Optimal debt with unobservable investments: Web-Appendix

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Appendix B: π as a function of k

In this appendix we discuss how our results change if we allow π to depend on k. Define $\pi(k)$ with $\pi(0) \geq 0$, $\pi' > 0$, $\pi'' < 0$, and $lim_{k\to\infty} y'(k) + \pi'(k) < 1$. A small investment is now associated with a small π, which in turn may affect E's incentives to repay his debt. We show that a debt contract, though not necessarily a simple debt contract, remains optimal.

B.1 Optimality of a debt contract

Proposition 2 holds as stated, with π replaced by $\pi(k^*(W, T, \beta))$. The only part of the proof affected by this change is step 3. By construction, switching from (W, T, β) to (W, T^1, β^1) (with π replaced by $\pi(k^*(W, T, \beta)))$ leaves E's payoff unchanged for all $(R, \widehat{R})$, as long as $k^*(W, T^1, \beta^1) = k^*(W, T, \beta)$. However, if π is not constant, then switching to (W, T^1, β^1) is not payoff-neutral for E if $k \neq k^*(W, T, \beta)$, and $k^*(W, T^1, \beta^1))$ may differ from $k^*(W, T, \beta))$. We can show that (W, T^1, β^1) nevertheless Pareto-dominates (W, T, β).

Let $u^1(R, \widehat{R}, k) = R - T^1(\widehat{R}) + \beta(\widehat{R})\pi(k)$, and define $u^0(R, \widehat{R}, k)$ analogously for contract (W, T, β). For $\widehat{R} \in \rho_a$, we have

$$u^1(R, \widehat{R}, k) = R - \widehat{R} + \left[\beta(\widehat{R}) + \frac{\widehat{R} - T(\widehat{R})}{\pi(k^*)}\right]\pi(k) = u^0(R, \widehat{R}, k) - [\widehat{R} - T(\widehat{R})]\left(1 - \frac{\pi(k)}{\pi(k^*)}\right).$$

Similarly, for $\widehat{R} \in \rho_b$,

$$u^1(R, \widehat{R}, k) = R - T(\widehat{R}) - [1 - \beta(\widehat{R})]\pi(k^*) + \pi(k) = u^0(R, \widehat{R}, k) - [\pi(k^*) - \pi(k)][1 - \beta(\widehat{R})].$$

For $\widehat{R} \notin \rho$, by definition $u^1(R, \widehat{R}, k) = u^0(R, \widehat{R}, k)$. If E chooses $k < k^*$, then $\pi(k) < \pi(k^*)$ and hence $u^1(R, \widehat{R}, k) < u^0(R, \widehat{R}, k)$ for all $\widehat{R} \in \rho$. By definition of k^*, we then have

$$\begin{aligned} & \mathrm{E}_\theta[\max_{\widehat{R}} u^1(R(k^*, \theta), \widehat{R}, k^*)] - k^* = \mathrm{E}_\theta[\max_{\widehat{R}} u^0(R(k^*, \theta), \widehat{R}, k^*)] - k^* \\ \geq\ & \mathrm{E}_\theta[\max_{\widehat{R}} u^0(R(k, \theta), \widehat{R}, k)] - k \geq \mathrm{E}_\theta[\max_{\widehat{R}} u^1(R(k, \theta), \widehat{R}, k)] - k. \end{aligned}$$

(Notice that we are not assuming truthtelling for any $k \neq k^*$ under either contract.) This means that E would never choose $k < k^*$ under (W, T^1, β^1); hence $k^*(W, T^1, \beta^1) \geq k^*(W, T, \beta)$, possibly with strict inequality since for $k > k^*$ the second inequality above is reversed. The contract (W, T^1, β^1) is incentive compatible if for all R and $\widehat{R} < R$,

$$u^1(R, R, k) - u^1(R, \widehat{R}, k) = \widehat{R} - R + [\beta^1(R) - \beta^1(\widehat{R})]\pi(k) \geq 0. \tag{B1}$$

The term $\beta^1(R) - \beta^1(\widehat{R})$ must be nonnegative. Suppose not: then $\beta^1(R) < \beta^1(\widehat{R})$ would imply $\beta^1(R) < 1$ and therefore $T^1(R) = R$, as well as $T^1(R) < T^1(\widehat{R}) < R$, a contradiction. Hence, since (B1) holds for $k = k^*$, it must also hold for any larger k.

Step 4 of Proposition 2 can be applied to show that (W, T^1, β^1) must satisfy (7) and (8). I's expected stage-4 payoff can then be written as

$$\int_{\underline{\theta}}^{D/y(k)} y(k)\theta f(\theta)\mathrm{d}\theta + [1 - F(D/(y/k))], \tag{B2}$$

which is increasing in $y(k)$. Thus, if I's payoff is higher with (W, T^1, β^1) than with (W, T, β) for $k = k^*$, the same must be true for any larger k. As before, E can appropriate this increase by designing a new contract (W, T^2, β^2).

B.2 Investment incentives and the optimal contract

Suppose E and I write a simple debt contract $(W, T, \bar{\beta})$, where $\bar{\beta}(R) = 1 - (D - R)/\pi(k_0)$ and E and I expect E to choose k_0. Clearly, we can restrict our attention to contracts that set $W = k_0$. Define $u(R, \widehat{R}, k) = R - T(\widehat{R}) + \beta(\widehat{R})\pi(k)$. If E invests k, for $\widehat{R} \geq D$ we have $u(R, \widehat{R}, k) = R - D + \pi(k)$, and for $\widehat{R} < D$

$$u(R, \widehat{R}, k) = R - \widehat{R} + \left(1 - \frac{D - \widehat{R}}{\pi(k_0)}\right)\pi(k) = R - D + \pi(k) + (D - \widehat{R})\left(1 - \frac{\pi(k)}{\pi(k_0)}\right). \tag{B3}$$

Since $\pi' > 0$, inspection of (B3) shows that the contract induces truthtelling if $k \geq k_0$. If $k < k_0$, however, E would announce $\widehat{R} = 0$ and not make any payment to I. For the contract to be feasible, therefore, E must not have an incentive to choose any $k < k_0$.

If in stage 2 E invests k_0, he subsequently has an incentive to report his funds truthfully, and thus his expected payoff as of stage 2 is

$$y(k_0) - D + \pi(k_0) \tag{B4}$$

(recall that $W = k_0$). If he invests $k < k_0$, he will not repay anything in stage 4, and thus his expected payoff in stage 2 is

$$y(k) + \pi(k) + k_0 - k - \frac{D}{\pi(k_0)}\pi(k), \tag{B5}$$

which coincides with (B4) for $k = k_0$. Under our assumptions, (B5) has a unique maximum in k for given k_0; denote it by $\kappa(k_0)$. I would not agree to lend k_0 if he expected E subsequently to choose $k < k_0$; thus a simple debt contract is feasible only if $\kappa(k_0) \geq k_0$. Define the first-best investment as $k^{FB} = \arg\max y(k) + \pi(k) - k$. Since k^{FB} maximizes the first four terms in (B5), it follows that $\kappa(k^{FB}) < k^{FB}$. This means that no simple debt contract can induce E to choose k^{FB}; and by continuity, the same holds for all $k \in [\bar{k}, k^{FB}]$ for some $\bar{k} < k^{FB}$.

Denote by k^{SB} the solution to $\max_k y(k) - D(k) + \pi(k)$, where $D(k)$ solves (A4). If $\bar{k} \geq k^{SB}$, then the results of Section 5 continue to hold: A simple debt contract with $W = k^{SB}$ and $\bar{\beta} = 1 - (D - R)/\pi(k^{SB})$ induces the choice of $k^{SB} < k^{FB}$, and is optimal.

If $\bar{k} < k^{SB}$, it may be optimal to write a non-simple debt contract, such as of the form described in Propositions 4 and 5, to induce E to choose $k > \bar{k}$. As in Sections 6 and 7, however, both the benefit and the cost of using a non-simple contract are of first-order magnitude. If the cost of liquidating with higher probability exceeds the benefit of investing $k > \bar{k}$ even at the margin, then the optimal contract is again a simple debt contract, with $W = \bar{k}$ and $\bar{\beta} = 1 - (D - R)/\pi(\bar{k})$.[1]

[1] To illustrate, let $y(k) = \sqrt{k}$, $\pi(k) = \alpha y(k)$ for $\alpha > 0$, and assume that θ is uniformly distributed over [0,2]. Then $\bar{k} \geq k^{SB}$, and a simple debt contract is optimal if and only if $\alpha \geq 3/2$. If α is much smaller than 3/2, then a contract of the form (19) is preferred to a simple debt contract (but it is not necessarily the optimal contract); whereas if α is not much smaller than 3/2, a simple debt contract where E invests $\bar{k} < k^{SB}$ is preferred.