x and evaluating at $x_{B,0}^y$ to solve the smooth-pasting condition gives the endogenous default boundary (43).

A.3 Proof of Theorem 3

Let \bar{x}_B denote the exogenous default boundary. The problem for the principal order term $\tilde{d}_0^y(x)$ is the same as before, replacing $x_{B,0}^y$ with \bar{x}_B . Likewise, the solution to the problem is simply equation (40) with $x_{B,0}^y$ replaced by \bar{x}_B . I begin with a preliminary useful lemma.

Lemma 8 *The following identity holds:*

$$\mathcal{L}_{r+m}^y \left\{ \frac{1}{g + \gamma_1 y - \frac{1}{2}y} \ln \left(\frac{x}{\bar{x}_B} \right) \left(\frac{x}{\bar{x}_B} \right)^{\gamma_1} \right\} = \left(\frac{x}{\bar{x}_B} \right)^{\gamma_1} \tag{103}$$

Proof. By direct computation, I show that:

$$\begin{aligned}
& \mathcal{L}_{r+m}^y \left\{ \frac{1}{g + \gamma_1 y - \frac{1}{2}y} \ln \left(\frac{x}{\bar{x}_B} \right) \left(\frac{x}{\bar{x}_B} \right)^{\gamma_1} \right\} \\
&= \frac{1}{g + \gamma_1 y - \frac{1}{2}y} \left\{ \begin{aligned} & g \left(\frac{x}{\bar{x}_B} \right)^{\gamma_1} \left[1 + \gamma_1 \ln \left(\frac{x}{\bar{x}_B} \right) \right] \\ & + \frac{1}{2}y \left(\frac{x}{\bar{x}_B} \right)^{\gamma_1} \left[2\gamma_1 - 1 + \gamma_1 (\gamma_1 - 1) \ln \left(\frac{x}{\bar{x}_B} \right) \right] \\ & - (r + m) \left(\frac{x}{\bar{x}_B} \right)^{\gamma_1} \ln \left(\frac{x}{\bar{x}_B} \right) \end{aligned} \right\} \\
&= \left(\frac{x}{\bar{x}_B} \right)^{\gamma_1} \frac{g + \gamma_1 y - \frac{1}{2}y}{g + \gamma_1 y - \frac{1}{2}y} + \ln \left(\frac{x}{\bar{x}_B} \right) \frac{g\gamma_1 + \frac{1}{2}y\gamma_1(\gamma_1 - 1) - (r + m)}{g + \gamma_1 y - \frac{1}{2}y} \left(\frac{x}{\bar{x}_B} \right)^{\gamma_1} \\
&= \left(\frac{x}{\bar{x}_B} \right)^{\gamma_1},
\end{aligned}$$

due the definition of γ_1 . ■

I next compute an explicit expression for the gamma and vega of the principal order term, that is the vega in the constant volatility model with exogenous default boundary.

Lemma 9 *The gamma of the debt principal order term in the exogenous default model is given by:*

$$\frac{\partial^2 \tilde{d}_0^y}{\partial x^2} = \left[m(1 - \xi)U(\bar{x}_B) - \frac{c + mp}{r + m} \right] \gamma_1 (\gamma_1 - 1) \left\{ \frac{x}{\bar{x}_B} \right\}^{\gamma_1 - 2}. \quad (104)$$

The vega of the principal order term is given by:

$$\frac{\partial \tilde{d}_0^y}{\partial y} = - \left[m(1 - \xi)U(\bar{x}_B) - \frac{c + mp}{r + m} \right] \frac{\gamma_1 (\gamma_1 - 1)}{2(g + \gamma_1 y - \frac{1}{2}y)} \ln \left(\frac{x}{\bar{x}_B} \right) \left(\frac{x}{\bar{x}_B} \right)^{\gamma_1}. \quad (105)$$

Proof. The gamma of the principal order term can be calculated directly by differentiating equation (40) with respect to x . By the symmetry of partial derivatives, If I differentiate the ODE and boundary condition for \tilde{d}_0^y with respect to y , I get the following problem to solve for vega:

$$\mathcal{L}_{r+m}^y \frac{\partial \tilde{d}_0^y}{\partial y} + \frac{1}{2}x^2 \frac{\partial^2 \tilde{d}_0^y}{\partial y^2} = 0 \quad (106a)$$

$$\frac{\partial \tilde{d}_0^y}{\partial y}(\bar{x}_B) = 0 \quad (106b)$$

Since the gamma remains bounded as $x \rightarrow \bar{x}_B$, equation (105) clearly satisfies the boundary condition. It remains to check that it satisfies the differential equation. This is a consequence of the lemma above. Define:

$$A(\bar{x}_B) = m(1 - \xi)U(\bar{x}_B) - \frac{c + mp}{r + m}. \quad (107)$$

Then I can show:

$$\begin{aligned}
& \mathcal{L}_{r+m}^y \left\{ -A(\bar{x}_B) \frac{\gamma_1(\gamma_1 - 1)}{2(g + \gamma_1 y - \frac{1}{2}y)} \ln\left(\frac{x}{\bar{x}_B}\right) \left(\frac{x}{\bar{x}_B}\right)^{\gamma_1} \right\} \\
&= -\frac{1}{2}A(\bar{x}_B) \gamma_1(\gamma_1 - 1) \mathcal{L}_{r+m}^y \frac{1}{g + \gamma_1 y - \frac{1}{2}y} \ln\left(\frac{x}{\bar{x}_B}\right) \left(\frac{x}{\bar{x}_B}\right)^{\gamma_1} \tag{108}
\end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{2}A(\bar{x}_B) \gamma_1(\gamma_1 - 1) \left(\frac{x}{\bar{x}_B}\right)^{\gamma_1} \\
&= -\frac{1}{2}x^2 \frac{\partial^2 \tilde{d}_0^y}{\partial y^2}, \tag{109}
\end{aligned}$$

as desired. ■

Note that I could have simply calculated the vega of the principal order term directly through differentiation, recognizing that:

$$\begin{aligned}
\frac{\partial}{\partial y} \left(\frac{x}{\bar{x}_B}\right)^{\gamma_1} &= \left(\frac{x}{\bar{x}_B}\right)^{\gamma_1} \ln\left(\frac{x}{\bar{x}_B}\right) \frac{d\gamma_1}{dy} \\
&= -\frac{1}{2} \frac{\gamma_1(\gamma_1 - 1)}{g + \gamma_1 y - \frac{1}{2}y} \ln\left(\frac{x}{\bar{x}_B}\right) \left(\frac{x}{\bar{x}_B}\right)^{\gamma_1}, \tag{110}
\end{aligned}$$

where $d\gamma_1/dy$ is computed through implicit differentiation of equation (41). However, the proof given above illustrates the usefulness of the first lemma, which I will now further exploit.

Lemma 10 *The first-order debt correction term in the exogenous default model with $\rho_Y = 0$ is given explicitly by:*

$$\frac{\ln(x/\bar{x}_B)}{2(g + \gamma_1 y - \frac{1}{2}y)} A^\delta y \left[\frac{\partial \tilde{d}_0^y}{\partial y} + \frac{1}{2(g + \gamma_1 y - \frac{1}{2}y)^2} y x^2 \frac{\partial^2 \tilde{d}_0^y}{\partial y^2} \right] \tag{111}$$

Proof. I will break the calculation into parts. First, note that:

$$\begin{aligned}
& \mathcal{L}_{r+m}^y \frac{\ln(x/\bar{x}_B)}{2(g + \gamma_1 y - \frac{1}{2}y)} A^\delta y \frac{\partial \tilde{d}_0^y}{\partial y} \\
&= -\frac{1}{2}A(\bar{x}_B) \frac{\gamma_1(\gamma_1 - 1)}{2(g + \gamma_1 y - \frac{1}{2}y)^2} A^\delta y \mathcal{L}_{r+m}^y \ln\left(\frac{x}{\bar{x}_B}\right)^2 \left(\frac{x}{\bar{x}_B}\right)^{\gamma_1} \tag{112}
\end{aligned}$$

Then:

$$\begin{aligned}
\mathcal{L}_{r+m}^y \ln \left(\frac{x}{\bar{x}_B} \right)^2 \left(\frac{x}{\bar{x}_B} \right)^{\gamma_1} &= g \left(\frac{x}{\bar{x}_B} \right)^{\gamma_1} \left[2 \ln \left(\frac{x}{\bar{x}_B} \right) + \gamma_1 \ln \left(\frac{x}{\bar{x}_B} \right)^2 \right] \\
&\quad + \frac{1}{2} y \left(\frac{x}{\bar{x}_B} \right)^{\gamma_1} \left[2 + 2(2\gamma_1 - 1) \ln \left(\frac{x}{\bar{x}_B} \right) + \gamma_1 (\gamma_1 - 1) \ln \left(\frac{x}{\bar{x}_B} \right)^2 \right] \\
&\quad - (r+m) \ln \left(\frac{x}{\bar{x}_B} \right)^2 \left(\frac{x}{\bar{x}_B} \right)^{\gamma_1} \\
&= y \left(\frac{x}{\bar{x}_B} \right)^{\gamma_1} + 2 \ln \left(\frac{x}{\bar{x}_B} \right) \left(\frac{x}{\bar{x}_B} \right)^{\gamma_1} \left(g + \gamma_1 y - \frac{1}{2} y \right) \\
&\quad + \ln \left(\frac{x}{\bar{x}_B} \right)^2 \mathcal{L}_{r+m}^y \left(\frac{x}{\bar{x}_B} \right)^{\gamma_1} \tag{113}
\end{aligned}$$

$$= y \left(\frac{x}{\bar{x}_B} \right)^{\gamma_1} + 2 \ln \left(\frac{x}{\bar{x}_B} \right) \left(\frac{x}{\bar{x}_B} \right)^{\gamma_1} \left(g + \gamma_1 y - \frac{1}{2} y \right) \tag{114}$$

Therefore:

$$\begin{aligned}
&\mathcal{L}_{r+m}^y \frac{\ln(x/\bar{x}_B)}{2(g + \gamma_1 y - \frac{1}{2}y)} A^\delta y \frac{\partial \tilde{d}_0^y}{\partial y} \\
&= -\frac{1}{2} A^\delta y A(\bar{x}_B) \frac{\gamma_1(\gamma_1 - 1)}{2} \frac{1}{(g + \gamma_1 y - \frac{1}{2}y)^2} y \left(\frac{x}{\bar{x}_B} \right)^{\gamma_1} \\
&\quad - A^\delta y A(\bar{x}_B) \frac{\gamma_1(\gamma_1 - 1)}{2} \frac{1}{g + \gamma_1 y - \frac{1}{2}y} \ln \left(\frac{x}{\bar{x}_B} \right) \left(\frac{x}{\bar{x}_B} \right)^{\gamma_1} \\
&= -\frac{1}{2} A(\bar{x}_B) \frac{\gamma_1(\gamma_1 - 1)}{2} A^\delta y \frac{1}{(g + \gamma_1 y - \frac{1}{2}y)^2} y \left(\frac{x}{\bar{x}_B} \right)^{\gamma_1} + A^\delta y \frac{\partial \tilde{d}_0^y}{\partial y} \tag{115}
\end{aligned}$$

Finally, I show that:

$$\begin{aligned}
&\mathcal{L}_{r+m}^y \frac{\ln(x/\bar{x}_B)}{2(g + \gamma_1 y - \frac{1}{2}y)^3} A^\delta y \frac{1}{2} y x^2 \frac{\partial^2 \tilde{d}_0^y}{\partial y^2} \\
&= \frac{1}{2} A^\delta y A(\bar{x}_B) \frac{\gamma_1(\gamma_1 - 1)}{2} \frac{1}{(g + \gamma_1 y - \frac{1}{2}y)^2} y \mathcal{L}_{r+m}^y \frac{1}{(g + \gamma_1 y - \frac{1}{2}y)} \ln \left(\frac{x}{\bar{x}_B} \right) \left(\frac{x}{\bar{x}_B} \right)^{\gamma_1} \\
&= \frac{1}{2} A^\delta y A(\bar{x}_B) \frac{\gamma_1(\gamma_1 - 1)}{2} \frac{1}{(g + \gamma_1 y - \frac{1}{2}y)^2} y \left(\frac{x}{\bar{x}_B} \right)^{\gamma_1}. \tag{116}
\end{aligned}$$

This completes the proof since:

$$\mathcal{L}_{r+m}^y \frac{\ln(x/\bar{x}_B)}{2(g + \gamma_1 y - \frac{1}{2}y)} A^\delta y \left[\frac{\partial \tilde{d}_0^y}{\partial y} + \frac{1}{2(g + \gamma_1 y - \frac{1}{2}y)^2} y x^2 \frac{\partial^2 \tilde{d}_0^y}{\partial y^2} \right] = A^\delta y \frac{\partial \tilde{d}_0^y}{\partial y} \tag{117}$$

■

Note that an explicit expression can be derived for the case where $\rho_Y \neq 0$ as well. However, the computations and ultimate expression are quite messy and not very intuitive. I therefore omit them.

B Contingent Claims of Growth Firms

In this appendix, I provide the problems to solve for the equity and debt values of growth firms and provide the details of how to solve for them by regular perturbation.

B.1 Value of Assets in Place

The value of assets in place for a growth firm is given by:

$$U_t^a(x) = \mathbb{E}_t^* \left[\int_t^\infty e^{-r(s-t)} (1 - \phi) K_a X_s ds \mid X_t = x \right] = \frac{(1 - \phi) K_a x}{r - g}. \quad (118)$$

This is simply the Gordon growth formula for an unlevered firm with tax rate ϕ .

B.2 Equity Valuation of Growth Firms

The equity valuation of the growth firm is given by the following optimal stopping problem under the risk-neutral measure:

$$E_\delta^a(x, y) = \sup_{\tau', \tau'' \in \mathcal{T}} \mathbb{E}_t^* \left[\int_t^{\tau' \wedge \tau''} e^{-r(s-t)} \left\{ (1 - \phi) (K_a X_s - C) + \tilde{d}_\delta^a(x, y) - p \right\} ds \right] + \mathbb{I}[\tau' < \tau''] (E_\delta(X_{\tau'}, Y_{\tau'}) - I), \quad (119)$$

where τ' is the stopping time which denotes investment and τ'' is the stopping time which denotes default. Upon defaulting, equityholders again receive nothing. At the investment threshold however, the equity value equals to the equity value a mature firm minus the cost of investment. Recall that by assumption, the cost of investment must be borne by equityholders, i.e., the firm cannot issue new debt to finance the purchase of additional capital.

By the same logic used to prove Theorem 1, we can characterize the equity valuation as a free boundary problem.

Theorem 11 *The equity value of a growth firm $E_\delta^a(x, y)$ is the solution to:*

$$DIV_\delta^a(x, y) + \left(\mathcal{L}_r^y + \sqrt{\delta} \mathcal{M}_1^y + \delta \mathcal{M}_2^y \right) E_\delta^a = 0 \quad \text{for } x \in (x_{B,\delta}^a(y), x_{I,\delta}^a(y)) \quad (120a)$$

$$E_\delta^a(x_B^a(y), y) = 0 \quad (120b)$$

$$E_\delta^a(x_I^a(y), y) = E_\delta(x_I^a(y), y) - I \quad (120c)$$

$$\frac{\partial E_\delta^a}{\partial x} \Big|_{x=x_{B,\delta}^a(y)} = 0 \quad (120d)$$

$$\frac{\partial E_\delta^a}{\partial y} \Big|_{x=x_{B,\delta}^a(y)} = 0 \quad (120e)$$

$$\frac{\partial E_\delta^a}{\partial x} \Big|_{x=x_{I,\delta}^a(y)} = \frac{\partial E_\delta}{\partial x} \Big|_{x=x_{I,\delta}^a(y)} \quad (120f)$$

$$\frac{\partial E_\delta^a}{\partial y} \Big|_{x=x_{I,\delta}^a(y)} = \frac{\partial E_\delta}{\partial y} \Big|_{x=x_{I,\delta}^a(y)} \quad (120g)$$

where

$$DIV_\delta^a(x, y) = (1 - \phi)(K_a x - C) + \tilde{d}_\delta^a(x, y) - p \quad (121)$$

and $x_B^a(y)$ and $x_I^a(y)$ are free boundaries to be determined and

Proof. The proof follows the same logic as the proof of Theorem 1. ■

The free boundary problem is similar to the one specified for the equity value of mature firms. At the default boundary, the equity value must be equal to zero and the derivatives must be continuous. However, now there is no limiting condition as $x \rightarrow \infty$. Instead, there are additional value-matching and smooth-pasting conditions. At the investment threshold, the equity value must be equal to the equity of a mature firm minus the cost of investment. Once again, optimality in conjunction with the nature of a regular diffusion requires that the derivatives of the valuation be continuous across this barrier. Finally, the dividend to the equityholders now reflects the lower level of capital and the valuation of newly issued debt for young firms.

B.3 Debt Valuation of Growth Firms

Let τ^I denote the optimal stopping time for investment and τ^D the optimal stopping time for default:

$$\tau^I = \min \{t : (X_t, Y_t) = (x_{I,\delta}^a(Y_t), Y_t)\} \quad (122)$$

$$\tau^B = \min \{t : (X_t, Y_t) = (x_{B,\delta}^a(Y_t), Y_t)\} \quad (123)$$

Existing debt is simply rolled over when the firm invests in capital. Therefore, the value of a vintage of debt at time t issued at date 0 is given by:

$$d_\delta^a(t) = \mathbb{E}_t^* \left[\begin{aligned} & \int_t^{\tau^I \wedge \tau^D} \{e^{-r(s-t)} e^{-ms} (c + mp)\} ds \\ & + \mathbb{I}[\tau^D < \tau^I] e^{-r(\tau^D-t)} \left(e^{-m\tau^D} p/P \right) (1 - \xi) U_{\tau^D} \\ & + \mathbb{I}[\tau^I < \tau^D] e^{-r(\tau^I-t)} d_\delta(\tau^I) \end{aligned} \right] \quad (124)$$

Upon default, debtholders receive a fraction of the value of the assets in place (minus bankruptcy costs) according to their vintage. At the investment threshold, the debt value equals the value of debt of identical vintage in a mature firm.

Noting that $p/P = m$ and $d_\delta(\tau^I) = e^{-m\tau^I} \tilde{d}_\delta(X_{\tau^I}, Y_{\tau^I})$, it follows that:

$$e^{mt} d_\delta^a(t) = \mathbb{E}_t^* \left[\begin{array}{l} \int_t^{\tau^I \wedge \tau^D} \{e^{-(r+m)(s-t)} (c + mp)\} ds \\ + \mathbb{I}[\tau^D < \tau^I] e^{-(r+m)(\tau^D-t)} m(1-\xi) U_{\tau^D}^a \\ + \mathbb{I}[\tau^I < \tau^D] e^{-(r+m)(\tau^I-t)} \tilde{d}_\delta(X_{\tau^I}, Y_{\tau^I}) \end{array} \right], \quad (125)$$

so that by Feynman-Kac the following theorem holds.

Theorem 12 *The value of the date 0 debt vintage at time t for a young firm is given by $d^a(t) = e^{-mt} \tilde{d}^a(X_t, Y_t)$ where $\tilde{d}^a(X_t, Y_t)$ is the value of the newly issued debt and satisfies:*

$$c + mp + \left(\mathcal{L}_{r+m}^y + \sqrt{\delta} \mathcal{M}_1^y + \delta \mathcal{M}_2^y \right) \tilde{d}_\delta^a = 0 \quad \text{for } x \in (x_{B,\delta}^a(y), x_{I,\delta}^a(y)) \quad (126a)$$

$$\tilde{d}_\delta^a(x_B^a(y), y) = m(1-\xi) U^a(x_B^a(y)) \quad (126b)$$

$$\tilde{d}_\delta^a(x_I^a(y), y) = \tilde{d}_\delta^a(x_I^a(y), y) \quad (126c)$$

The total value of debt $D^a = \tilde{d}^a/m$.

I now expand the contingent claims and default boundaries in powers of $\sqrt{\delta}$:

$$E_\delta^a(x, y) = E_0^{a,y}(x) + \sqrt{\delta} E_1^{a,y}(x) + \delta E_2^{a,y}(x) + \dots \quad (127a)$$

$$\tilde{d}_\delta^a(x, y) = \tilde{d}_0^{a,y}(x) + \sqrt{\delta} \tilde{d}_1^{a,y}(x) + \delta \tilde{d}_2^{a,y}(x) + \dots \quad (127b)$$

$$x_B^a(y) = x_{B,0}^{a,y} + \sqrt{\delta} x_{B,1}^{a,y} + \delta x_{B,2}^{a,y} + \dots \quad (127c)$$

These are substituted into the problems above. Boundary conditions are expanded with Taylor series. Finally, contingent claims of mature firms are also written using asymptotic expansions.

B.4 Principal Order Terms

The principal order terms reflect the valuations and default boundary in the constant volatility case. The problem for debt is:

$$c + mp + \mathcal{L}_{r+m}^y \tilde{d}_0^{a,y} = 0 \quad \text{for } x \in (x_{B,0}^{a,y}, x_{I,0}^{a,y}) \quad (128a)$$

$$\tilde{d}_0^{a,y}(x_{B,0}^{a,y}) = m(1-\xi) U^a(x_{B,0}^{a,y}) \quad (128b)$$

$$\tilde{d}_0^{a,y}(x_{I,0}^{a,y}) = \tilde{d}_0^y(x_{I,0}^{a,y}) \quad (128c)$$

and the problem for equity is:

$$(1 - \phi)(xK_a - C) + \tilde{d}_0^{a,y}(x) - p + \mathcal{L}_r^y E_0^{a,y} = 0 \quad \text{or } x \in (x_{B,0}^{a,y}, x_{I,0}^{a,y}) \quad (129a)$$

$$E_0^{a,y}(x_{B,0}^{a,y}) = 0 \quad (129b)$$

$$E_0^{a,y}(x_{I,0}^{a,y}) = E_0^y(x_{I,0}^{a,y}) - I \quad (129c)$$

$$\frac{\partial E_0^{a,y}}{\partial x} \Big|_{x=x_{B,0}^{a,y}} = 0 \quad (129d)$$

$$\frac{\partial E_0^{a,y}}{\partial x} \Big|_{x=x_{I,0}^{a,y}} = \frac{\partial E_0^y}{\partial x} \Big|_{x=x_{I,0}^{a,y}} \quad (129e)$$

I follow an indirect approach to solve for $E_0^{a,y}$ as in appendix A.2. I introduce the sum of debt and equity values $V_0^{a,y}$, which is the solution to:

$$(1 - \phi)K_a x + \phi C + \mathcal{L}_r^y V_0^{a,y} = 0 \quad \text{for } x \in (x_{B,0}^{a,y}, x_{I,0}^{a,y}) \quad (130a)$$

$$V_0^{a,y}(x_{B,0}^y) = (1 - \xi)U^a(x_{B,0}^y) \quad (130b)$$

$$V_0^{a,y}(x_{I,0}^y) = V_0^y(x_{I,0}^y) \quad (130c)$$

Then $E_0^{a,y} = V_0^{a,y} - \tilde{d}_0^{a,y}/m$ and the default/investment boundaries are found by applying the two smooth-pasting conditions.

B.5 First-Order Correction Terms

Finally, the first-order corrections once again reflect comparative statics in the constant volatility case as well as boundary corrections. The first-order correction for debt is:

$$\mathcal{L}_{r+m}^y \sqrt{\delta} \tilde{d}_1^{a,y} = \left(A^\delta y \frac{\partial \tilde{d}_0^{a,y}}{\partial y} - B^\delta y x \frac{\partial^2 \tilde{d}_0^{a,y}}{\partial x \partial y} \right) \quad \text{for } x \in (x_{B,0}^{a,y}, x_{I,0}^{a,y}) \quad (131a)$$

$$\sqrt{\delta} \tilde{d}_1^{a,y}(x_{B,0}^{a,y}) = \sqrt{\delta} x_{B,1}^{a,y} \left[\frac{m(1 - \xi)(1 - \phi)}{r - g} - \frac{\partial \tilde{d}_0^{a,y}}{\partial x}(x_{B,0}^{a,y}) \right] \quad (131b)$$

$$\sqrt{\delta} \tilde{d}_1^{a,y}(x_{I,0}^{a,y}) = \sqrt{\delta} x_{I,1}^{a,y} \left[\frac{\partial \tilde{d}_0^y}{\partial x}(x_{I,0}^{a,y}) - \frac{\partial \tilde{d}_0^{a,y}}{\partial x}(x_{I,0}^{a,y}) \right] + \sqrt{\delta} \tilde{d}_1^y(x_{I,0}^{a,y}) \quad (131c)$$

The first-order correction for equity is:

$$\mathcal{L}_r^y \sqrt{\delta} E_1^{a,y} = \left(A^\delta y \frac{\partial E_0^{a,y}}{\partial y} - B^\delta y x \frac{\partial^2 E_0^{a,y}}{\partial x \partial y} - \sqrt{\delta} \tilde{d}_1^{a,y} \right) \quad \text{for } x \in (x_{B,0}^{a,y}, x_{I,0}^{a,y}) \quad (132a)$$

$$\sqrt{\delta} E_1^{a,y}(x_{B,0}^{a,y}) = 0 \quad (132b)$$

$$\sqrt{\delta} E_1^{a,y}(x_{I,0}^{a,y}) = \sqrt{\delta} E_1^y(x_{I,0}^{a,y}) \quad (132c)$$

Note that I used the smooth-pasting condition of the principal order term at the default and investment boundaries to derive the final two equations. Finally, the corrections in the

default/investment barriers must satisfy:

$$\sqrt{\delta} x_{B,1}^{a,y} \frac{\partial^2 E_0^{a,y}}{\partial x^2} (x_{B,0}^{a,y}) = -\sqrt{\delta} \frac{\partial E_1^{a,y}}{\partial x} (x_{B,0}^{a,y}) \quad (133)$$

$$\sqrt{\delta} x_{I,1}^{a,y} \frac{\partial^2 E_0^{a,y}}{\partial x^2} (x_{I,0}^{a,y}) = -\sqrt{\delta} \frac{\partial E_1^{a,y}}{\partial x} (x_{I,0}^{a,y}) \quad (134)$$

All of this together solves a system of equations which can be solved numerically in MATLAB.

C Volatility Time Scales

This technical appendix elaborates on the discussion of volatility time scales in section 4.1 and specifically on the role of δ in controlling the rate of mean-reversion. The discussion follows closely the textbook treatment provided in Fouque, Papanicolaou, Sircar, and Solna (2011). I define the infinitesimal generator of a time-homogenous, ergodic Markov process Y_t to be:

$$\mathcal{L}h(y) = \lim_{t \rightarrow 0} \frac{\mathbf{P}_t h(y) - h(y)}{t}, \quad (135)$$

where:

$$\mathbf{P}_t h(y) = \mathbb{E}[h(Y_t) | Y_0 = y]. \quad (136)$$

For example, the infinitesimal generator of the Cox-Ingersoll-Ross process of equation (2) is given by:

$$\mathcal{M}_2^y = (\theta_Y - y) \frac{\partial}{\partial y} + \frac{1}{2} \nu_Y^2 y \frac{\partial^2}{\partial y^2}. \quad (137)$$

In general, the infinitesimal generator of a regular diffusion can be found by considering Ito's formula and the backwards Kolmogorov equation. To find the invariant distribution of the process Y_t , which exists by ergodicity, I look for a distribution Λ for Y_0 which satisfies for any bounded h :

$$\frac{d}{dt} \int \mathbb{E}[h(Y_t) | Y_0 = y] d\Lambda(y) = 0, \quad (138)$$

where the integral is taken of the state space on which the Markov process is defined. By the backward Kolmogorov equation for a time-homogenous Markov process:

$$\frac{d}{dt} \mathbf{P}_t h(y) = \mathcal{L} \mathbf{P}_t h(y), \quad (139)$$

it follows that:

$$\begin{aligned} \frac{d}{dt} \int \mathbb{E}[h(Y_t) | Y_0 = y] d\Lambda(y) &= \int \frac{d}{dt} \mathbf{P}_t h(y) d\Lambda(y) \\ &= \int \mathcal{L} \mathbf{P}_t h(y) d\Lambda(y) \\ &= \int \mathbf{P}_t h(y) \mathcal{L}^* d\Lambda(y), \end{aligned} \quad (140)$$

where \mathcal{L}^* is the adjoint operator of \mathcal{L} defined uniquely by:

$$\int \alpha(y) \mathcal{L}\beta(y) dy = \int \beta(y) \mathcal{L}^*\alpha(y) dy \quad (141)$$

for test functions α, β . The invariant distribution, therefore, is the solution to the equation:

$$\mathcal{L}^*\Lambda = 0, \quad (142)$$

since the relation above must hold for all h . I denote integration with respect to the invariant distribution by $\langle h \rangle$ and suppose that the invariant distribution has a mean.

Now define the process Y_t^δ according to the infinitesimal generator $\delta\mathcal{L}$. For a CIR process, this procedure then gives a new process which is given explicitly in equation (30). The adjoint of the operator $\delta\mathcal{L}$ is clearly given by $\delta\mathcal{L}^*$. Therefore, it is immediately clear that the invariant distribution of the process Y_t^δ is independent of the choice of δ , as described in section 4.1. This indicates that, in the long-run, the choice of δ does not affect the degree of variability in the process Y_t^δ .

Now I suppose that the process Y_t is reversible, or that the operator \mathcal{L} has a discrete spectrum and that zero is an isolated eigenvalue. This allows the formation of an orthonormal basis of $L^2(\Lambda)$ and to consider the eigenfunction expansion of a function $h(y)$ by:

$$h(y) = \sum_{n=0}^{\infty} d_n \psi_n(y) \quad (143)$$

where each ψ_n satisfies:

$$\mathcal{L}\psi_n = \lambda_n \psi_n \quad (144)$$

and where $0 = \lambda_0 > \lambda_1 > \dots$. Each constant satisfies $d_n = \langle h\psi_n \rangle$ and in particular $d_0 = \langle h \rangle$, since the eigenfunction associated with the zero eigenvalue is simply $\psi_0 = 1$. Next consider the backwards Kolmogorov equation:

$$\frac{d}{dt} \mathbf{P}_t h(y) = \mathcal{L} \mathbf{P}_t h(y) \quad (145)$$

and look for a solution of the form:

$$\mathbf{P}_t h(y) = \sum_{n=0}^{\infty} z_n(t) \psi_n(y). \quad (146)$$

Substituting this expression into the backwards Kolmogorov equation and using equation (144) gives an ODE for each $z_n(t)$:

$$z'_n(t) = \lambda_n z_n(t), \quad (147)$$

with initial condition $z_n(0) = d_n$. Solving, this implies that:

$$\mathbf{P}_t h(y) = \sum_{n=0}^{\infty} c_n e^{\lambda_n t} \psi_n(y). \quad (148)$$

From this, it is possible to show that:

$$|\mathbf{P}_t g(y) - \langle h \rangle| \leq C e^{\lambda_1 t}, \quad (149)$$

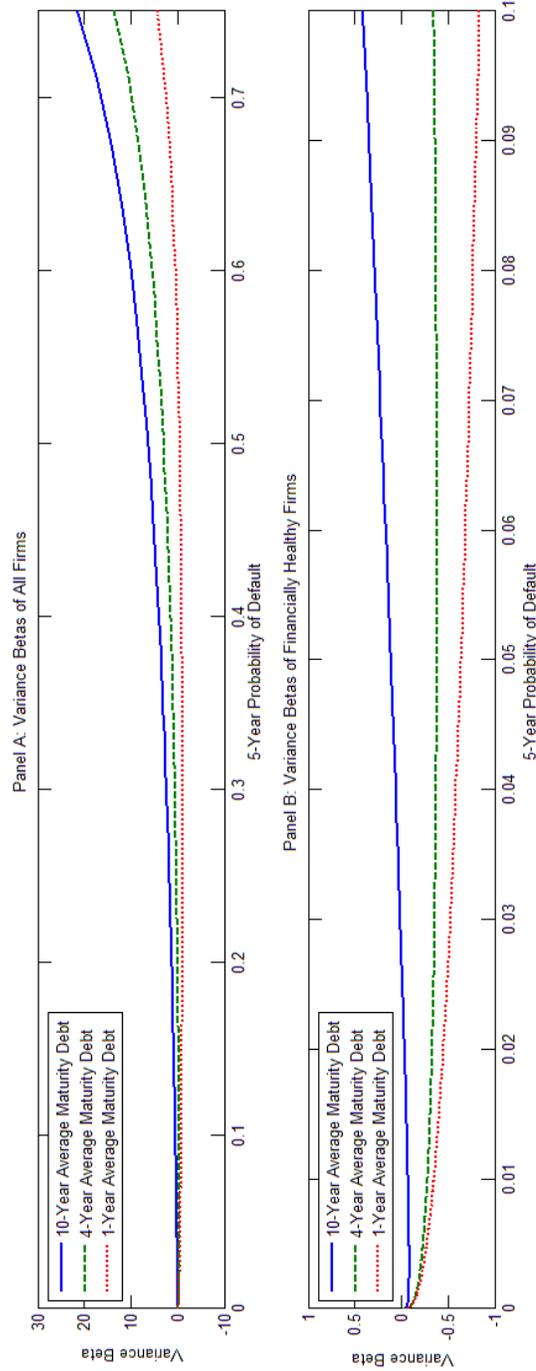
for some constant C . In words, the spectral gap, defined as the magnitude of the first negative eigenvalue, controls the rate of mean reversion of the process to the invariant distribution.

Finally, it is trivial to see that the eigenfunctions/eigenvalues of the infinitesimal generator $\delta\mathcal{L}$ are given by:

$$\mathcal{L}\delta\psi_n = \lambda_n \delta\psi_n. \quad (150)$$

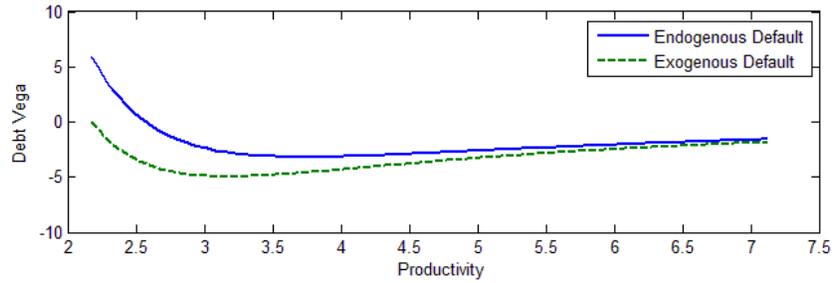
That is, the spectrum of $\delta\mathcal{L}$ is simply a scaling of the spectrum of \mathcal{L} according to the parameter δ . But this implies that the spectral gap of the process Y_t^δ is proportional to δ and, therefore, that δ controls the rate of mean-reversion of the process.

Figure 1: Variance Betas and Financial Distress



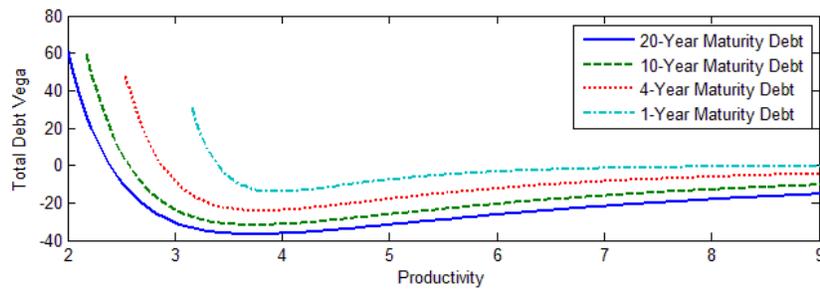
Note: This figure illustrates how equity variance betas vary with the average debt maturity of the firm and financial health of the firm, as measured by the 5-year cumulative probability of default. Firms are mature. The riskless rate, corporate tax rate, risk-neutral rate of productivity growth, and asset risk premium are set according to Table 2. Total principal equals 43.3 and the coupon rate equals 8.168%. Volatility of the productivity process equals 22%. Panel A shows the variance risk premia for all firms, while Panel B zooms in on financially healthy firms.

Figure 2: Debt Vegas: Endogenous and Exogenous Default



Note: This figure illustrates how debt vegas vary with productivity in the constant volatility model under endogenous and exogenous default assumptions. Firms are mature. The riskless rate, corporate tax rate, risk-neutral rate of productivity growth, and asset risk premium are set according to Table 2. Total principal equals 43.3 and the coupon rate equals 8.168%. Current volatility of the productivity process equals 22%.

Figure 3: Total Debt Vegas and Average Debt Maturity



Note: This figure illustrates how total debt vegas in the constant volatility model vary with the average debt maturity and productivity of the firm. Firms are mature. The riskless rate, corporate tax rate, risk-neutral rate of productivity growth, and asset risk premium are set according to Table 2. Total principal equals 43.3 and the coupon rate equals 8.168%. Current volatility of the productivity process equals 22%.