Lecture 11

Dynamic Asset Pricing Models - II

Fixing the C-CAPM

The risk-premium puzzle is a big drag on structural models, like the C-CAPM, which are loved by economists. A lot of efforts to salvage them:

• (1) Power utility function is too strict, since EIS and $\gamma$ are unrealistically linked. Epstein-Zin-Weil’s effort. Hansen, Heaton and Li (2007).


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**Habit Formation**

- Constantinides (1990): Average Joe’s utility depends on the difference between current consumption and the “habit”:
  \[ U(c) = E_t \sum_i \beta_i [(c_{t+i} - x_{t+i})^{1-\gamma}]/(1-\gamma), \quad \lambda > 0. \]

Let the habit depend on lagged consumption (an *internal* habit):

\[ x_{t+1} = \lambda c_{t+i} \]

\[ U'(c_{t+i}) = E_t \{ \beta_i [(c_{t+i} - \lambda c_{t+i})^{\gamma} + \beta^{i+1} (c_{t+i+1} - \lambda c_{t+i})^{\gamma} - \lambda \} \}

\[ = \beta_i c_{t+i}^{\gamma} ((c_{t+i}/c_{t+i-1})^{\gamma} - \lambda) E_t[((c_{t+i-1}/c_{t+i})^{\gamma} - \lambda)^{\gamma}] \]

\[ = \beta_i c_{t+i}^{\gamma} (g_{t+i-1}^{\gamma} - \lambda \beta g_{t+i}^{\gamma} E_t[(g_{t+i-1}^{\gamma} - \lambda)^{\gamma}) (g_{t+i-1} = c_{t+i}/c_{t+i-1}) \]

If we assume that $g_{t+i}$ is iid (not a realistic assumption):

\[ E_t[(g_{t+i-1}^{\gamma} - \lambda)^{\gamma}] = E_t[(g_{t-i}^{\gamma} - \lambda)^{\gamma}] = E[(g_{t}^{\gamma} - \lambda)^{\gamma}] = \Theta \text{ (a constant)} \]
• Then, recall that \( m_{t+1} = \frac{U'(c_{t+1})}{U'(c_t)} \)

\[
m_{t+1} = \beta g_t^{\gamma - 1} \{(g_{t+1} - \lambda \beta g_{t+1}^{\gamma - 1} \Theta) - (g_t - \lambda \beta g_t^{\gamma - 1} \Theta)\} \]

Compare this to \( m_{t+1} \) from the standard model:

\[
m_{t+1} = \beta g_{t+1}^{\gamma - 1} \]

• Now, since \( \lambda > 0 \), and \( (g_{t+1} - \lambda) < 1 \) is more often than \( g_{t+1} < 1 \). Thus, we do not need a large risk-aversion coefficient to amplify the variation of consumption.

Note: If \( \lambda = 1 \) and the habit is fixed, the utility function becomes:

\[
U(c) = E_t \sum_i \beta^i [(C_{t+i} - X_{t+i})^{1-\gamma} - 1] / (1 - \gamma) \quad x: \text{fixed subsistence level (habit).}
RRA = \gamma / (1 - x/c_t) \quad (\text{if } x/c_t = 0.8, \text{RRA} = 5 \gamma.)
\]

• But, risk aversion is now time-varying. This utility function makes the agent very averse to consumption risk. It helps to address the low risk-free rate.

• **Campbell and Cochrane’s (1999):** Take on Abel’s (1990) formulation of habit formation. Let \( U(c) \) be

\[
U(c) = E_t \sum_i \beta^i [(C_{t+i} - X_{t+i})^{1-\gamma} - 1] / (1 - \gamma),
\]

where \( x_i \) is the level of habit.

CC work with “surplus consumption ratio” – their state variable:

\[
S_t = (C_t - X_t)/C_t
\]

CC define the habit \( x_t \) as external: Average Joe looks at aggregate consumption (the “Joneses”) to determine his level of happiness:

\[
S^a_t = [(C^a_t - X_t)/C^a_t)
\]

Now, \( m_{t+1} = \beta[(C_{t+1}/C_t)(S_{t+1}/S_t)]^{\gamma} \)

CC assume \( s^a_t = \log(S_t) \) follows an heteroscedastic AR(1) process:

\[
s^a_t = (1-\phi) s^a_t + \phi s^a_{t-1} + \lambda s^a_t (c^a_t - c^a_{t-1} - g)
\]

CC assume \( \Delta c_t = \log(c_t/c_{t-1}) \sim \text{iid lognormal } (g, \sigma^2) \).
Now, the risk-free rate:
\[ r_f = -\ln(\beta) + \gamma g - \gamma(1-\varphi)(s_t-s) - \gamma^2 \sigma^2/2 \left[ 1 + \lambda(s_t) \right]^2 \]

CC have to choose \( \lambda(s_t) \) to proceed. They use a simple threshold function – i.e., with two (implicit) surplus states.

• CC explain the volatility puzzle and the low interest rate puzzle. But they also need a high risk aversion coefficient to match the equity premium (average risk aversion 80, in low surplus state gets to 100!).

• CC find that agents fear stocks primarily because they do badly in recessions (times of low surplus consumption ratios), not because stock returns are correlated with declines in wealth or consumption.

• CC conclude that habit formation, or some other device is needed to generate time-varying countercyclical risk premia along with relatively constant risk-free rates.

Separating EIS (\( \psi \)) and \( \gamma \)

• Epstein and Zin (1991) – Weil (1989): They work with a recursive utility function:
\[ U_t = \{(1-\delta)C_t^{(1-\gamma)/\theta} + \delta(E_t[U_{t+1}^{(1-\gamma)/\theta}])^{(1-\gamma)\theta} \}^{\theta/(1-\gamma)} \]
where \( \theta = (1-\gamma)/(1-1/\psi) \). When \( \theta=1 \), the recursion becomes linear (usual power utility model).

• The intertemporal budget constraint is
\[ W_{t+1} = (1+R_{m,t+1}) (W_t - C_t) \quad (W_t = \text{Average Joe’s wealth.}) \]

• Assuming lognormality and homoscedasticity:
\[ r_{f,t+1} = -\ln \delta + (\theta-1)/2 \sigma_m^2 - \theta/(2\psi^2) \sigma_c^2 + 1/\psi E_t[\ln(c_{t+1}) - \ln(c_t)] \]
• Then, the risk premium for asset i is:
  \[ E[r_{i,t+1}] - r_{f,t+1} = \frac{\theta}{\psi} \sigma_{ic} - (1 - \theta) \sigma_{im} - \sigma_{c}^2/2 \]

The Epstein-Zin-Weil model nests the static CAPM (\(\theta=0\)) and the C-CAPM with power utility (\(\theta=1\)).

• Campbell (1993) introduces a log-linearized budget constraint and after a lot of substitutions, we get:
  \[ E[r_{i,t+1}] - r_{f,t+1} = \gamma \sigma_{im} + (1 - \gamma) \sigma_{ih} - \sigma_{i}^2/2, \]

where \(\sigma_{ih}\) is the covariance of asset i with “news” about future returns on the market –see CLM.

• Static CAPM?
  - \(\gamma = 1\) (no realistic)
  - \(\sigma_{ih} = 0\) (no realistic)
  - \(R_{m,t}\) follows a univariate process, then future returns are perfectly correlated with current return. (Maybe?).

• Findings: Equity premium estimates using this model are lower \(\gamma\). But, with an unrealistic consumption volatility.
More General Utility Functions

• Introduce nonseparabilities: consumption and some other good.

• Usually done using Cobb-Douglas utility functions. Then, we have an easy to work marginal utility function:

\[ u'(C_t, X_t) = C_t^{-\gamma_1} X_t^{-\gamma_2} \]

where \( X_t \) represents some other good.

• Now, we have an easy to work Euler’s equation:

\[ 1 = E_t[\beta(C_{t+1}/C_t)^{-\gamma_1}(X_{t+1}/X_t)^{-\gamma_2}(1+R_{t+1})]. \]

Assuming joint lognormality and homoscedasticity:

\[ E_t[r_{t+1}] = \mu_i + \gamma_1 E_t[\Delta c_{t+1}] + \gamma_2 E_t[\Delta x_{t+1}], \]

Other goods: leisure - Eichenbaum, Hansen and Singleton (1988)  
government spending – Aschauer (1985)  
stock of durable goods – Startz (1989)

Findings: None of the \( X_t \) variables significantly improves the CAPM.

• Surprised? Not really. The proposed \( X_t \) variables do not have enough variability to explain the variability of excess returns.
Ex-ante ≠ Ex-post: Survey Evidence

- C-CAPM: $E[r_t] - r_{ft} = \text{Cov}(r_t, \Delta c_t) \gamma = \text{Corr}(r_t, \Delta c_t) \sigma_c \sigma_m \gamma$

$E[r_t] - r_{ft} = .06 = \text{Corr}(r_t, \Delta c_t) \sigma_c \sigma_m \gamma = 0.17 \times 0.33 \times 0.14 \times 0.01 \times \gamma$
=> Requires risk aversion, $\gamma$, to be very high: $\gamma \geq 43!$

- Soderling (2006) tests the C-CAPM using ex-ante data:
  - For $E[r_t] - r_{ft}$, survey data (Livingston survey (1952-2005))
  - For $\sigma_c$, survey data (SPF: survey of professional forecasters)
  - For $\sigma_m$, no survey data. Expectations extracted from S&P options.
  - For $\text{Corr}(r_t, \Delta c_t)$, no expectations data.

- This approach avoids the usual “joint test:” C-CAPM and RE.

Data:
- SPF: survey of economic experts about probabilities of GDP growth rates –assume results carry over to consumption growth.

Example: Excess returns survey

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<thead>
<tr>
<th></th>
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</thead>
<tbody>
<tr>
<td>Survey expectations</td>
<td>-1.1</td>
<td>1.5</td>
<td>-0.4</td>
</tr>
<tr>
<td>S&amp;P Industrials, ex post</td>
<td>3.2</td>
<td></td>
<td></td>
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<tr>
<td>S&amp;P 500, ex post</td>
<td></td>
<td>5.3</td>
<td></td>
</tr>
<tr>
<td>S&amp;P combined, ex post</td>
<td></td>
<td></td>
<td>3.8</td>
</tr>
<tr>
<td>Dividend yield S&amp;P 500</td>
<td>4.0</td>
<td>2.1</td>
<td>3.5</td>
</tr>
</tbody>
</table>
• $E[r_t] - r_{f,t} \approx 3\%-3.5\%$ (about half the historical mean excess return)

• Volatile market, hard to “learn.” Test for zero average error.
  Average forecast error = 3%, SD of forecast error = 16%, $T=100$
  \[ t = 3/(16/100^{1/2}) = 1.9 \]

• Ex-ante estimates:
  - $E[r_t] - r_{f,t} \approx 0.5 \times 0.06 = 0.03$
  - $\sigma_c \approx 0.6 \times 0.01 = 0.006$
  - $\sigma_m \approx 1.15 \times 0.14 = 0.161$
  - $\text{Corr}(r_t, \Delta c_t) = ?$

$E[r_t] - r_{f,t} = 0.03 = \text{Corr}(r_t, \Delta c_t) \sigma_c \sigma_m \gamma = 0.17 \times 0.33 \times 0.006 \times \gamma$

  \[ \Rightarrow \text{Now, } \gamma \text{ needs to be } 0.7 \text{ of } \gamma \text{ in ex-post data (30 instead of 43)} \]

• Better for C-CAPM, but still a very high $\gamma$
Learning

• Cogley and Sargent (2008): Follow Friedman and Schwartz (1963), where the U.S. 1930 depression created a pessimist mood, seriously affected estimates of expected returns. (The “depression generation.”)
• Siegel (1992) presents supporting evidence: 1802-1925 Equity premium was 2.0%; 1926-1990 Equity premium was 5.9%.

• CS present a learning model to explain the equity premium.
• Assumptions:
  – Consumption is driven by a two-state Markov process.
  – Representative agent is a Bayesian learner.
  – Initial beliefs from the 1930s are very pessimistic.
  – Learning is slow.
  – Asset pricing is distorted by these beliefs for a long time.

• Related work: Cecchetti, Lam, Mark (2000).
  Equity premium due to distorted beliefs. CLM (2000) have no learning. Agents are naturally and permanently pessimistic.

• Under Euler’s equation we have:
  \[ p_t = E_t^s [m_{t+1} x_{t+1}] \]
  If \( E_t^s \) is the expectation implied by the true transition probabilities \( F \), the agent has rational expectations. Call this \( E_t^a \). The equity premium will be small in that case.

  CLM (2000) show that \( E_t^s \neq E_t^a \) explain equity premium. But, we have permanently distorted beliefs.

  CLM (2000) pessimistic views come from agents assigning a larger probability to the bad (“depression”) state.
CS model consumption as:

$$\Delta \ln(C_t) = \mu(S_t) + \varepsilon_t,$$

where $S_t$ is a state (H, L) variable. $S_t$ moves according to the transition matrix $F$, with elements $F_{ij} = \text{Prob}[S_{t+1} = j | S_t = i]$ - a Hamilton process.

<table>
<thead>
<tr>
<th>Table 1: Maximum Likelihood Estimates of the Consumption Process</th>
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</thead>
<tbody>
<tr>
<td>$F_{hh}$</td>
</tr>
<tr>
<td>----------</td>
</tr>
<tr>
<td>Estimate</td>
</tr>
<tr>
<td>Standard Error</td>
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</tbody>
</table>

Note: Reproduced from Cecchetti, et. al. (2000)

Note: $F_{ll}$, the probability that a contraction will continue is estimated at 0.515 with a standard error of 0.264. A 90% C.I.: [.079, 0.951], which implies that contractions could have a median durations ranging from 3 months to 13 years.

=>$> even with 100 years of data, big model uncertainty persists.

• Back to CS
• CS allows Average Joe to learn their way out of their pessimism.

• Simplify model by setting $\varepsilon_t=0$ (simpler learning problem), but assume:

  $$g_t = \begin{cases} 
  1+\mu(h)/100 & \text{if } S_t=H 
  
  1+\mu(l)/100 & \text{if } S_t=L
  \end{cases}$$

• CS also take CLM’s $\mu(h)$ and $\mu(l)$ –known by the agents- and $F_{ll}$ and $F_{hh}$ –unknown. Average Joe applies Bayes’s theorem to learn $F_{ij}$.

• Average Joe uses a beta-binomial probability model for learning about $\Delta \ln(C_t)$. A binomial likelihood is a good candidate, a beta density is the conjugate prior for a binomial likelihood.

• Average Joe has independent beta priors over $(F_{hh}, F_{ll})$.

• Average Joe counts the number of transitions from state $i$ to state $j$ ($n_t^{ij}$) through date $t$ and learns. The agent has a prior ($n_0^{ij}$).
• Average Joe has independent beta priors over \((F_{hh}, F_{ll})\).

• Average Joe counts the number of transitions from state i to state j \((n_{ij}^t)\) through date \(t\) and learns. The agent has a prior \((n_{ij}^0)\). Then:

\[
p(F_{hh}, F_{ll}) = p(F_{hh})p(F_{ll}),
\]

where

\[
p(F_{hh}) \propto F_{hh}^{n^h_{hh} - 1}(1 - F_{hh})^{n^l_{hh} - 1},
\]

\[
p(F_{ll}) \propto F_{ll}^{n^l_{ll} - 1}(1 - F_{ll})^{n^h_{ll} - 1}.
\]

• Average Joe counts the number of transitions from state i to state j \((n_{ij}^t)\) through date \(t\) and learns. The agent has a prior \((n_{ij}^0)\).

• The updating (learning) rule:

\[
\begin{align*}
    n_{t+1}^{ij} &= n_{t}^{ij} + 1 & \text{if } s_{t+1} = j \text{ and } s_t = i, \\
    n_{t+1}^{ij} &= n_{t}^{ij} & \text{otherwise}.
\end{align*}
\]

• The date-\(t\) estimate of \(F\) is formed from the counters:

\[
F_t = \begin{bmatrix}
    n^{hh}_{t} & n^{hl}_{t} \\
    n^{lh}_{t} & n^{ll}_{t}
\end{bmatrix},
\]

• The agent makes decisions based on the values in this \(F\) matrix. The values are treated as constants when making decisions, but are random variables until convergence to RE equilibrium.

• CS need a starting point, where agents are “pessimistic” (or with “shattered” beliefs, because of the depression). Think of a “worst case” \(F=FWC\). (The WC model should be hard to reject in a training sample \(T_0\)).

Note: This is partial equilibrium, as many in the asset pricing literature. The agent cannot affect the system by changing beliefs. Hence, there is no active learning incentive. Agents “wait.” Learn about \(F_{hl}\) complicated.
**Estimation**

- CS draw 1,000 consumption growth paths of 70 years each, assuming the true Markov chain given by F and \(\mu(h)\) and \(\mu(l)\).

- Pessimistic Average Joe is endowed with \(F^{WC}\). Then, he determines asset prices using Euler’s equation and applies Bayes rule each period.

\[
P_t(S_t=i,C_t) = \beta \sum_j F_{ij}(t) (g_{j,t+1})^{-\gamma} [P_{t+1}(S_{t+1}=j; g_{j,t}+1C_t) + g_{j,t+1}C_t]
\]

Findings: Two anomalies are explained

1. High market premium, but low risk aversion (from surveys) – i.e, the equity premium.
2. Ex-post arbitrage opportunities: Euler equations hold ex-ante, but with respect to the agents’ subjective F, not the ex-post realized frequencies.

Note: Slow convergence is critical to explain anomalies.
Survival

- Ex post (observed) $E_t[r_{t+1} - r_f] > ex \ ante \ E_t[r_{t+1} - r_f]$
- Ex post $E_t[r_{t+1} - r_f]$ decreases with crash frequency
- Cross-sectional implications
  - Low risk markets have higher ex post premium

- International evidence: From Goetzmann and Jorion (1999)
  - U.S. market shows the maximum equity risk premium (5.48%)
  - 6 of 21 markets experienced no interruption from the 1920's.
  - 8 had a temporary closure and 7 suffered a long-term closure.
  - The “Non-U.S. Survived markets” returned 4.52%, while the “Non-U.S. All markets” returned 3.84%.
  - Survival bias is around .60%
  - The GJ evidence points more towards a “good draw” for the U.S.

![Fig.1. Real Returns on Global Stock Markets](image)