Lecture 9

Conditional CAPM

The CAPM Revisited

• Let’s rewrite the CAPM DGP:

\[ R_{i,t} - r_f = \alpha_{i,t} + \beta_{i,t} (R_{m,t} - r_f) + \epsilon_{i,t} \]

\[ \beta_{i} = \text{Cov}(R_{i,t}, R_{m,t}) / \text{Var}(R_{m,t}) \]

• The CAPM can be written in terms of cross sectional returns. That is the SML:

\[ E[R_{i,t} - r_f] = \gamma_0 + \gamma_1 \beta_{i} \]

There is a linear constant relation between \( E[R_{i,t} - r_f] \) and \( \beta_{i} \).

• This version of the CAPM is called the static CAPM, since \( \beta_{i} \) is constant, or unconditional CAPM, since conditional information plays no role in determining excess returns.
• Q: Is beta unresponsive to (conditioning) information?
  • Suppose that in January we have information about asset i’s next dividend. Suppose this was true for every stock. Then, what should the risk/return tradeoff look like over the course of a year?

• Time-varying expected returns are possible.
• Q: What about time-varying risk premia?

• Other problems with an unconditional CAPM:
  – Leverage causes equity betas to rise during a recession (affects asset betas to a lesser extent).
  – Firms with different types of assets will be affected by the business cycle in different ways.
  – Technology changes.
  – Consumers’ tastes change.
  – One period model, with multi-period agents.

• In particular, the unconditional CAPM does not describe well the CS of average stock returns: The SML fails in the CS.

• The CAPM does not explain why, over the last forty years:
  - small stocks outperform large stocks (the “size effect”).
  
  - firms with high book-to-market (B/M) ratios outperform those with low B/M ratios (the “value premium”).
  
  -stocks with high prior returns during the past year continue to outperform those with low prior returns (‘momentum’).
The Conditional CAPM

• We have discussed a lot of anomalies that reject CAPM. Recall that some of the “anomaly” variables seemed related to $\beta$.

• Simple idea (“trick”) to “rescue” the CAPM: The ‘anomaly’ variables proxy for time-varying market risk exposures:

$$R_{i,t} - r_f = \alpha_{i,t} + \beta_{i,t} (R_{m,t} - r_f) + \epsilon_{i,t}$$

$$\beta_{i,t} = \frac{\text{Cov}(R_{i,t}, R_{m,t})}{\text{Var}(R_{m,t})} = \frac{\text{Cov}(R_{i,t}, R_{m,t}|I_t)}{\text{Var}(R_{m,t}|I_t)}$$

where $I_t$ represents the information set available at time $t$. (Note, the conditional cross-sectional CAPM notation used $I_{t-1}$ to represent the information set available at time $t$. Accordingly, they also use $\beta_{i,t-1}$.

$=> \beta_{i,t-1}$ is time varying. Conditional information can affect $\beta_{i,t-1}$.

• In the SML formulation of the CAPM (and using usual notation):

$$R_{i,t} - r_f = \gamma_{0,t-1} + \gamma_{1,t-1} \beta_{i,t-1} + \epsilon_{i,t}$$

• The SML is used to explain CS returns. Taking expectations:

$$E[R_{i,t} - r_f] = E[\gamma_{0,t-1}] + E[\gamma_{1,t-1}] E[\beta_{i,t-1}] + \text{Cov}(\gamma_{1,t-1}, \beta_{i,t-1})$$

If the $\text{Cov}(\gamma_{1,t-1}, \beta_{i,t-1})=0$ (or a linear function of the expected beta) for asset $i$, then we have the static CAPM back: expected returns are a linear function of the expected beta.

• In general, $\text{Cov}(\gamma_{1,t-1}, \beta_{i,t-1})\neq 0$. During bad economic times, the expected market risk premium is relatively high, more leveraged firms are likely to face more financial difficulties and have higher conditional betas.
Given $I_{t-1}$, $\text{Cov}(\gamma_{1,t-1}, \beta_{i,t-1}) = 0$ is testable.

- This is the base for conditional CAPM testing.

- Q: But, what is the right conditioning information set, $I_{t-1}$?
  Usually, papers condition on observables.
  - Estimation error and Roll’s critique are still alive.
  - If the variables in $I_{t-1}$ are chosen according to previous research, data mining problems are also alive and well.

- Q: How do we model $\beta_{i,t-1}$ –actually, how do we model $\text{Cov}(\gamma_{1,t-1}, \beta_{i,t-1})$?
  A great source of papers. The conditional CAPM is an ad-hoc attempt to explain anomalies. (Moreover, in general, theory does not tell us much about functional forms or conditioning variables.)

  $\Rightarrow$ It is up to the researchers to come up with $\beta_{i,t-1} = f(Z_{t-1})$.

- There are two usual approaches to model $\beta_{i,t-1}$:
  1. Time-series, where the dynamics of $\beta_{i,t-1}$ are specified by a time series model.
  2. Exogenous driving variables: $\beta_{i,t-1} = f(Z_{t})$, where $Z_{t}$ is an exogenous variable (say D/P, size, etc.). In general, $f(.)$ is linear.

  **Example:**
  \[
  \beta_{i,t-1} = \beta_{i,0} + \beta_{i,1} Z_{t}
  \]
  \[
  R_{i,t} = \alpha_{i} + (\beta_{i,0} + \beta_{i,1} Z_{t}) R_{m,t} + \varepsilon_{i,t}
  = \alpha_{i} + \beta_{i,0} R_{m,t} + \beta_{i,1} Z_{t} R_{m,t} + \varepsilon_{i,t}
  \]
  Now we have a multifactor model: easy to estimate and to test.

  Testing the conditional CAPM: $H_0: \beta_{i,1} = 0$. (A t-test would do it.)

  **Note:** An application of this example is the up-$\beta$ and down-$\beta$:
  \[
  Z_{t} = 1 \quad \text{if} \ g(R_{m,t-1}) > 0 \quad \text{-say,} \ g(R_{m,t-1}) = R_{m,t-1}
  \]
  \[
  Z_{t} = 0 \quad \text{otherwise.}
  \]
Conditional vs. Unconditional CAPM

- The conditional CAPM says that expected returns are proportional to conditional betas: \( E[R_{i,t}|I_{t-1}] = \beta_{i,t-1} \gamma_{t-1} \).

- Taking unconditional expectations:
  \[
  E[R_{t,t}] = E[\beta_{i,t-1}] E[\gamma_{t-1}] + \text{Cov}(\gamma_{t-1}, \beta_{i,t-1}) = \beta \gamma + \text{Cov}(\gamma_{t-1}, \beta_{i,t-1})
  \]

- The asset’s unconditional alpha is defined as:
  \[
  \alpha_u = E[R_{i,t}] - \beta_u \gamma
  \]

- Substituting for \( E[R_{i,t}] \) yields:
  \[
  \alpha_u = \gamma (\beta - \beta_u) + \text{cov}(\beta_{i,t-1}, \gamma_{t-1}).
  \]

- Note: Under some conditions, discussed below, a stock’s \( \beta_u \) and its expected conditional beta (\( \beta \)) will be similar.

- It can be shown (see Lewellen and Nagel (2006)):
  \[
  \alpha_u = [1 - \gamma^2/\sigma_m^2] \text{cov}(\beta_{t-1}, \gamma_{t-1}) - \gamma/\sigma_m^2 \text{cov}(\beta_{t-1}, (\gamma_{t-1} - \gamma)^2) - \gamma/\sigma_m^2 \text{cov}(\beta_{t-1}, \sigma_{m,t}^2)
  \]

- Some implications:
  - It is well known that the conditional CAPM could hold perfectly, period-by-period, even though stocks are mispriced by the unconditional CAPM. Jensen (1968), Dybvig and Ross (1985), and Jagannathan and Wang (1996).

  - A stock’s conditional alpha (or pricing error) might be zero, when its \( \alpha_u \) is not, if its beta changes through time and is correlated with the equity premium or with conditional market volatility.

  - That is, the market portfolio might be conditionally MV efficient in every period but, at the same time, not on the unconditional MV efficient frontier. Hansen and Richard (1987).
Application 1: International CAPM

• From the CAPM DGP, the International CAPM can be written:
  \[ R_{i,t} = \alpha_i + \beta_i R_{w,t} + \epsilon_{i,t} \]
  \[ \beta_i = \frac{\text{Cov}(R_{i,t}, R_{w,t})}{\text{Var}(R_{w,t})} \]

• Using a bivariate GARCH model, we can make \( \beta \) time varying:
  \[ \beta_{i,t} = \frac{\text{Cov}_t(R_{i,t}, R_{w,t})}{\text{Var}_t(R_{w,t})} \]

• A model for the World factor is needed. Usually, an AR(p) model:
  \[ R_{w,t} = \delta_0 + \delta_1 R_{w,t-1} + \epsilon_{w,t} \]
  where \( \epsilon_{w,t} \) and \( \epsilon_{i,t} \) follow a bivariate GARCH model.

Mark (1988) and Ng (1991) find significant time-variation in \( \beta_{i,t} \).
• **Braun, Nelson and Sunier** (1995): Use an E-GARCH framework, where $\beta_{i,t}$ also respond asymmetrically to positive versus negative domestic ($\epsilon_{i,t}$) or world news ($\epsilon_{w,t}$).

$$R_{i,t} = \alpha_i + \beta_i(\epsilon_{i,t}, \epsilon_{w,t}) R_{w,t} + \epsilon_{i,t}$$

They find no significant time-variation evidence for their version of $\beta_{i,t}$.

• **Ramchand and Susmel** (1998): use a SWARCH model, where $\beta_{i,t}$ is state dependent:

$$R_{i,t} = \alpha_i + (\beta_{i,0} + \beta_{i,1} S_t) R_{w,t} + \epsilon_{i,t},$$

where $\epsilon_{i,t}$ follows a SWARCH model.

Strong evidence for state dependent $\beta_{i,t}$ in Pacific and North American markets, not that significant in European markets.

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**US: World Beta-varying coefficients using SWARCH model**

![Graph](image-url)
• **Bekaert and Harvey** (1995): Study a conditional version of the ICAPM for emerging markets’ stocks, where beta is conditioned on an unobservable state variable that takes on the value of zero or one.

\[ R_{it} = \alpha + \beta_1 (1-S_t) R_{m,t-1} + \beta_2 S_t R_{w,t-1} + \epsilon_{w,t} \]

where \( S_t \) is an unobservable state variable, which they considered linked to the degree of the emerging market’s integration with a world benchmark.

They find evidence for time variation on \( \beta_1 \) and \( \beta_2 \), somewhat consistent with partial integration.

**Note:** These International CAPM papers do not use exogenous observable information. These papers focus on the time-series side of expected returns. They provide a very simple way of constructing time-varying betas.

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**Application 2: CS returns**

• **Ferson and Harvey** (1993): Attempt to explain the CS expected returns across world stock markets.

  • FH make \( \alpha_{it} \) and \( \beta_{it} \) linear function of variables such as dividend yields and the slope of the term structure.

\[ R_{it} = (\alpha_{0i} + \alpha'_{1i} Z_{t-1} + \alpha'_{2i} A_{it-1}) + (\beta_{0i} + \beta'_{1i} Z_{t-1} + \beta'_{2i} A_{it-1}) R_{m,t} + \epsilon_{it} \]

  \( Z_{t-1} \): global variables (“instruments”) that affect all assets –say, interest rates, world and national factors.

  \( A_{it-1} \): asset specific variables (“instruments”) –say, P/E, D/P, volatility.

**Note:** “Instruments,” since they are pre-determined at \( t \).
• FH find several instruments to be significant – i.e., $\text{Cov}(\gamma_{1,t-1}, \beta_{i,t-1}) \neq 0$.
  - Betas are time-varying, mostly due to local variables: $E/P$, inflation, long-term interest rates.
  - Alphas are also time-varying, due to: $E/P$, $P/CF$, $P/BV$, volatility, inflation, long-term interest rates, and the term spread.
  - Economic significance: typical abnormal return (in response to 1σ change in X) around 1-2% per month

Overall, however, the model explains a small percentage of the predicted time variation of stock returns.

Note: Ferson and Korajczyck (1995), though, using a similar model for the U.S. stock market, cannot reject the constant $\beta_i$ model.

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• Jagannathan and Wang (1996): Work with the SML to explain CS returns:
  
  $$E[R_{i,t} - r_f] = E[\gamma_{0,t-1}] + E[\gamma_{1,t-1}] E[\beta_{i,t-1}] + \text{Cov}(\gamma_{1,t-1}, \beta_{i,t-1})$$

• They decompose the conditional beta of any asset into 2 orthogonal components by projecting the conditional beta on the market risk premium.

  - For each asset $i$, JW define the beta-premium sensitivity as
    
    $$\upsilon_i = \text{Cov}(\gamma_{1,t-1}, \beta_{i,t-1}) / \text{Var}(\gamma_{1,t-1})$$
    
    $$\eta_{i,t-1} = \beta_{i,t-1} - E[\beta_{i,t-1}] - \upsilon_i (\gamma_{1,t-1} - E[\gamma_{1,t-1}])$$

    $\upsilon_i$ measures the sensitivity of the conditional beta to the market risk premium.

Then, rewriting the last equation as a regression:

$$\beta_{i,t-1} = E[\beta_{i,t-1}] - \upsilon_i (\gamma_{1,t-1} - E[\gamma_{1,t-1}]) + \eta_{i,t-1}$$

where $E[\eta_{i,t-1}] = E[\gamma_{1,t-1}, \eta_{i,t-1}] = 0$. 
Now, the conditional beta can be written in three parts:
- The expected (unconditional) beta.
- A random variable perfectly correlated with the conditional market risk premium.
- Something mean zero and uncorrelated with the conditional market risk premium.

Going back to the SML:

$$E[R_{i,t} - r_f] = E[\gamma_{0,t-1}] + E[\gamma_{1,t-1}] E[\beta_{i,t-1}] + \nu_i \text{Var}(\gamma_{1,t-1})$$

The unconditional expected return on any asset $i$ is a linear function of
- Expected beta
- Beta-prem sensitivity, the larger the sensitivity, the larger the variability of the “second part” of the conditional beta.

Note: The beta-prem sensitivity measures instability of $\beta_i$ over the business cycle. Stocks with $\beta_i$ that vary more over the cycle have higher $E[R_{i,t} - r_f]$.

- We are back to the Fama-MacBeth (1973) CS estimation.
- To estimate the model, we need to estimate:
  - Expected beta: $E[\beta_{i,t-1}]$
  - Estimates of beta-prem sensitivity: $\nu_i$.

- We can see $\eta$ does not affect expected returns, it affect $\beta_{i,t-1}$. Thus, we can concentrate on the first two parts of the conditional beta.

- We need to make assumptions about the stochastic process governing the joint temporal evolution of $\beta_{i,t-1}$ and $\gamma_{1,t-1}$. 
• Usually, the JW-type conditional CAPM is estimated using the following SML formulation:

\[ E[R_{i,t} - r_f] = \gamma_0 + \gamma_1 E[\beta_{i,t-1}] + \lambda_i \]

where \( E[\beta_{i,t-1}] \) will be an average beta for asset \( i \) and \( \lambda_i \) measures how the stock’s beta co-varies though time with the risk premium. Different assumptions will deliver different \( E[\beta_{i,t-1}] \) and \( \lambda_i \).

• Findings: JW find that the betas of small, high-B/M stocks vary over the business cycle in a way that, according to JW, largely explains why those stocks have positive unconditional alphas.

• Lettau and Ludvigson (2001), Santos and Veronesi (2005), and Lustig and Van Nieuwerburgh (2005) find similar results. All papers find a dramatic increase in \( R^2 \) for their conditional models.

• **Lettau and Ludvigson (2001):** Estimate how a stock consumption betas change with the consumption-to-wealth ratio, or CAY:

\[ \beta_{i,t} = \beta_i + \delta_i \text{CAY}_t \]

where \( \beta_i \) and \( \delta_i \) are estimated in the first-pass regression:

\[ R_{i,t} = \alpha_{i0} + \alpha_{i1} \text{CAY}_t + \beta_i \Delta c_i + \delta_i \text{CAY}_t \Delta c_i + e_{it}, \]

\( \text{CAY}_t \) is the consumption residuals from a Stock and Watson (1993) cointegrating regression, with assets \( (a_t) \) and labor income \( (y_t) \):

\( \text{CAY}_t = c_t - 0.31 a_t - 0.59 y_t - 0.60. \)

Then, substituting \( \beta_{i,t} \) into the unconditional relation gives:

\[ E[R_{i,t}] = \beta_i \gamma + \delta_i \text{cov}(\text{CAY}_t, \gamma_t). \]

**Note:** There are some econometric issues here. Wealth (human capital) is not observable. Stationarity of proxy is an empirical matter.

• LL call their model a conditional C-CAPM. (More on Lecture 10.)
• LL use as γ a market returns and Δyt or Δct to estimate the SML.

• They also include other variables in the SML to test their conditional C-CAPM: Size and B/M. (Traditional omitted variables test)

• Note: LL’s model implies that the slope on βi should be the average consumption-beta risk premium and the slope on δi should be \(\text{cov}(CAY_t, γ_t)\).

• Class comment: Check the last row (6) on Table 6, Panel B –taken from LL. No coefficient has a significant t-stat, but R² is huge (.78)! Multicollinearity problem? (Recall that multicollinearity affects the standard errors, but not the estimates. The estimates are unbiased)

\[
\begin{array}{cccccc}
\text{Row} & \text{Constant} & \hat{R}_m & \Delta \alpha & \hat{R}_m & \Delta \beta & \text{t} \\
\hline
1 & 1.12 & -0.60 & -0.70 & 1.12 & -0.60 & -0.70 \\
(0.77) & (-2.59) & (-3.67) & (0.77) & (-2.59) & (-3.67) \\
2 & 0.13 & -0.80 & 0.82 & -0.40 & -0.70 & 1.24 \\
(0.78) & (-1.28) & (1.00) & (-0.40) & (-0.70) & (2.00) \\
3 & 0.09 & 0.26 & 0.26 & 0.26 & 0.26 & 0.26 \\
(0.46) & (-0.26) & (0.26) & (-0.26) & (0.26) & (0.26) \\
4 & -0.93 & 0.10 & -0.93 & 0.10 & -0.93 & 0.10 \\
(0.73) & (-1.00) & (0.73) & (-1.00) & (0.73) & (0.73) \\
0.09 & -0.84 & 0.09 & -0.84 & 0.09 & -0.84 \\
(0.48) & (-0.90) & (0.48) & (-0.90) & (0.48) & (0.48) \\
5 & 0.30 & 0.04 & 0.30 & 0.04 & 0.30 & 0.04 \\
(0.04) & (0.58) & (0.04) & (0.58) & (0.04) & (0.58) \\
6 & 0.09 & -0.10 & 0.09 & -0.10 & 0.09 & -0.10 \\
(0.21) & (-1.01) & (0.21) & (-1.01) & (0.21) & (-1.01) \\
7 & 0.05 & -0.10 & 0.05 & -0.10 & 0.05 & -0.10 \\
(0.05) & (-0.10) & (0.05) & (-0.10) & (0.05) & (-0.10)
\end{array}
\]

\[
\begin{array}{cccccc}
\text{Row} & \text{Constant} & \hat{R}_m & \Delta \alpha & \hat{R}_m & \Delta \beta & \text{t} \\
\hline
1 & 1.12 & 1.17 & -0.20 & 1.12 & 1.17 & -0.20 \\
(0.08) & (1.08) & (0.08) & (1.08) & (0.08) & (1.08) \\
2 & 2.25 & 1.45 & 0.15 & 2.25 & 1.45 & 0.15 \\
(0.01) & (1.08) & (0.01) & (1.08) & (0.01) & (1.08) \\
3 & 1.91 & 0.30 & 1.91 & 0.30 & 1.91 & 0.30 \\
(0.21) & (0.45) & (0.21) & (0.45) & (0.21) & (0.45) \\
4 & 1.39 & 0.01 & 1.39 & 0.01 & 1.39 & 0.01 \\
(0.96) & (-0.19) & (0.96) & (-0.19) & (0.96) & (-0.19) \\
5 & 0.69 & 0.69 & 0.69 & 0.69 & 0.69 & 0.69 \\
(0.03) & (0.03) & (0.03) & (0.03) & (0.03) & (0.03) \\
6 & 0.99 & -0.02 & 0.99 & -0.02 & 0.99 & -0.02 \\
(0.03) & (0.03) & (0.03) & (0.03) & (0.03) & (0.03) \\
7 & 0.95 & -0.02 & 0.95 & -0.02 & 0.95 & -0.02 \\
(0.03) & (0.03) & (0.03) & (0.03) & (0.03) & (0.03) \\
\end{array}
\]

\[\text{Note: } \text{This table presents estimates of consumption-based Fama-French regressions using the variance on the Fama-French portfolio.} \]

\[\text{R}_m, \text{R}_f = \text{Fama-French portfolio.} \]

\[\text{R}_m, \text{R}_f = \text{Fama-French portfolio.} \]
Conditional CAPM: Does it Work?

• Lewellen and Nagel (2006): argue that variation in betas and the equity premium would have to be implausibly large to explain the asset pricing anomalies like momentum and the value premium.

• LN use a simple test of the conditional CAPM using direct estimates of conditional $\alpha$ and $\beta$ from short-window regressions –i.e., assuming that $\alpha$ and $\beta$ do not change in the estimation window. (Maybe, not a trivial assumption during some periods.)

• LN claim that they are avoiding the need to specify $I_t$.
• Fama and French (1993) methodology, adding momentum factor.
• LN estimate $\alpha$ and $\beta$ quarterly, semiannually, and annually.

Table 3
Average conditional alphas, 1964 – 2001
The table reports average conditional alphas for size, B/M, and momentum portfolios (% monthly). Alphas are estimated quarterly using daily returns, semiannually using daily and weekly returns, and annually using monthly returns. The portfolios are formed from all NYSE and Amex stocks on CRSP / Compustat. We begin with 25 size-B/M portfolios (5×5 sort, breakpoints determined by NYSE quantiles) and 10 return-sorted portfolios, all value weighted. 'Small' is the average of the five low-market-cap portfolios, 'Big' is the average of the five high-market-cap portfolios, and 'S-B' is their difference. Similarly, 'Growth' is the average of the five low-B/M portfolios, 'Value' is the average of the five high-B/M portfolios, and 'V-G' is their difference. Return-sorted portfolios are formed based on past 6-month returns. 'Losers' is the bottom decile, 'Winners' is the top decile, and 'W-L' is their difference. Bold denotes estimates greater than two standard errors from zero.

<table>
<thead>
<tr>
<th></th>
<th>Size</th>
<th>B/M</th>
<th>Momentum</th>
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<tr>
<td></td>
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<td>Big</td>
<td>S-B</td>
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<tr>
<td>Quarterly</td>
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<td>0.00</td>
<td>0.42</td>
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<td>0.00</td>
<td>0.26</td>
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<td>0.08</td>
<td>-0.14</td>
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<tr>
<td>Standard error</td>
<td></td>
<td></td>
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<tr>
<td>Quarterly</td>
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<td>0.06</td>
<td>0.22</td>
</tr>
<tr>
<td>Semiannual 1</td>
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<td>0.06</td>
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<tr>
<td>Semiannual 2</td>
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<td>0.23</td>
</tr>
<tr>
<td>Annual</td>
<td>0.26</td>
<td>0.07</td>
<td>0.29</td>
</tr>
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</table>

Quarterly and Semiannual 1 alphas are estimated from daily returns, Semiannual 2 alphas are estimated from weekly returns, and Annual alphas are estimated from monthly returns.
• **Findings:** The conditional CAPM performs nearly as poorly as the unconditional CAPM.
  - The conditional alphas (pricing errors) are significant.
  - The conditional betas change over time. But, not enough to explain unconditional alphas. (Not enough co-variation with the market risk premium or volatility.)

• LN have a final good insight on Conditional CAPM tests:
  - LN Conditional CAPM models estimate a restricted version of the SML, imposing a constraint on the slope of $\lambda_i$. The slope of $\lambda_i$ is equal to 1:
    \[
    E[R_{it} - r_f] = \gamma_0 + \gamma_1 E[\beta_{it-1}] + \lambda_i
    \]
    
In their tests, LN reject this restriction.