Core Allocations For a Salary-Adjustment Process With Worker Offers

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November 1997

Abstract

This paper extends the set of feasible algorithms for finding core allocations in a job-matching game. In Kelso and Crawford, Econometrica 1982, firms make offers to the workers. The reverse case of algorithms in which the workers make the offers was not analysed as it was expected to lead to technical difficulties. In fact, such algorithms can easily be constructed. They are similar to the ones described by Kelso and Crawford, and the same assumptions are critical.

Keywords: *matching, core, cooperative game theory, salary-adjustment process*

JEL code: *C78*

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1 Introduction

In their analysis of job-matching processes that lead to core allocations with non-homogeneous workers and firms, Kelso and Crawford (1982) restrict their attention to algorithms in which only the firms can make offers, and the workers can only accept or reject offers. This follows from the conjecture that the analogous algorithm in which workers make offers would not lead to clear results: "We have not considered the effects of reversing the roles of firms and workers in the adjustment process [...]. Because of the use made of our assumption that workers are indifferent about which other workers their firms hire, this seems to involve significantly greater difficulties in the non–separable case" (Kelso and Crawford (1982), fn. 4, p. 1494. See also Roth and Sotomayor (1990, p. 184)).

In fact, it is possible to design an algorithm that achieves exactly that: workers make offers to firms, and the algorithm leads to a 'worker-optimal' core allocation. The assumptions which are necessary are very similar to the ones necessary to characterise the analogous allocations using the algorithm with only the firms making offers. This emphasizes the importance of the "gross substitutability" assumption emphasized in Kelso and Crawford (1982).

2 Assumptions and Definitions

Expressions and notation are taken from Kelso and Crawford (1982) for simplicity. There is a set of firms $F, j \in F$, with production functions $y^j(C)$, where C is a set of workers from W. The firm's profit function is $\pi^j(C; s^j)$, where s^j is the vector of salaries s_{ij} paid to the workers $i \in C$. The workers have preferences over the firms and the salaries these firms pay. The preferences of worker i are expressed in the utility function $u^i(j; s_{ij})$. A matching is described by the correspondence $\mu : W \cup F \to W \cup F$.

The following assumptions are the 'No Free Lunch' and 'Gross Substitutability' assumptions from Kelso and Crawford (1982).

Assumption 1 (NFL) $y^j(\emptyset) = 0$ *for all firms* $j \in F$.

Assumption 2 (GS) *$M^j(s^j) := \{C \big| C \in \arg\max_C \pi^j(C; s^j)\}$ and $s^j \leq \tilde{s}^j, T^j(C^j) := \{i \big| i \in C^j$ and $\tilde{s}_{ij} = s_{ij}\}$. There exists $\tilde{C}^j \in M^j(\tilde{s}^j)$ such that $T^j(C^j) \subseteq \tilde{C}^j$.*

Each firm may employ several workers $i \in W$, but each worker is allowed to work for at most one firm. Workers are not interested in who else is being employed by the same firm.

Definition 1 *Individual rationality*

$\forall i : s_{i\mu(i)} \geq \sigma_{i\mu(i)}$, the worst outcome is $u^i(j; \sigma_{ij}) = u^i(\mu(i) = i; 0)$.

$\forall j : \pi^j \geq 0$, the worst outcome is $\pi^j(\mu(j) = j) = \pi^j(\emptyset) = 0$.

Definition 2 *Core allocations*

A (discrete) strict core allocation is an individually rational allocation $(\mu; C^j_\mu, s^j)$ such that there are no coalitions between firms and sets of workers (j, C) and (integer) salaries r^j that satisfy $u^i(j; r_{ij}) \geq u^i(\mu(i); s_{i\mu(i)})$ for all $i \in C$, and $\pi^j(C^j; r^j) \geq \pi^j(C^j_\mu; s^j_\mu)$, with strict inequality for at least one member of (j, C). A (discrete) core allocation is defined in the same way, except for all inequalities being strict.

3 The Salary-Adjustment Process With Worker Offers

In the first round all workers propose some ζ_{ij} to some firm. ζ_{ij} must be an exaggerated salary demand, which no firm will accept. The idea behind this is that the worker should not make initial demands too low, maybe below some stable assignment salary. By starting with much too high salary demands, only the number of periods the algorithm needs to find an equilibrium is higher, but under the given assumptions these extra rounds do not add any costs. For simplicity let ζ_{ij} be the highest profit that the firm can produce: $\zeta_{ij} = \max_{C \subseteq W} \pi^j(C, \sigma^j)$.

In all following rounds the workers who were not accepted in the preceding round are allowed to make offers to any firm, given the vector of permitted salaries $s_{ij}(t)$ in this round. They can either make an offer to the same firm that rejected the offer, demanding $s_{ij}(t+1) = s_{ij}(t) - 1$, or they can make an offer to another firm ℓ, demanding a permitted salary $s_{i\ell}(t+1) = s_{i\ell}(t)$. All workers who were accepted repeat their offer unchanged: $s_{ij}(t+1) = s_{ij}(t)$.

In each round the firms accept a preferred (profit–maximizing) subset of all offers they have (see Lemma 1 below.) Offers from earlier rounds may be rejected as well as new ones.

The algorithm stops at some round t^* if no offers are rejected.

The workers have preferences over the set of all combinations $(j; s_{ij})$ and, starting from ζ_{ij}, diminish their demanded utility levels after each rejection. In finite time they can reach their reservation salaries σ_{ij}, if enough rejections occur.

The firms have preferences over sets of workers and their demanded salaries, expressed in the profit function. The firms are only interested in maximizing profits. The individual marginal productivities of the workers and their salary demands are not of interest, only the absolute difference between these values does matter.

The acceptance and rejection rules of the firms need some comments. Without further restrictions it is not clear, why their situation is not worsened by following these rules, i.e. why doing so should not leave scope for coalitions later. One could imagine for instance that there are pairs of workers who together produce a high output, but not alone. Then in the process a single offer would not be accepted, but an incidentally pairwise offer in the same round would. This would allow for a coalition with this pair, if it is not matched to the firm. In order to exclude this, assumption (GS) is needed. With (GS) the firm will not incur any losses in the process by accepting in each round the profit-maximizing subset of the new and old offers. The proof that the (GS)-assumption is a sufficient condition for this is given below in Lemma 1.

The losses may not only stem from complementarities as described above, but also from the possibility that after rejecting an offer the same offer could be repeated with a lower salary demand. Both causes for losses are ruled out if the workers are "more substitutes than complements", i.e. by the competition between the workers for jobs that this substitutability produces.

4 Core Allocations

In this section two theorems are stated and proved. To simplify the comparison with the algorithm in Kelso and Crawford (1982), the formulations and proofs are as similar as possible in language and structure.

Lemma 1 *When assumption (GS) holds, the firms incur no losses in the process by accepting in each round the profit–maximizing subset of offers.*

Proof: The idea of the proof is that no firm when hearing an offer will ever regret to have rejected another offer in an earlier round. Since the firm can in every round reject every offer it has not yet rejected, there are no possible 'mistakes' if profits are maximized independently in each round. The proof is by contradiction, showing that no firm will be the first to feel such regrets. Suppose that in round τ firm j hears an offer $s_{\tau j}$ from w_τ, and regrets having rejected an offer $s_{\rho j}$ from w_ρ in a round $\rho < \tau$. Up to round $\tau - 1$ it did not regret the rejection, but when it hears w_τ's offer, it would like to add w_ρ with unchanged salary demand. During the adjustment process, the permitted salaries cannot have risen. All workers w_k who offer in round τ surely have demanded higher salaries at an earlier round: $s_{kj}(\rho) \geq s_{kj}(\tau)$ (In which round these earlier offers happened exactly does not matter, since the firm by hypothesis feels uncomfortable for the first time in round τ). Using the notation of assumption (GS), $\tilde{s}_{kj} \geq s_{kj}$ and $\tilde{s}_{\rho j} = s_{\rho j}$. Now, w_ρ is accepted when some coworkers are paid s_{ij}, but not when some of these coworkers are more expensive, contradicting (GS). ■

Theorem 1 *The salary-adjustment process converges in finite time to a core allocation in the discrete market for which it is defined.*

Proof: The proof follows from Steps 1–3

Step 1: In finite time, every worker is matched and the process stops.

This follows from the fact that after each rejection the demanded salaries are diminished. After a finite number of rounds (if all $\zeta_{ij} < \infty$) each worker will either have been accepted by some firm or the sinking salary demands will have reached the reservation salaries σ_{ij} (for all j). In the latter case a worker will be matched with himself: He will be unemployed.

Step 2: The salary-adjustment process converges to an individually rational allocation.

Say the process stops at round t^* with the matching μ. By voluntariness of the offers, $s_{i\mu(i)} \geq \sigma_{i\mu(i)}$ for all workers i who are matched to a firm. By the acceptance and rejection rules and by Lemma 1, for all firms $j \pi^j(C^j_\mu; s^j_\mu(t^*)) \geq 0$ (since $\pi^j(\emptyset) = 0$ can always be chosen.)

Step 3: The process converges to a core allocation in the discrete market for which it is defined.

By Step 1 the process converges to an equilibrium. Suppose this equilibrium μ is not a core allocation. Since by Step 2 μ is individually rational, there must be a coalition (j, C) and integer salaries r^j such that

$$u^i(j; r_{ij}) > u^i(\mu(i); s_{i\mu(i)}(t^*)) \forall i \in C, \qquad \text{and} \tag{1}$$

$$\pi^j(C, r^j) > \pi^j(C^j_\mu; s^j_\mu(t^*)). \tag{2}$$

Then j has never received an offer $s_{ij} \leq r_{ij}$ from i, which it clearly would have accepted. Since in the process the demanded salaries cannot rise, the permitted salary demands at round t^* satisfy $s_{ij}(t^*) \geq r_{ij}$.

Then $u^i(j; s_{ij}(t^*)) \geq u^i(j; r_{ij})$. Together with the hypothesis (1) a contradictory inequality follows:

$$u^i(j; s_{ij}(t^*)) > u^i(\mu(i); s_{i\mu(i)}(t^*)), \tag{3}$$

showing that μ could not have been an equilibrium. ■

Theorem 2 *The salary-adjustment process converges to a discrete strict core allocation that is at least as good for every worker as any other strict core allocation.*

Proof: For the proof that the salary-adjustment process converges to the core defined by weak dominance, the assumption is needed that in the core there is no indifference. That is, given any allocation in the core and the permitted salaries corresponding to it, there is no other allocation with the same permitted salaries that yields the same profits and utilities.

Suppose that given a core assignment a firm could form a coalition with some workers, netting a higher profit. All workers in this coalition who weren't contained in the original assignment of firm j must strictly prefer j to the firm they have been assigned to. Then Theorem 1 applies and the assignment cannot be an equilibrium.

To prove that for all w this core allocation is at least as good as any other core allocation, it can be shown that no firm would ever reject an offer s_{ij} that is part of any core allocation. This follows directly from Lemma 1, since each firm must in any core allocation have chosen a profit– maximizing set of workers. Therefore, all workers who are matched in any core allocation to a firm, will be accepted, if they demand the corresponding salary. And since the workers make offers following their preferences, the outcome will be for each worker the most preferred firm–salary combination in the core (yielding the highest utility). ■

References

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