Lecture 14
ARIMA – Identification, Estimation & Seasonalities

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ARMA Process

• We defined the ARMA(*p, q)* model:

$$\phi(L)(y_t - \mu) = \theta(L)\varepsilon_t$$

Let $x_t = y_t - \mu$.

Then, $$\phi(L)x_t = \theta(L)\varepsilon_t$$

$\Rightarrow x_t$ is a *demeaned* ARMA process.

• In this lecture, we will study:
- Identification of *p, q*.
- Estimation of ARMA(*p,q*)
- Non-stationarity of x_t.
- Differentiation issues – ARIMA(*p,d,q*)
- Seasonal behavior – SARIMA$(p,d,q)_S$

Autocovariance Function

• We define the autocovariance function, $\gamma(t\text{-}j)$ as: $\gamma(t-j)=E[y_t, y_{t-j}]$

• For an AR(p) process, WLOG with $\mu=0$ (or demeaned y_t), we get:

$$\gamma(t-j)=E[y_t, y_{t-j}]=E[(\phi_1 y_{t-1}y_{t-j}+\phi_2 y_{t-2}y_{t-j}+....+\phi_p y_{t-p}y_{t-j}+\varepsilon_t y_{t-j})]$$
$$=\phi_1\gamma(j-1)+\phi_2\gamma(j-2)+....+\phi_p\gamma(j-p)$$

<u>Notation</u>: $\gamma(k)$ or γ_k are commonly used. Sometimes, $\gamma(k)$ is referred as "covariance at lag k.

• The $\gamma(t\text{-}j)$ determine a system of equations:

$$\gamma(0)=E[y_t, y_t]=\phi_1\gamma(1)+\phi_2\gamma(2)+\phi_3\gamma(3)+....+\phi_p\gamma(p)+\sigma^2$$
$$\gamma(1)=E[y_t, y_{t-1}]=\phi_1\gamma(0)+\phi_2\gamma(1)+\phi_3\gamma(2)+....+\phi_p\gamma(p-1)$$
$$\gamma(2)=E[y_t, y_{t-1}]=\phi_1\gamma(1)+\phi_2\gamma(0)+\phi_3\gamma(1)+....+\phi_p\gamma(p-2)$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

Autocovariance Function

$$\begin{bmatrix} \gamma(0) & \gamma(1) & \cdots & \gamma(p-1) \\ \gamma(1) & \gamma(0) & \cdots & \gamma(p-2) \\ \vdots & \vdots & \cdots & \vdots \\ \gamma(p-1) & \gamma(p-2) & \cdots & \gamma(0) \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_p \end{bmatrix} = \begin{bmatrix} \gamma(1) \\ \gamma(2) \\ \vdots \\ \gamma(p) \end{bmatrix}$$

• These are the *Yule-Walker equations*, can be solved numerically. MM can be used (replace population moments with sample moments).

• Properties for a stationary time series

1. $\gamma(0) \geq 0$ (from definition of variance)
2. $\gamma(k) \leq \gamma(0)$ (from Cauchy-Schwarz)
3. $\gamma(k) = \gamma(-k)$ (from stationarity)
4. $\mathbf{\Gamma}$, the auto-correlation matrix, is psd ($a'\,\mathbf{\Gamma}\,a \geq 0$)

Moreover, any function γ: Z $\rightarrow R$ that satisfies (3) and (4) is the autocovariance of some stationary time series.

Autocovariance Function – Example: ARMA(1,1)

- For an ARMA(1,1) we have:.

$$\begin{aligned}\gamma_k &= E\left[(y_t-\mu)(y_{t-k}-\mu)\right]\\ &= E\left[\{\phi(y_{t-1}-\mu)+\varepsilon_t-\theta\varepsilon_{t-1}\}(y_{t-k}-\mu)\right]\\ &= \phi E\left[(y_{t-1}-\mu)(y_{t-k}-\mu)\right]+E\left[\varepsilon_t(y_{t-k}-\mu)\right]-\theta E\left[\varepsilon_{t-1}(y_{t-k}-\mu)\right]\\ &= \phi\gamma_{k-1}+E\left[\varepsilon_t(y_{t-k}-\mu)\right]-\theta E\left[\varepsilon_{t-1}(y_{t-k}-\mu)\right]\end{aligned}$$

$$\begin{aligned}\gamma_0 &= \phi\gamma_{-1}+\underbrace{E\left[\varepsilon_t(Y_t-\mu)\right]}_{\sigma^2}-\theta E\left[\varepsilon_{t-1}\underbrace{(y_t-\mu)}_{\phi(y_{t-1}-\mu)+\phi\varepsilon_t-\theta\varepsilon_{t-1}}\right]\\ &= \phi\gamma_1+\sigma^2-\theta E\left[\varepsilon_{t-1}[\phi\underbrace{(y_{t-1}-\mu)}_{\phi(y_{t-2}-\mu)+\phi\varepsilon_{t-1}-\theta\varepsilon_{t-2}}+\varepsilon_t-\theta\varepsilon_{t-1}]\right]\\ &= \phi\gamma_1+\sigma^2-\theta\left(\phi\sigma^2-\theta\sigma^2\right)\end{aligned}$$

Autocovariance Function – Example: ARMA(1,1)

- Similarly:

$$\begin{aligned}\gamma_1 &= \phi\gamma_0+\underbrace{E\left[\varepsilon_t(y_{t-1}-\mu)\right]}_{0}-\theta E\left[\varepsilon_{t-1}\underbrace{(y_{t-1}-\mu)}_{\phi(y_{t-2}-\mu)+\varepsilon_{t-1}-\theta\varepsilon_{t-2}}\right]\\ &= \phi\gamma_0-\theta\sigma^2\end{aligned}$$

- Two equations for γ_0 and γ_1:

$$\Rightarrow \gamma_0 = \phi\left(\phi\gamma_0-\theta\sigma^2\right)+\sigma^2\left(1-\phi\theta+\theta^2\right)$$

$$\gamma_0 = \frac{1+\theta^2-2\phi\theta}{1-\phi^2}$$

- In general: $\gamma_k = \phi\gamma_{k-1} = \phi^{k-1}\gamma_1, \quad k>1$

Autocorrelation Function (ACF)

• Now, we define the autocorrelation function (ACF):

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \frac{\text{covariance at lag k}}{\text{variance}}$$

The ACF lies between -1 and +1, with $\rho(0)=1$.

• Dividing the autocovariance system by $\gamma(0)$, we get:

$$\begin{bmatrix} \rho(0) & \rho(1) & \cdots & \rho(p-1) \\ \rho(1) & \rho(0) & \cdots & \rho(p-2) \\ \vdots & \vdots & \cdots & \vdots \\ \rho(p-1) & \rho(p-2) & \cdots & \rho(0) \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_p \end{bmatrix} = \begin{bmatrix} \rho(1) \\ \rho(2) \\ \vdots \\ \rho(p) \end{bmatrix}$$

• These are "Yule-Walker" equations, which can be solved numerically.

Autocorrelation Function (ACF)

• Easy estimation: Use sample moments to estimate $\gamma(k)$ and plug in formula:

$$r_k = \frac{\sum (Y_t - \bar{Y})(Y_{t+k} - \bar{Y})}{\sum (Y_t - \bar{Y})^2}$$

• Distribution: For a linear, stationary, $y_t = \mu + \Sigma_i \psi_j \varepsilon_{t-j}$, with $E[\varepsilon_t^4] < \infty$:

$\mathbf{r} \xrightarrow{d} N\boldsymbol{\rho}, \mathbf{V}/T)$, $\mathbf{V}$ is the covariance matrix with elements:

$$w_{ij} = \sum_{k=1}^{\infty} \{\rho(k+i) + \rho(k-i) - 2\rho(i)\rho(k)\} \times \{\rho(k+j) + \rho(k-j) - 2\rho(j)\rho(k)\}$$

If $\rho_k = 0$, for all $k \neq 0$, $\mathbf{V} = \mathbf{I}$ $\Rightarrow$ Var[r(k)] = $1/T$.

• The standard errors under H_0, 1/sqrt(T), are sometimes referred as *Bartlett's SE*.

Autocorrelation Function (ACF)

• Example: Sample ACF for an MA(1) process.

$\rho(0) = 1$,

$\rho(k) = \theta/(1+\theta^2)$, for $k = 1,-1$

$\rho(k) = 0$ for $|k|>1$.

$$V_{1,1} = \sum_{h=1}^{\infty} (\rho(h+1) + \rho(h-1) - 2\rho(1)\rho(h))^2 = (\rho(0) - 2\rho(1)^2)^2 + \rho(1)^2$$

$$V_{2,2} = \sum_{h=1}^{\infty} (\rho(h+2) + \rho(h-2) - 2\rho(2)\rho(h))^2 = \sum_{h=-1}^{1} \rho(h)^2.$$

And if $\hat{\rho}$ is the sample ACF from a realization of this MA(1) process, then with probability 0.95,

$$|\hat{\rho}(h) - \rho(h)| \leq 1.96\sqrt{\frac{V_{hh}}{n}}.$$

Autocorrelation Function (ACF)

• Example: Sample ACF for an MA(q) process:

$$y_t = \mu + \varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2} + \ldots + \theta_q\varepsilon_{t-q}$$

$\gamma(0) = E[y_t\, y_t] = \sigma^2 (1 + \theta_1^2 + \theta_2^2 + \ldots + \theta_q^2)$

$\gamma(1) = E[y_t\, y_{t-1}] = \sigma^2 (\theta_1 + \theta_2\, \theta_3 + \ldots + \theta_q\, \theta_{q-1})$

$\gamma(2) = E[y_t\, y_{t-2}] = \sigma^2 (\theta_2 + \theta_3\, \theta_1 + \ldots + \theta_q\, \theta_{q-1})$

$\gamma(q) = \theta_q$

In general, $\gamma(k) = \sigma^2 \sum_{j=k}^{q} \theta_j \theta_{j-k}$ $\quad k \leq q \quad$ (with $\theta_0 = 1$).

$= 0$ otherwise.

Then, $\rho(k) = \sigma^2 \dfrac{\sum_{j=k}^{q} \theta_j \theta_{j-k}}{(1 + \theta_1^2 + \theta_2^2 + \ldots + \theta_q^2)}$ $\quad k \leq q$

$= 0$ otherwise.

Autocorrelation Function - Example

• Example: US Monthly Returns (1800 – 2013, T=2534)

Autocorrelations and Ljung Box Tests (SAS: Check for White Noise)

To Lag	Chi-Square	DF	Pr > ChiSq	Autocorrelations					
6	38.29	6	<.0001	0.096	-0.017	-0.044	0.012	0.054	-0.025
12	46.89	12	<.0001	0.013	0.032	0.033	0.032	0.000	0.006
18	75.07	18	<.0001	-0.042	-0.084	-0.009	-0.024	0.030	0.028
24	89.77	24	<.0001	-0.009	-0.054	-0.036	-0.020	-0.027	0.017

SE(r_k) = 1/sqrt(T) = 1/sqrt(2534)≈.02.

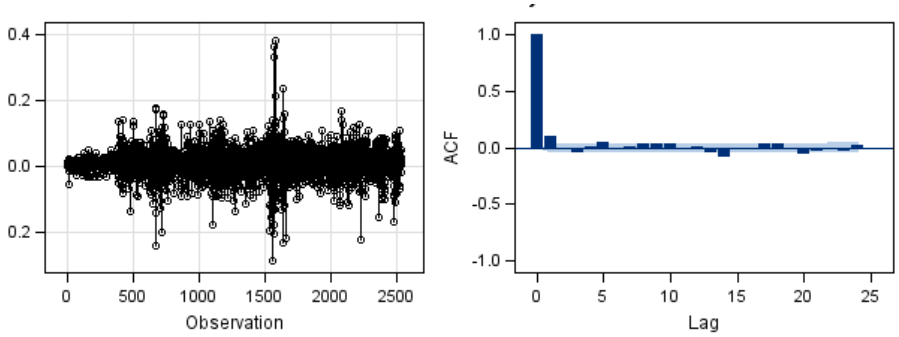
• Note: Small correlations, but because T is large, many of them significant (even at k=20 months). Joint significant too!

ACF: Correlogram

• The sample correlogram is the plot of the ACF against k.

As the ACF lies between -1 and +1, the correlogram also lies between these values.

• Example: Correlogram for US Monthly Returns (1800 – 2013)



ACF: Joint Significance Tests

• Recall the Q statistic as:

$$Q = T \sum_{k=1}^{m} \hat{\rho}_k^2$$

It can be used to determine if the first *m* sample ACFs are jointly equal to zero. Under H_0: $\rho_1 = \rho_2 = ... = \rho_m = 0$, then Q converges in distribution to a $\chi^2(m)$.

• The Ljung-Box statistic is similar, with the same asymptotic distribution, but with better small sample properties:

$$LB = T(T+2) \sum_{k=1}^{m} \left(\frac{\hat{\rho}_k^2}{(T-k)} \right)$$

Partial ACF

• The Partial Autocorrelation Function (PACF) is similar to the ACF. It measures correlation between observations that are *k* time periods apart, after controlling for correlations at intermediate lags.

<u>Definition</u>: The PACF of a stationary time series $\{y_t\}$ is
$\phi_{11} = \text{Corr}(y_t, y_{t-1}) = \rho(1)$
$\phi_{hh} = \text{Corr}(y_t - E[y_t \mid F_{t-1}], y_{t-h} - E[y_{t-h-1} \mid F_{t-1}])$ for $h = 2, 3, ...$

This removes the linear effects of $y_{t-1}, y_{t--2}, ... y_{t-h}$.

• The PACF Φ_{hh} is also the last coefficient in the best linear prediction of y_t given $y_{t-1}, y_{t--2}, ..., y_{t-h}$: $\mathbf{\Gamma}_h \boldsymbol{\phi}_h = \gamma(k)$

where $\boldsymbol{\phi}_h = (\phi_{h1}, \phi_{h2}, ..., \phi_{hh})$.

Partial ACF

Example: AR(p) process:

$$y_t = \mu + \phi_1 y_{t-1} + \phi_2 y_{t-2} + ... + \phi_p y_{t-p} + \varepsilon_t$$

$$E[y_t \mid F_{t-1}] = \mu + \phi_1 y_{t-1} + \phi_2 y_{t-2} + ... + \phi_p y_{t-p}$$

Then, $\phi_{hh} = \phi_h$ if $1 \leq h \leq p$
$= 0$ otherwise

• Estimation: $\hat{\phi}_h = [\hat{\boldsymbol{\Gamma}}]^{-1}\hat{\boldsymbol{\gamma}}(k)$ $\Rightarrow$ a recursive system.

A recursive algorithm, Durbin-Levinson, can be used.

• We can also produce a *partial correlogram*, which is used in Box-Jenkins methodology (covered later).

Inverse ACF (IACF)

• The IACF of the ARMA(*p,q*) model

$$\phi(L) y_t = \theta(L) \varepsilon_t$$

is defined to be (assuming invertibility) the ACF of the *inverse* (or *dual*) process

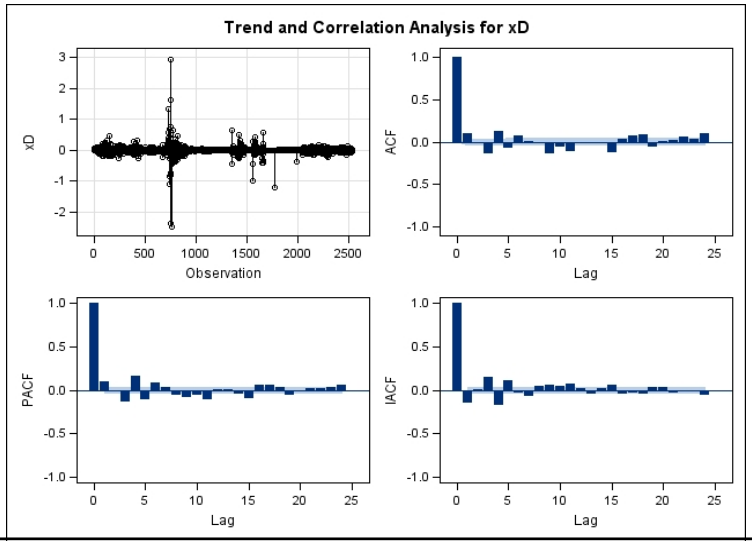
$$\theta(L) y_t^{-1} = \phi(L) \varepsilon_t$$

• The IACF has the same property as the PACF: AR(*p*) is characterized by an IACF that is nonzero at lag p but zero for larger lags.

• The IACF can also be used to detect over-differencing. If the data come from a nonstationary or nearly nonstationary model, the IACF has the characteristics of a noninvertible moving-average.

ACF, Partial ACF & IACF: Example

Example: Monthly USD/GBP 1st differences (1800-2013)



Non-Stationary Time Series Models

• The ACF is as a rough indicator of whether a trend is present in a series. A slow decay in ACF is indicative of a large characteristic root; a true unit root process, or a trend stationary process.

• Formal tests can help to determine whether a system contains a trend and whether the trend is deterministic or stochastic.

• We will analyze two situations faced in ARMA models:
(1) Deterministic trend – Simple model: $y_t = a + \beta\, t + \varepsilon_t$.
– Solution: *Detrending* –i.e., regress y_t on t. Then, keep residuals for further modeling.

(2) Stochastic trend – Simple model: $y_t = c + y_{t-1} + \varepsilon_t$.
– Solution: *Differencing* –i.e., apply $\Delta = (1 - L)$ operator to y_t . Then, use Δy_t for further modeling.

Non-Stationary Models: Deterministic Trend

• Suppose we have the following model:

$$y_t = \alpha + \beta t + \varepsilon_t. \qquad \Rightarrow \Delta y_t = y_t - y_{t-1}$$
$$= \beta t - \beta (t-1) + \varepsilon_t - \varepsilon_{t-1}$$
$$= \beta + \varepsilon_t - \varepsilon_{t-1}$$
$$\Rightarrow E[\Delta y_t] = \beta$$

• $\{y_t\}$ will show only temporary departures from the trend line $\alpha+\beta t$. This type of model is called a trend stationary (TS) model.

• If a series has a deterministic time trend, then we simply regress y_t on an intercept and a time trend ($t = 1, 2, \ldots, T$) and save the residuals. The residuals are the *detrended* y_t series.

• If y_t is stochastic, we do not necessarily get stationary series.

Non-Stationary Models: Deterministic Trend

• Many economic series exhibit "exponential trend/growth". They grow over time like an exponential function over time instead of a linear function. In this cases, it is common to work with logs

$$\ln(y_t) = \alpha + \beta t + \varepsilon_t.$$

$\Rightarrow$ The average growth rate is: $E[\Delta \ln(y_t)] = \beta$

• We can have a more general model:

$$y_t = \alpha + \phi_1 y_{t-1} + \ldots + \phi_p y_{t-p} + \beta_1 t + \beta_2 t^2 + \ldots + \beta_{2k} t^k + \varepsilon_t,$$

• Estimation:
- OLS.
- Frish-Waugh method:
  (1) Detrend y_t, get the residuals;
  (2) Use residuals to estimate the AR(p) model.

Non-Stationary Models: Deterministic Trend

• This model has a short memory.

• If a shock (big ε_t) hits y_t, it goes back to trend level in a short time. Thus, the best forecasts are not affected.

• In practice, it is not a popular model. A more realistic model involves stochastic (local) trend.

Non-Stationary Models: Stochastic Trend

• The more modern approach is to consider trends in time series as a variable.

• A variable trend exists when a trend changes in an unpredictable way. Therefore, it is considered as *stochastic*.

• Recall the AR(1) model: $y_t = c + \phi y_{t-1} + \varepsilon_t$.

• As long as $|\phi| < 1$, everything is fine: OLS is consistent, t-stats are asymptotically normal, etc.

• Now consider the extreme case where $\phi = 1$, $\Rightarrow y_t = c + y_{t-1} + \varepsilon_t$.

• Where is the (stochastic) trend? No t term.

Non-Stationary Models: Stochastic Trend

• Let us replace recursively the lag of y_t on the right-hand side:

$$
\begin{aligned}
y_t &= \mu + y_{t-1} + \varepsilon_t \\
&= \mu + (\mu + y_{t-2} + \varepsilon_{t-1}) + \varepsilon_t \\
&\ldots \\
&= y_0 + t\,\mu + \Sigma_{i=0 \text{ to } t}\, \varepsilon_{t-i}
\end{aligned}
$$

$\downarrow$

Deterministic trend

• This is what we call a "*random walk with drift*". The series grows with *t*.

• Each ε_t shock represents a shift in the intercept. All values of $\{\varepsilon_t\}$ have a 1 as coefficient $\Rightarrow$ each shock never vanishes (permanent).

Non-Stationary Models: Stochastic Trend

• y_t is said to have a *stochastic trend* (ST), since each ε_t shock gives a permanent and random change in the conditional mean of the series.

• For these situations, we use *Autoregressive Integrated Moving Average (ARIMA)* models.

• Q: Deterministic or Stochastic Trend?
They appear similar: Both lead to growth over time. The difference is how we think of ε_t. Should a shock today affect y_{t+1}?

– TS: $y_{t+1} = c + \beta(t+1) + \varepsilon_{t+1}$ $\Rightarrow$ ε_t does not affect y_{t+1}.

– ST: $y_{t+1} = c + y_t + \varepsilon_{t+1} = c + [c + y_{t-1} + \varepsilon_t] + \varepsilon_{t+1}$ $\Rightarrow$ ε_t affects y_{t+1}.

(In fact, the shock will have a permanent impact.)

ARIMA(p,d,q) Models

• For *p, d, q* ≥ 0, we say that a time series $\{y_t\}$ is an *ARIMA (p,d,q) process* if $w_t = \Delta^d y_t = (1 - L)^d y_t$ is ARMA(*p,q*). That is,

$$\phi(L)(1-L)^d y_t = \theta(L)\varepsilon_t$$

• Applying the $(1 - L)$ operator to a time series is called *differencing*.

• Notation: If y_t is non-stationary, but $\Delta^d y_t$ is stationary, then y_t is integrated of order *d*, or I*(d)*. A time series with unit root is I(1). A stationary time series is I(0).

• Examples:
Example 1: RW: $y_t = y_{t-1} + \varepsilon_t$.
y_t is non-stationary, but $(1 - L)\, y_t = \varepsilon_t \Rightarrow$ white noise!
Now, $y_t \sim$ ARIMA(0,1,0).

ARIMA(p,d,q) Models

Example 2: AR(1) with time trend: $y_t = \mu + \delta\, t + \phi\, y_{t-1} + \varepsilon_t$.
y_t is non-stationary, but $w_t = (1 - L)\, y_t = \delta + \phi\, w_{t-1} + \varepsilon_t - \varepsilon_{t-1}$
Now, $y_t \sim$ ARIMA(1,1,1).

• We call both process first difference stationary.

• Note:
− Example 1: Differencing a series with a unit root in the AR part of the model reduces the AR order.

− Example 2: Differencing can introduce an extra MA structure. We introduced non-invertibility. This happens when we difference a TS series. Detrending should be used in these cases.

ARIMA(p,d,q) Models

• In practice:
A root near 1 of the AR polynomial ⇒ differencing
A root near 1 of the MA polynomial ⇒ over-differencing

• In general, we have the following results:
- Too little differencing: not stationary.
- Too much differencing: extra dependence introduced.

• Finding the right *d* is crucial. For identifying preliminary values of *d*:
- Use a time plot.
- Check for slowly decaying (persistent) ACF/PACF.

Example: ACF & Partial ACF

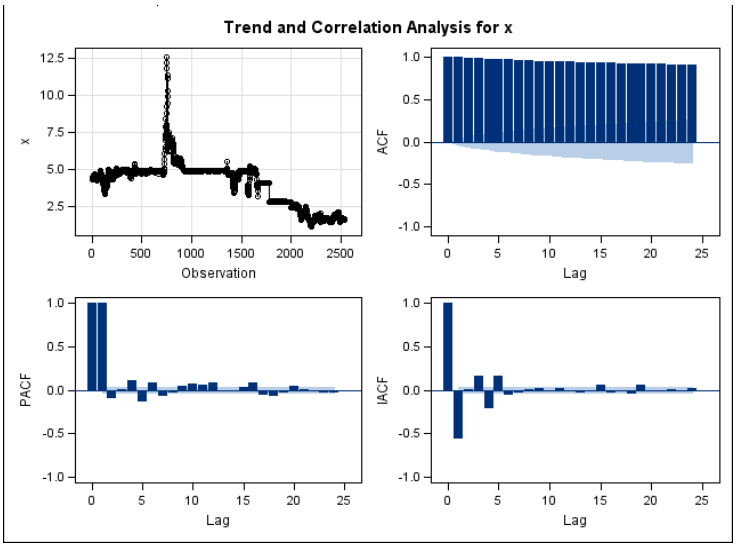
Example: Monthly USD/GBP levels (1800-2013)

Autocorrelation Check for White Noise

To Lag	Chi-Square	DF	Pr > ChiSq	----------Autocorrelations----------					
6	9999.99	6	<.0001	0.995	0.989	0.984	0.979	0.974	0.969
12	9999.99	12	<.0001	0.963	0.957	0.952	0.947	0.943	0.940
18	9999.99	18	<.0001	0.937	0.934	0.931	0.929	0.927	0.924
24	9999.99	24	<.0001	0.920	0.917	0.914	0.911	0.907	0.903

Example: ACF & Partial ACF

Example: Monthly USD/GBP levels (1800-2013)



ARIMA(p,d,q) Models – Random Walk

• A *random walk* (RW) is defined as a process where the current value of a variable is composed of the past value plus an error term defined as a white noise (a normal variable with zero mean and variance one).

• RW is an ARIMA(0,1,0) process

$$y_t = y_{t-1} + \varepsilon_t \Rightarrow \Delta y_t = (1-L)y_t = \varepsilon_t, \qquad \varepsilon_t \sim WN(0,\sigma^2)$$

• Popular model. Used to explain the behavior of financial assets, unpredictable movements (Brownian motions, drunk persons).

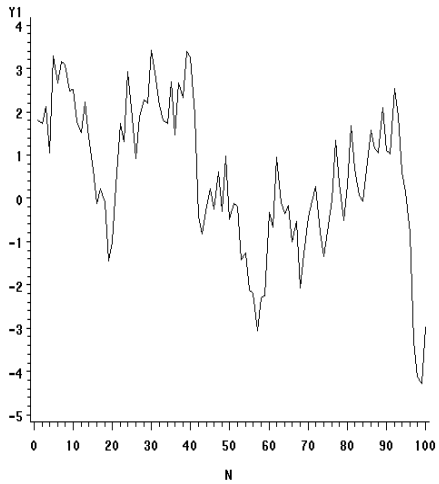
• It is a special case (limiting) of an AR(1) process.

• Implication: $E[y_{t+1} \mid F_t] = y_t$ $\Rightarrow \Delta y_t$ is absolutely random.

• Thus, a RW is nonstationary, and its variance increases with *t*.

ARIMA(p,d,q) Models – Random Walk

• Examples:



ARIMA(p,d,q) Models – RW with Drift

• Change in Y_t is partially deterministic and partially stochastic.

$$y_t - y_{t-1} = \Delta y_t = \mu + \varepsilon_t$$

• It can also be written as

$$y_t = y_0 + t\,\mu + \Sigma_{i=0}\, \varepsilon_{t-i}$$

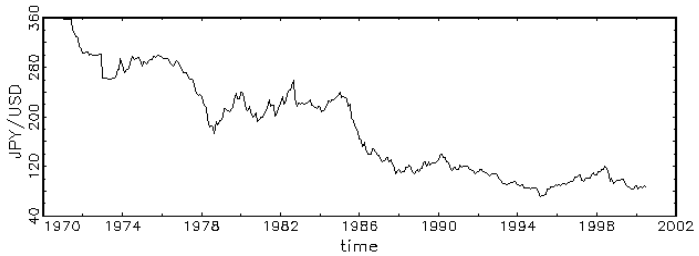
Deterministic part (trend) | Accumulation of errors (shocks) – stochastic part

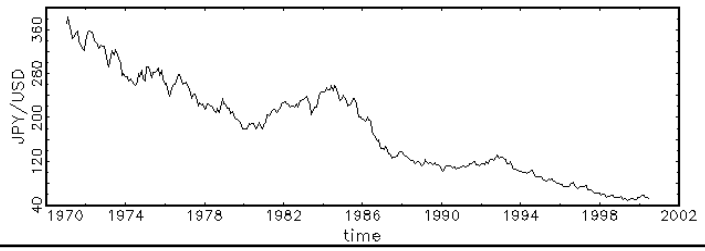
$\Rightarrow$ ε_t has a permanent effect on the mean of y_t.

• $E[y_t] = y_0 + t\,\mu$ (Unconditional forecast)
$E[y_{t+s} \mid y_t] = y_t + s\,\mu$ (Conditional forecast)

ARIMA(p,d,q) Models – RW with Drift

• Two series: 1) True JPY/USD 1997-2000 series; 2) A simulated RW (same drift and variance). Can you pick the true series?





ARIMA Models: Box-Jenkins

• An effective procedure for building empirical time series models is the Box-Jenkins approach, which consists of three stages:
(1) Model specification or identification (of ARIMA order),
(2) Estimation
(3) Diagnostics testing.

• Two main approaches to (1) Identification.
- *Correlation approach*, mainly based on ACF & PACF.

- *Information criteria*, based on the *maximized likelihood* (x2) plus a *penalty function*. For example, model selection based on the AIC.

ARIMA Models: Box-Jenkins

• We have a family of ARIMA models, indexed by *p, q,* and *d*.
Q: How do we select one?

• Box-Jenkins Approach
1) Make sure data is stationary –check a time plot. If not, differentiate.
2) Using ACF & PACF, guess small values for *p* & *q*.
3) Estimate order *p, q*.
4) Run diagnostic tests on residuals.

Q: Are they white noise? If not, add lags (*p* or *q*, or both).

• If order choice not clear, use AIC, AIC Corrected (AICc), BIC, or HQC (Hannan and Quinn (1979)).

• Value parsimony. When in doubt, keep it simple (KISS).

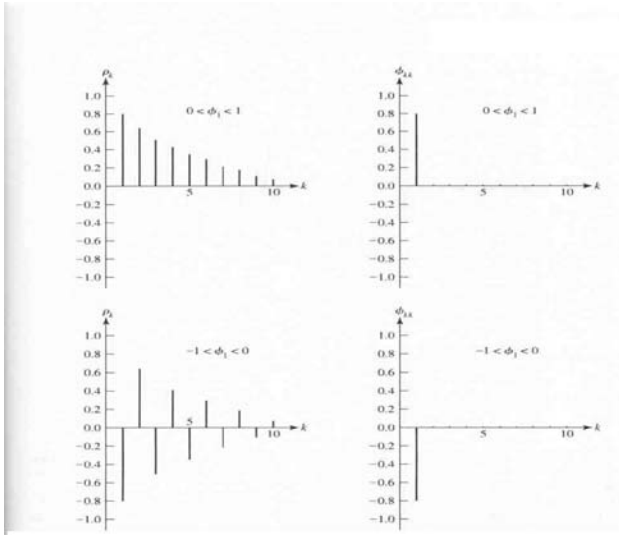
ARIMA Models: Identification - Correlations

• Correlation approach.
Basic tools: sample ACF and sample PACF.
- ACF identifies order of MA: Non-zero at lag *q;* zero for lags > *q*.
- PACF identifies order of AR: Non-zero at lag *p;* zero for lags >*p*.
- All other cases, try ARMA(*p,q*) with $p > 0$ and $q > 0$.

For $p > 0$ and $q > 0$:

	AR(p)	MA(q)	ARMA(p, q)
ACF	Tails off	Cuts off after lag q	Tails off
PACF	Cuts off after lag p	Tails off	Tails off
IACF	Cuts off after lag p	Tails off	Tails off

ARIMA Models: Identification – AR(1)



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ARIMA Models: Identification – AR(2)

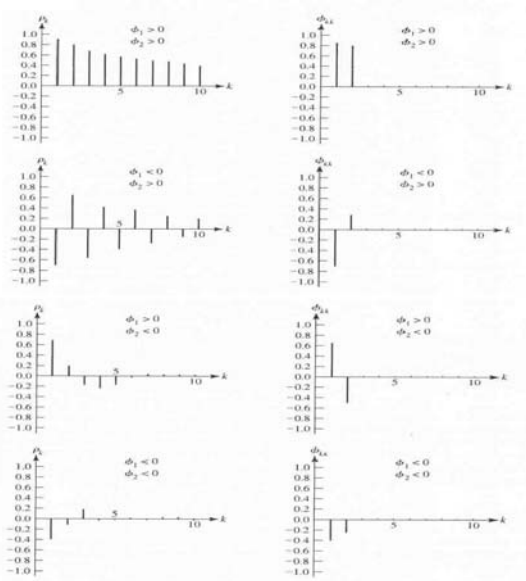

FIGURE ACF and PACF of AR(2) process: $(1 - \phi_1 B - \phi_2 B^2)\dot{Z}_t = a_t$.

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ARIMA Models: Identification – MA(1)

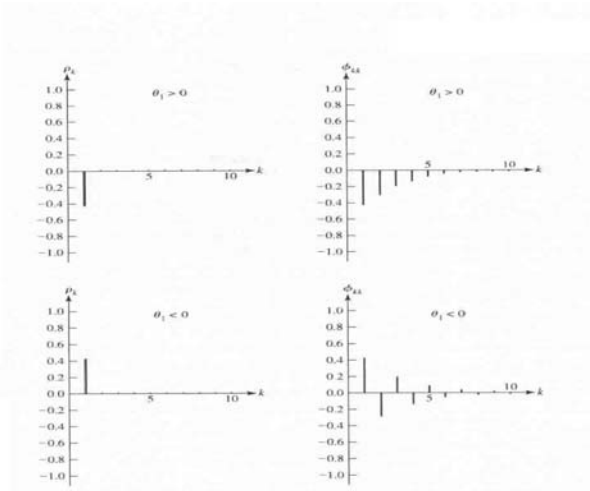

FIGURE ACF and PACF of MA(1) processes: $Z_t = (1 - \theta_1 B)a_t$.

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ARIMA Models: Identification – MA(2)

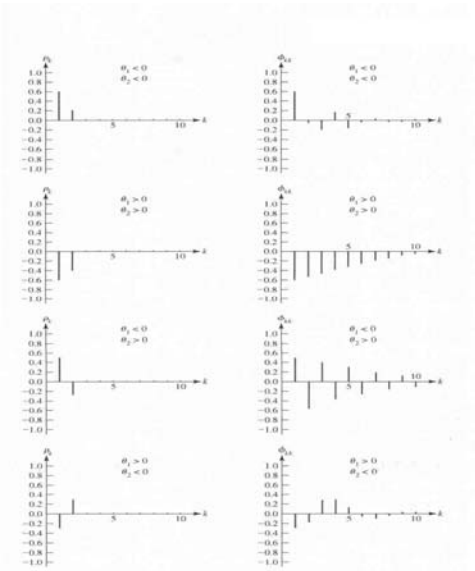

FIGURE ACF and PACF of MA(2) processes: $Z_t = (1 - \theta_1 B - \theta_2 B^2)a_t$.

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ARIMA Models: Identification – ARMA(1,1)

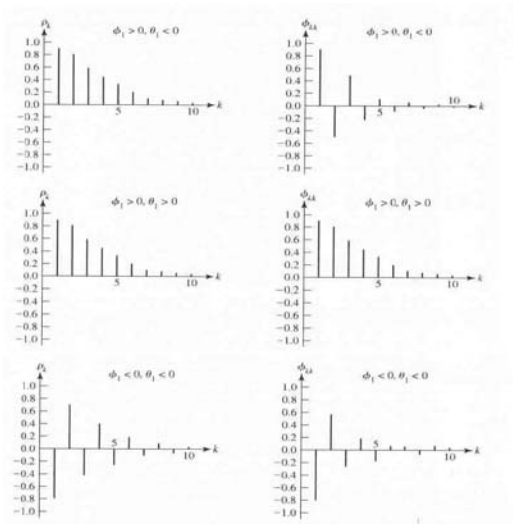

FIGURE ACF and PACF of ARMA(1, 1) model $(1 - \phi_1 B)\tilde{Z}_t = (1 - \theta_1 B)a_t$.

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ARIMA Models: Identification – ARMA(1,1)

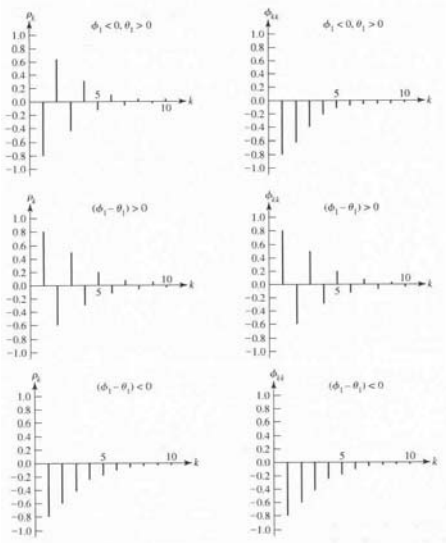

FIGURE (*Continued*)

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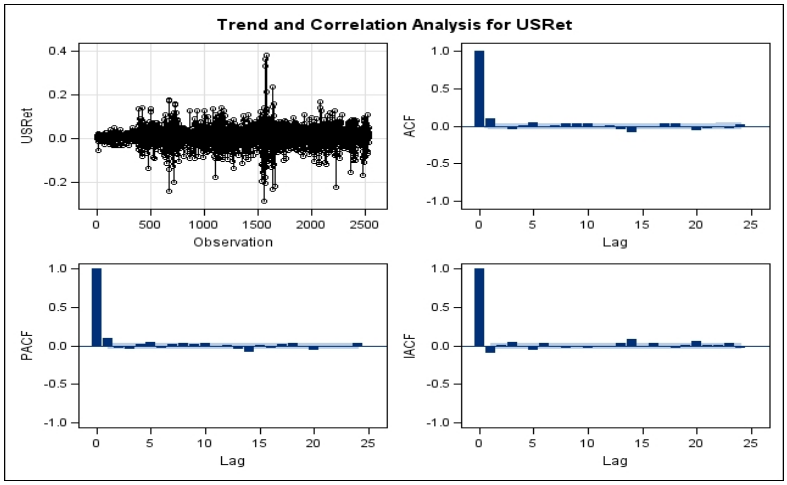
ARIMA Models: Identification – ARMA(1,1)

• Example: Monthly US Returns (1800 - 2013).



• Note: Identification is not clear.

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ARIMA Model: Identification - IC

• Popular information criteria

• AIC = -2 ln(L=likelihood) + 2 M, M: fitted model's parameters

$$\ln L = -\frac{T}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\underbrace{S(\phi_p, \theta_q, \mu)}_{SS\,\mathrm{Re}\,sidual} \qquad \left(\text{assuming } \varepsilon_t \overset{i.i.d}{\sim} N(0,\sigma^2)\right).$$

$$\ln \hat{L} == -\frac{T}{2}\ln\hat{\sigma}^2 - \underbrace{\frac{T}{2}(1+\ln 2\pi)}_{\text{constant}}$$

$$AIC = T \ln \hat{\sigma}^2 + 2M$$

• $BIC = T\ln\hat{\sigma}^2 + M\ln T$

• $HQC = T\ln\hat{\sigma}^2 + 2M\ln(\ln T)$

ARIMA Model: Identification - IC

• There is an AIC corrected statistic, that corrects AIC for finite sample sizes:

$$AICc = T \ln \hat{\sigma}^2 + \frac{2M(M+1)}{T-M-1}$$

Burnham & Anderson (2002) recommend using AICc, rather than AIC, if T is small or M is large. Since AICc converges to AIC as T gets large, B&A advocate the AICc.

• For AR(p), other criteria are possible: Akaike's final prediction error (FPE), Akaike's BIC, Parzen's CAT.

• Hannan and Rissannen's (1982) *minic*: Calculate the BIC for different p's (estimated first) and different q's. Select the best model.

Note: Box, Jenkins, and Reinsel (1994) proposed using the AIC above.

ARIMA Model: Identification - IC

• Example: Monthly US Returns (1800 - 2013) Hannan and Rissannen (1982)'s minic.

Minimum Information Criterion

Lags	**MA 0**	**MA 1**	**MA 2**	**MA 3**	**MA 4**	**MA 5**
AR 0	-6.1889	**-6.19573**	-6.19273	-6.19177	-6.18872	-6.18886
AR 1	-6.19511	-6.193	-6.19001	-6.18929	-6.18632	-6.18678
AR 2	-6.19271	-6.18993	-6.1911	-6.18802	-6.18536	-6.1839
AR 3	-6.19121	-6.18916	-6.18801	-6.18562	-6.18256	-6.18082
AR 4	-6.18853	-6.18609	-6.18523	-6.18254	-6.17983	-6.17774
AR 5	-6.18794	-6.18671	-6.18408	-6.18099	-6.1779	-6.17564

• Note: Best Model is ARMA(0,1).

ARIMA Model: Identification - IC

• There is no agreement on which criteria is best. The AIC is the most popular, but others are also used. (Diebold, for instance, recommends the BIC.)

• Asymptotically, the BIC is consistent –i.e., it selects the true model if, among other assumptions, the true model is among the candidate models considered. For example, Hannan (1980) shows that in the case of common roots in the AR and MA polynomials, the BIC (& HQC) still select the correct orders *p and q* consistently. But, it is not efficient.

• The AIC is not consistent, generally producing too large a model, but is more efficient –i.e., when the true model is not in the candidate model set the AIC asymptotically chooses whichever model minimizes the MSE/MSPE.

ARIMA Process – Estimation

• We assume:
- The model order (d, p and q) is known. Make sure y_t is I(0).
- The data has zero mean (μ=0). If this is not reasonable, demean y .

Fit a zero-mean ARMA model to the demeaned y_t:

$$\phi(L)(y_t - \bar{y}) = \theta(L)\varepsilon_t$$

• Several ways to estimate an ARMA(p, q) model:

1) *MLE*. Assume a distribution, usually normal, and do ML.
2) *Yule-Walker for* ARMA*(p,q)*. Method of moments. Not efficient.
3) *Innovations algorithm for MA(q)*.
4) *Hannan-Rissanen algorithm for* ARMA(p, q).

ARIMA Process – MLE

• Steps:

1) Assume a distribution for the errors. Typically, *.i.i.d.* normal, say:

$$\varepsilon_t \sim i.i.d.\ N(0,\sigma^2)$$

2) Write down the joint pdf for ε_t: $f(\varepsilon_1, ..., \varepsilon_T) = f(\varepsilon_1) ... f(\varepsilon_T)$

<u>Note</u>: we are not writing the joint pdf in terms of the y_t's, as a multiplication of the marginal pdfs because of the dependency in y_t.

3) Get ε_t. For the general stationary ARMA(*p,q*) model:

$$\varepsilon_t = y_t - \phi_1 y_{t-1} - \cdots - \phi_p y_{t-p} + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q}$$

(if $\mu \neq 0$, demean y_t.)

4) The joint pdf for $\{\varepsilon_1. ..., \varepsilon_T)$ is:

$$f\left(\varepsilon_1, \cdots, \varepsilon_T \middle| \mu, \phi, \theta, \sigma^2\right) = \left(2\pi\sigma^2\right)^{-T/2} \exp\left\{-\frac{1}{2\sigma^2}\sum_{t=1}^{n} \varepsilon_t^2\right\}$$

ARIMA Process – MLE

• Steps:

5) Let $Y = (y_1, \ldots, y_T)$ and assume that initial conditions $Y_* = (y_{1-p}, \ldots, y_0)'$ and $\varepsilon_* = (\varepsilon_{1-q}, \ldots, \varepsilon_0)'$ are known.

6) The conditional log-likelihood function is given by

$$\ln L\left(\mu, \phi, \theta, \sigma^2\right) = -\frac{T}{2}\ln\left(2\pi\sigma^2\right) - \frac{S_*(\mu, \phi, \theta)}{2\sigma^2}$$

where $S_*(\mu, \phi, \theta) = \sum_{t=1}^{n} \varepsilon_t^2\left(\mu, \phi, \theta \middle| Y, Y_*, \varepsilon_*\right)$ is the conditional SS.

<u>Note</u>: Usual Initial conditions: $y_* = \bar{y}$ and $\varepsilon_* = E[\varepsilon_t] = 0.$

• Numerical optimization problem. Initial values ($\mathbf{y}_*$) matter.

ARIMA Process – MLE: AR(1)

• Example: AR(1)

$$y_t = \phi y_{t-1} + \varepsilon_t, \qquad \varepsilon_t \overset{i.i.d.}{\sim} N\left(0, \sigma^2\right).$$

- Write down joint for $\boldsymbol{\varepsilon}_t$

$$f\left(\varepsilon_1, \cdots, \varepsilon_n\right) = \left(2\pi\sigma\right)^{-n/2} \exp\left\{-\frac{1}{2\sigma^2}\sum_{t=1}^{n}\varepsilon_t^2\right\}$$

- Solve for ε_t:

$$\begin{aligned}
&Y_1 = \phi Y_0 + \varepsilon_1 \rightarrow \text{Let' s take } Y_0 = 0 \\
&Y_2 = \phi Y_1 + \varepsilon_2 \Rightarrow \varepsilon_2 = Y_2 - \phi Y_1 \\
&Y_3 = \phi Y_2 + \varepsilon_3 \Rightarrow \varepsilon_3 = Y_3 - \phi Y_2 \\
&\vdots \\
&Y_n = \phi Y_{n-1} + \varepsilon_n \Rightarrow \varepsilon_n = Y_n - \phi Y_{n-1}
\end{aligned}$$

ARIMA Process – MLE: AR(1)

• Example:

- To change the joint from $\boldsymbol{\varepsilon}_t$ to $\mathbf{y}_t$, we need the Jacobian y_{t-2}

$$|J| = \begin{vmatrix} \frac{\partial\varepsilon_2}{\partial Y_2} & \frac{\partial\varepsilon_2}{\partial Y_3} & \cdots & \frac{\partial\varepsilon_2}{\partial Y_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial\varepsilon_n}{\partial Y_2} & \frac{\partial\varepsilon_n}{\partial Y_3} & \cdots & \frac{\partial\varepsilon_n}{\partial Y_n} \end{vmatrix} = \begin{vmatrix} 1 & 0 & \cdots & 0 \\ -\phi & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{vmatrix} = 1$$

$$f\left(Y_2, \cdots, Y_n \middle| Y_1\right) = f\left(\varepsilon_2, \cdots, \varepsilon_n\right)|J| = f\left(\varepsilon_2, \cdots, \varepsilon_n\right)$$

- Then, the likelihood function can be written as

$$L\left(\phi, \sigma_a^2\right) = f\left(Y_1, \cdots, Y_n\right) = f\left(Y_1\right) f\left(Y_2, \cdots, Y_n \middle| Y_1\right) = f\left(Y_1\right) f\left(\varepsilon_2, \cdots, \varepsilon_n\right)$$

$$= \left(\frac{1}{2\pi\gamma_0}\right)^{1/2} e^{-\frac{(Y_1-0)^2}{2\gamma_0}} \left(\frac{1}{2\pi\sigma^2}\right)^{(T-1)/2} e^{-\frac{1}{2\sigma^2}\sum_{t=2}^{T}(Y_t-\phi Y_{t-1})^2}, \quad \text{where } Y_1 \sim N\left(0, \gamma_0 = \frac{\sigma^2}{1-\phi^2}\right).$$

ARIMA Process – MLE: AR(1)

• Example:

- Then,

$$L(\phi,\sigma^2)=\frac{\sqrt{1-\phi^2}}{(2\pi\sigma^2)^{n/2}}\exp\left\{-\frac{1}{2\sigma^2}\left[\sum_{t=2}^{n}(Y_t-\phi Y_{t-1})^2+(1-\phi^2)Y_1^2\right]\right\}$$

- Then, the log likelihood function:

$$\ln L(\phi,\sigma^2)=-\frac{n}{2}\ln 2\pi-\frac{n}{2}\ln\sigma^2-\frac{1}{2}\ln(1-\phi^2)-$$

$$-\frac{1}{2\sigma^2}\underbrace{\left[\underbrace{\sum_{t=2}^{n}(Y_t-\phi Y_{t-1})^2}_{S_*(\phi)}+(1-\phi^2)Y_1^2\right]}_{S(\phi)}$$

- $S^*(\phi)$ is the conditional SS and $S(\phi)$ is the unconditional SS.

ARIMA Process – MLE: AR(1)

• Example:

- F.o.c.'s:

$$\frac{\partial \ln L(\phi,\sigma^2)}{\partial\phi}=0$$

$$\frac{\partial \ln L(\phi,\sigma^2)}{\partial\sigma}=0$$

Note: If we neglect $ln(1-\phi^2)$, then MLE = Conditional LSE.

$$\max_{\phi} L(\phi,\sigma^2)=\min S(\phi).$$

If we neglect both $ln(1-\phi^2)$ and $(1-\phi^2)Y_1^2$, then

$$\max_{\phi} L(\phi,\sigma^2)=\min S(\phi_*).$$

ARIMA Process – Yule-Walker

2) *Yule-Walker for AR(p)*: Regress y_t against $y_{t-1}, y_{t-2}, \ldots, y_{t-p}$
- *Yule-Walker for* ARMA*(p,q):* Method of moments. Not efficient.

Example: For an AR(*p*), we the Yule-Walker equations are

$$\begin{bmatrix} \gamma(0) & \gamma(1) & \cdots & \gamma(p-1) \\ \gamma(1) & \gamma(0) & \cdots & \gamma(p-2) \\ \vdots & \vdots & \cdots & \vdots \\ \gamma(p-1) & \gamma(p-2) & \cdots & \gamma(0) \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_p \end{bmatrix} = \begin{bmatrix} \gamma(1) \\ \gamma(2) \\ \vdots \\ \gamma(p) \end{bmatrix}$$

• MM Estimation: Equate sample moments to population moments, and solve the equation. In this case, we use:

$$E(Y_t) = \frac{1}{T}\sum_{t=1}^{T} Y_t \Rightarrow \mu = \bar{Y}$$

$$E[(Y_t - \mu)(Y_{t-k} - \mu)) = \frac{1}{T}\sum_{t=1}^{T}(Y_t - \mu)(Y_{t-k} - \mu)) \quad \Rightarrow \gamma_k = \hat{\gamma}_k \quad (\&\rho_k = \hat{\rho}_k)$$

ARIMA Process – Yule-Walker

• Then, the Yule-Walker estimator for ϕ is given by solving:

$$\begin{bmatrix} \hat{\gamma}(0) & \hat{\gamma}(1) & \cdots & \hat{\gamma}(p-1) \\ \hat{\gamma}(1) & \hat{\gamma}(0) & \cdots & \hat{\gamma}(p-2) \\ \vdots & \vdots & \cdots & \vdots \\ \hat{\gamma}(p-1) & \hat{\gamma}(p-2) & \cdots & \hat{\gamma}(0) \end{bmatrix} \begin{bmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \\ \vdots \\ \hat{\phi}_p \end{bmatrix} = \begin{bmatrix} \hat{\gamma}(1) \\ \hat{\gamma}(2) \\ \vdots \\ \hat{\gamma}(p) \end{bmatrix} \quad \Rightarrow \hat{\phi} : \begin{cases} \hat{\phi} = \hat{\Gamma}_p^{-1}\hat{\gamma}_p \\ \sigma^2 = \hat{\gamma}_0 - \hat{\phi}'\hat{\gamma}_p \end{cases}$$

• If $\hat{\gamma}_0 > 0$, then $\hat{\Gamma}_m$ is nonsingular.
• If $\{Y_t\}$ is an AR(*p*) process,

$$\hat{\phi} \xrightarrow{d} N\left(\phi, \frac{\sigma^2}{T}\Gamma_p^{-1}\right)$$

$$\hat{\sigma}^2 \xrightarrow{p} \sigma^2$$

$$\hat{\phi}_{kk} \xrightarrow{d} N\left(0, \frac{1}{T}\right) \text{ for k > p.}$$

• Thus, we can use the sample PACF to test for AR order, and we can calculate approximated C.I. for ϕ.

ARIMA Process – Yule-Walker

• Distribution:

If y_t is an AR(p) process, and T is large,

$$\sqrt{T}\left(\hat{\phi}-\phi\right) \overset{approx.}{\sim} N\left(0, \hat{\sigma}^2 \hat{\Gamma}_p^{-1}\right)$$

•100(1−α)% approximate C.I. for ϕ_j is $\hat{\phi}_j \pm z_{\alpha/2} \dfrac{\hat{\sigma}}{\sqrt{T}} \left(\hat{\Gamma}_p^{-1}\right)_{jj}^{1/2}$

• Note: The Yule-Walker algorithm requires $\mathbf{\Gamma}^{-1}$.

• For AR(p). The *Levinson-Durbin (LD) algorithm* avoids $\mathbf{\Gamma}^{-1}$. It is a recursive linear algebra prediction algorithm. It takes advantage that $\mathbf{\Gamma}$ is a symmetric matrix, with a constant diagonal (Toeplitz matrix). Use LD replacing γ with $\hat{\gamma}$.

• Side effect of LD: automatic calculation of PACF and MSPE.

ARIMA Process – Yule-Walker: AR(1)

Example: AR(1) (MM) estimation:

$$y_t = \phi y_{t-1} + \varepsilon_t$$

It is known that $\rho_1 = \phi$. Then, the MME of ϕ is

$$\Rightarrow \rho_1 = \hat{\rho}_1$$

$$\hat{\phi} = \hat{\rho}_1 = \frac{\sum_{t=1}^{T}\left(Y_t - \bar{Y}\right)\left(Y_{t-1} - \bar{Y}\right)}{\sum_{t=1}^{T}\left(Y_t - \bar{Y}\right)^2}$$

• Also, σ^2 is unknown: $\gamma_0 = \dfrac{\sigma^2}{\left(1-\phi^2\right)} \Rightarrow \hat{\sigma}^2 = \hat{\gamma}_0\left(1-\hat{\phi}^2\right)$

ARIMA Process – Yule-Walker: MA(1)

Example: MA(1) (MM) estimation:

$$y_t = \varepsilon_t - \theta\varepsilon_{t-1}$$

Again using the autocorrelation of the series at lag 1,

$$\rho_1 = -\frac{\theta}{(1+\theta^2)} = \hat{\rho}_1$$

$$\theta^2\hat{\rho}_1 + \theta + \hat{\rho}_1 = 0$$

$$\hat{\theta}_{1,2} = \frac{-1 \pm \sqrt{1-4\hat{\rho}_1^2}}{2\hat{\rho}_1}$$

• Choose the root satisfying the invertibility condition. For real roots:

$$1 - 4\hat{\rho}_1^2 \geq 0 \Rightarrow 0.25 \geq \hat{\rho}_1^2 \Rightarrow -0.5 \leq \hat{\rho}_1 \leq 0.5$$

If $\hat{\rho}_1 = \pm 0.5$, unique real roots but non-invertible.

If $|\hat{\rho}_1| < 0.5$, unique real roots and invertible.

ARIMA Process – Yule-Walker

• Remarks

- The MMEs for MA and ARMA models are complicated.

- In general, regardless of AR, MA or ARMA models, the MMEs are sensitive to rounding errors. They are usually used to provide initial estimates needed for a more efficient nonlinear estimation method.

- The moment estimators are not recommended for final estimation results and should not be used if the process is close to being nonstationary or noninvertible.

ARIMA Process – Estimation Hannan-Rissanen

4) *Hannan-Rissanen algorithm for* ARMA(*p,q*)

Steps:
1. Estimate high-order AR.
2. Use Step (1) to estimate (unobserved) noise ε_t
3. Regress y_t against y_{t-1}, y_{t-2}, ..., y_{t-p}, $\hat{\varepsilon}_{t-1}$, ... , $\hat{\varepsilon}_{t-q}$
4. Get new estimates of ε_t. Repeat Step (3).

ARFIMA Process: Fractional Integration

- Consider a simple ARIMA model: $(1-L)^d y_t = \varepsilon_t$

- We went over two cases for d =0 & 1. Granger and Joyeaux (1980) consider the model where $0 \leq d \leq 1$.

- Using the binomial series expansion of $(1 - L)^d$:

$$(1-L)^d = \sum_{k=0}^{\infty}\binom{d}{k}(-L)^k = \sum_{k=0}^{\infty}\frac{d!}{(d-k)!k!}(-L)^k = \sum_{k=0}^{\infty}\frac{\prod_{a=0}^{k-1}(d-a)}{k!}(-L)^k$$
$$= 1 - dL + \frac{d(d-1)L^2}{2!} - \ldots$$

- We can invert $(1 - L)^d \Rightarrow (1-L)^{-d} = 1 + dL + \frac{(d+1)dL^2}{2!} + \ldots$

- In the ARIMA(0,*d*,1): $y_t = (1-L)^{-d}\varepsilon_t = \varepsilon_t + d\varepsilon_{t-1} + \frac{(d+1)d\varepsilon_{t-2}}{2!} + \ldots$

ARFIMA Process: Fractional Integration

• In the ARIMA(0, *d,* 1): $y_t = (1-L)^{-d}\varepsilon_t = \varepsilon_t + d\varepsilon_{t-1} + \frac{(d+1)d\varepsilon_{t-2}}{2!} + \ldots$

• The above MA can be approximated by:

$$y_t = (1-L)^{-d}\varepsilon_t \approx \varepsilon_t + (1+1)^d\varepsilon_{t-1} + (1+2)^{d-1}\varepsilon_{t-2} + \ldots = \sum_{j=0}^{\infty}\beta_j\varepsilon_{t-j}$$

where $\beta_0 = 1$. Convergence depends on *d*.

• The series is covariance stationary if $d < 1/2$.

• ARFIMA models have slow (hyperbolic) decay patterns in the ACF. This type of slow decay patterns also show *long memory* for shocks.

ARFIMA Process: Estimation

• Estimation is complicated. Many methods have been proposed. The majority of them are two-steps procedures. First, we estimate *d*. Then, we fit a traditional ARMA process to the transformed.

Popular estimation methods:
– Based on the log periodogram regressions, due to Geweke and Porter-Hudak (1983), GPH.
– Rescaled range (RR), due to Hurst (1951) & modified by Lo (1991).
– Approximated ML (AML), due to Beran (1995). In this case, all parameters are estimated simulateneously.

ARFIMA Process: Remarks

• In a general review paper, Granger (1999) concludes that *ARFIMA* processes may fall into the *empty box* category – i.e., models with stochastic properties that do not mimic the properties of the data.

• Leybourne, Harris, and McCabe (2003) find some forecasting power for long series. Bhardwaj and Swanson (2004) find ARFIMA useful at longer forecast horizons.

ARFIMA Process: Example

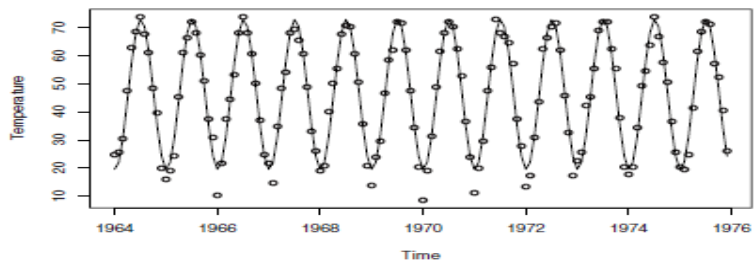
• From Bhardwaj and Swanson (2004)

Table 2: Analysis of U.S. S&P500 Daily Absolute Returns (*)

Estimation Scheme and Forecast Horizon	ARFIMA Model	d	non-ARFIMA Model	DM	ENC-t	DM Best vs. RW
1 day ahead, recursive	WHI (1,1)	0.41 (0.0001)	ARMA(4,2)	-1.18	0.47	-13.64
5 day ahead, recursive	GPH (1,2)	0.57 (0.0011)	ARMA(4,2)	-0.71	1.75	-10.10
20 day ahead, recursive	GPH (1,2)	0.57 (0.0011)	ARMA(4,2)	-0.68	2.91	-5.96
120 day ahead, recursive	GPH (1,2)	0.57 (0.0011)	ARMA(4,2)	0.38	7.52	-6.33
240 day ahead, recursive	GPH (1,2)	0.57 (0.0011)	ARMA(4,2)	0.52	10.22	-6.16
1 day ahead, rolling	RR (1,1)	0.25 (0.0009)	ARMA(4,2)	2.02	4.56	-12.44
5 day ahead, rolling	GPH (1,2)	0.55 (0.0044)	ARMA(4,2)	-2.28	0.26	-10.24
20 day ahead, rolling	GPH (1,2)	0.55 (0.0044)	ARMA(4,2)	-2.44	0.79	-5.91
120 day ahead, rolling	GPH (1,2)	0.55 (0.0044)	ARMA(4,2)	-4.07	0.09	-6.32
240 day ahead, rolling	RR (1,1)	0.25 (0.0009)	ARMA(4,2)	-2.62	2.72	-5.90

SARIMA Process: Seasonal Time Series

• A time series repeats itself after a regular period of time.



• "Business cycle effects" in macroeconomics "time of the day" in trading patterns, "Monday effect" for stock returns, "9 to 5 effect" for electricity demand, etc.

• The smallest time period for this repetitive phenomenon is called a seasonal period, s.

Seasonal Time Series

• Two types of seasonal behavior:
- Deterministic – Usual treatment: Build a deterministic function,

$$f(t) = f(t + k \times s), \qquad k = 0, \pm 1, \pm 2, \cdots$$

We can include seasonal (means) dummies. Instead of dummies, trigonometric functions (sum of cosine curves) can be used. A linear time trend is often included in both cases.

-Stochastic – Usual treatment: SARIMA model. For example:

$$(1 - L^s)(1 - L)^d y_t = (1 - \theta L)(1 - \Theta L^s)\varepsilon_t$$

where s the seasonal periodicity or frequency of y_t.

Seasonal Time Series - SARIMA

• For stochastic seasonality, we use the Seasonal ARIMA model. In general, we have the SARIMA$(P, D, Q)_s$

$$\Phi_P(B^s)(1-B^s)^D y_t = \theta_0 + \Theta_Q(B^s)\varepsilon_t$$

where θ_0 is constant and

$$\Phi_P(B^s) = 1 - \Phi_1 B^s - \Phi_2 B^{2s} - \cdots - \Phi_P B^{sP}$$

$$\Theta_Q(B^s) = 1 - \Theta_1 B^s - \Theta_2 B^{2s} - \cdots - \Theta_Q B^{sQ}$$

<u>Example 1</u>: SARIMA$(0,0,1)_{12}$= SMA$(1)_{12}$

$$y_t = \theta_0 + \varepsilon_t - \Theta\varepsilon_{t-12}$$

- Invertibility Condition: $|\Theta| < 1$.
- $E(y_t) = \theta_0$.
- $Var(y_t) = (1+\Theta^2)\sigma^2$

$$ACF: \rho_k = \begin{cases} \dfrac{-\Theta}{1+\Theta^2}, & |k| = 12 \\ 0, & \text{otherwise} \end{cases}$$

Seasonal Time Series - SARIMA

<u>Example 1</u>: SARIMA$(1,0,0)_{12}$= AR$(1)_{12}$

$$(1-\Phi B^{12})y_t = \theta_0 + \varepsilon_t$$

- This is a simple seasonal AR model.
- Stationarity Condition: $|\Phi|<1$.

$$E(Y_t) = \frac{\theta_0}{1-\Phi}$$

$$Var(Y_t) = \frac{\sigma_a^2}{1-\Phi^2}$$

$$ACF: \rho_{12k} = \Phi^k, k = 0, \pm 1, \pm 2, \cdots$$

• When $\Phi = 1$, the series is non-stationary. To test for a unit root, consider seasonal unit root tests.

Seasonal Time Series – Multiplicative SARIMA

• A special, parsimonious class of seasonal time series models that is commonly used in practice is the multiplicative seasonal model *ARIMA(p, d, q)(P,D,Q)*$_s$.

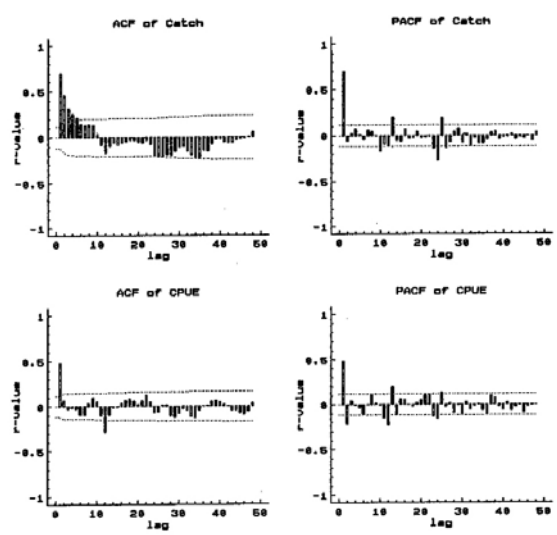
$$\phi_p(L)\Phi_P(L^s)(1-L)^d(1-L^s)^D y_t = \theta_0 + \theta_q(L)\Theta_Q(L^s)\varepsilon_t$$

where all zeros of $\phi(L)$; $\Phi(L^s)$; $\theta(L)$ and $\Theta(L^s)$ lie outside the unit circle. Of course, there are no common factors between $\phi(L)\Phi(L^s)$ and $\theta(L)\Theta(L^s)$.

• When $\Phi = 1$, the series is non-stationary. To test for a unit root, consider seasonal unit root tests.

Seasonal Time Series – Multiplicative SARIMA

• The ACF and PACF can be used to discover seasonal patterns.



Seasonal Time Series – Multiplicative SARIMA

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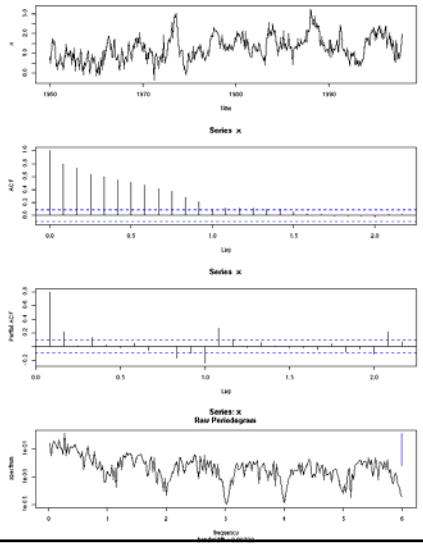

Seasonal Time Series – Multiplicative SARIMA

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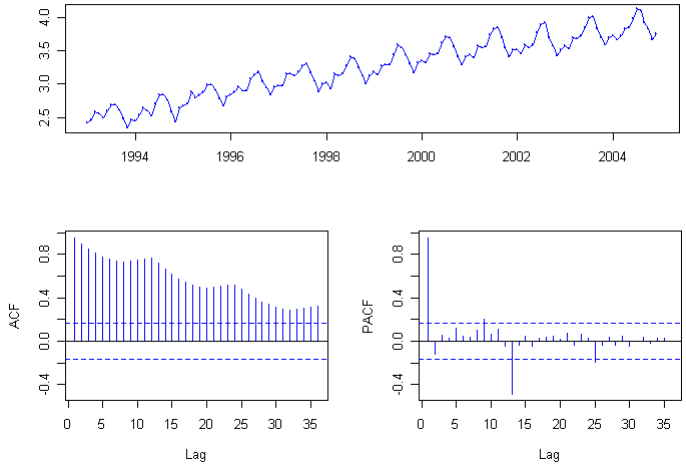

Seasonal Time Series – Multiplicative SARIMA

• We usually work with the RHS variable, $W_t = (1 - B)(1 - B^{12})y_t$

$$W_t = (1 - \theta L)(1 - \Theta L^{12})a_t$$
$$W_t = a_t - \theta a_{t-1} - \Theta a_{t-12} + \theta\Theta a_{t-13}$$
$$W_t \sim I(0)$$

$$\gamma_k = \begin{cases} (1+\theta^2)(1+\Theta^2)\sigma^2, & k = 0 \\ -\theta(1+\Theta^2)\sigma^2, & |k| = 1 \\ -\Theta(1+\theta^2)\sigma^2, & |k| = 12 \\ \theta\Theta\sigma^2, & |k| = 11,13 \\ 0, & otherwise \end{cases} \qquad \rho_k = \begin{cases} \dfrac{-\theta}{(1+\theta^2)}, & |k| = 1 \\ \dfrac{-\Theta}{(1+\Theta^2)}, & |k| = 12 \\ \dfrac{\theta\Theta}{(1+\theta^2)(1+\Theta^2)}, & |k| = 11,13 \\ 0, & otherwise \end{cases}.$$

Seasonal Time Series – Multiplicative SARIMA

• Most used seasonal model in practice: SARIMA(0,1,1)(0,1,1)$_{12}$

$$(1 - L)(1 - L^{12})y_t = (1 - \theta L)(1 - \Theta L^{12})\varepsilon_t$$

where $|\theta|<1$ and $|\Theta|<1$.

• This model is the most used seasonal model in practice. It was used by Box and Jenkins (1976) for modeling the well-known monthly series of airline passengers. It is called the *airline model*.

• We usually work with the RHS variable, $W_t = (1 - L)(1 - L^{12})y_t$.

$(1 - L)$: "regular" difference

$(1 - L^{12})$: "seasonal" difference.

Seasonal Time Series – Seasonal Unit Roots

• If a series has seasonal unit roots, then standard ADF test statistic do not have the same distribution as for non-seasonal series.

• Furthermore, seasonally adjusting series which contain seasonal unit roots can alias the seasonal roots to the zero frequency, so there is a number of reasons why economists are interested in seasonal unit roots.

• See Hylleberg, S., Engle, R.F., Granger, C. W. J., and Yoo, B. S., Seasonal integration and cointegration,(1990, *Journal of Econometrics*).

Non-Stationarity in Variance

• Stationarity in mean does not imply stationarity in variance

• Non-stationarity in mean implies non-stationarity in variance.

• If the mean function is time dependent:

1. The variance, $Var(y_t)$ is time dependent.
2. $Var[y_t]$ is unbounded as $t \to \infty$.
3. Autocovariance functions and ACFs are also time dependent.
4. If t is large wrt y_0, then $\rho_k \approx 1$.

• It is common to use variance stabilizing transformations: Find a function $G(.)$ so that the transformed series $G(y_t)$ has a constant variance. For example, the Box-Cox transformation:

$$T(Y_t) = \frac{Y_t^{\lambda} - 1}{\lambda}$$

Variance Stabilizing Transformation - Remarks

• Variance stabilizing transformation is only for positive series. If a series has negative values, then we need to add each value with a positive number so that all the values in the series are positive.

• Then, we can search for any need for transformation.

• It should be performed before any other analysis, such as differencing.

• Not only stabilize the variance, but we tend to find that it also improves the approximation of the distribution by Normal distribution.