Lecture 9
Models for Censored and Truncated Data – Truncated Regression and Sample Selection

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Censored and Truncated Data: Definitions

• Y is **censored** when we observe X for all observations, but we only know the true value of Y for a restricted range of observations. Values of Y in a certain range are reported as a single value or there is significant clustering around a value, say 0.

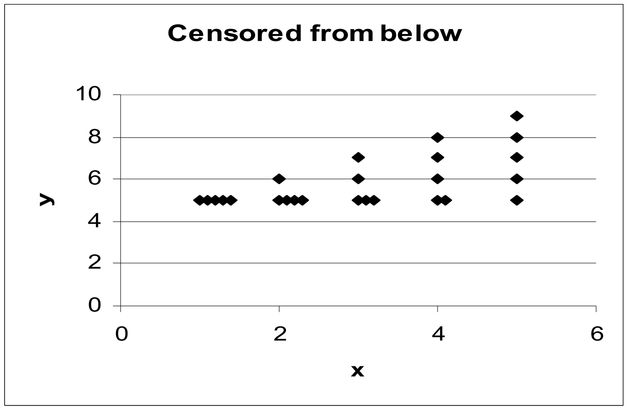
- If $Y = k$ or $Y > k$ for all $Y \Rightarrow Y$ is *censored from below* or *left-censored.*

- If $Y = k$ or $Y < k$ for all $Y \Rightarrow Y$ is *censored from above* or *right-censored.*

We usually think of an uncensored Y, Y^*, the true value of Y when the censoring mechanism is not applied. We typically have all the observations for $\{Y,X\}$, but not $\{Y^*, X\}$.

• Y is **truncated** when we only observe X for observations where Y would not be censored. We do not have a full sample for $\{Y, X\}$, we exclude observations based on characteristics of Y.

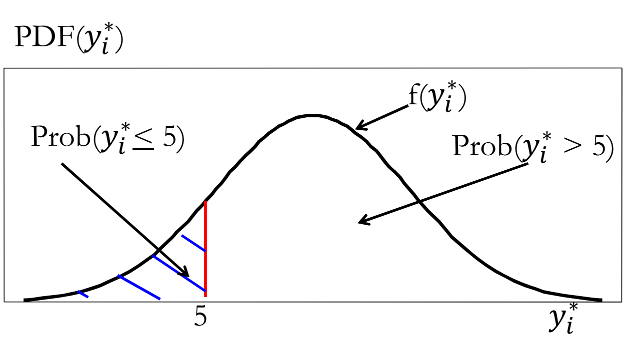
Censored from below: Example



• If $Y \leq 5$, we do not know its exact value.

Example: A Central Bank intervenes if the exchange rate, Y, hits the band's lower limit. If $Y \leq \bar{E} \Rightarrow Y = \bar{E}$.

Censored from below: Example



• The pdf of the observable variable, y, is a mixture of discrete (prob. mass at Y=5) and continuous (Prob[Y^*>5]) distributions.

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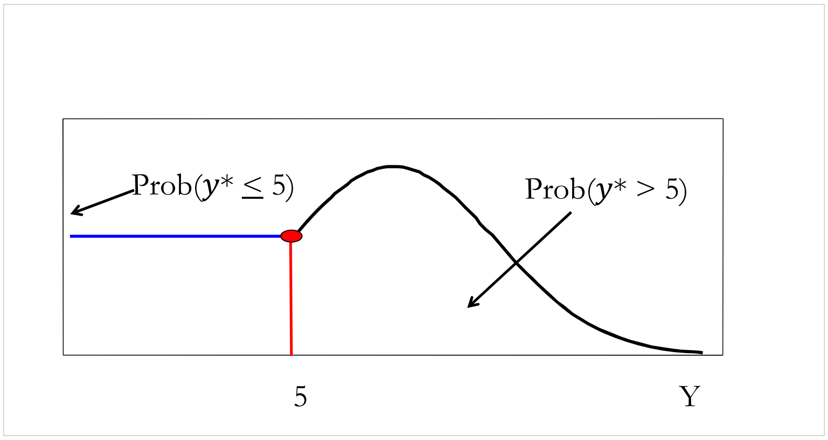
Censored from below: Example



• Under censoring we assign the full probability in the censored region to the censoring point, 5.

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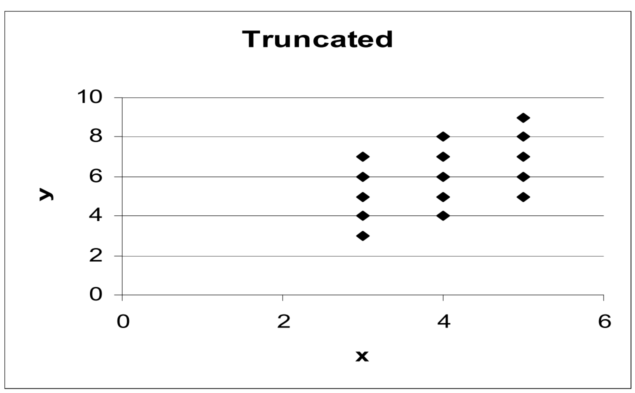
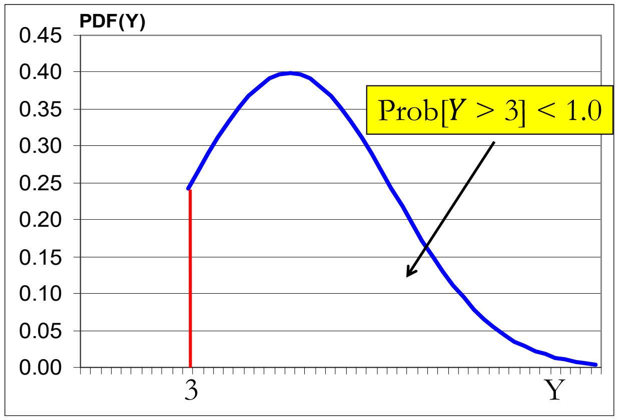
Truncated Data: Example



• If $Y < 3$, the value of X (or Y) is unknown. *(Truncation from below.)*

Example: If a family's income is below certain level, we have no information about the family's characteristics.

Truncated Data: Example



• Under data censoring, the censored distribution is a combination of a pmf plus a pdf. They add up to 1. We have a different situation under truncation. To create a pdf for Y we will use a conditional pdf.

Truncated regression

• Truncated regression is different from censored regression in the following way:

Censored regressions: The dependent variable may be censored, but you can include the censored observations in the regression

Truncated regressions: A subset of observations are dropped, thus, only the truncated data are available for the regression.

• Q: Why do we have truncation?

(1) ***Truncation by survey design***: Studies of poverty. By survey's design, families whose incomes are greater than that threshold are dropped from the sample.

(2) **Incidental Truncation**: Wage offer married women. Only those who are working have wage information. It is the people's decision, not the survey's design, that determines the sample selection.

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Truncation and OLS

Q: What happens when we apply OLS to a truncated data?

- Suppose that you consider the following regression:

$$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i = \boldsymbol{x}_i'\beta + \varepsilon_i,$$

- We have a random sample of size N. All CLM assumptions are satisfied. (The most important assumption is (**A2**) E$[\varepsilon_i|\boldsymbol{x}_i] = 0$.)

- Instead of using all the N observations, we use a subsample. Then, run OLS using this sub-sample (truncated sample) only.

• Q: Under what conditions, does **sample selection** matter to OLS?

(A) OLS is Unbiased

(A-1) Sample selection is randomly done.

(A-2) Sample selection is determined solely by the value of ***x*-variable**. For example, suppose that x is age. Then if you select sample if age is greater than 20 years old, this OLS is unbiased.

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Truncation and OLS

(B) OLS is Biased

(B-1) Sample selection is determined by the value of ***y*-variable**.

Example: We are studying the determinants of hedging, y. We select the sample if y is greater than certain threshold. Then this OLS is biased.

(B-2) Sample selection is correlated with ε_i.

Example: We run a wage regression $y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$, where ε_i contains unobserved ability. If sample is selected based on the unobserved ability, this OLS is biased.

- In practice, this situation happens when the selection is based on the survey participant's decision. Since the decision to participate is likely to be based on unobserved factors which are contained in ε_i, the selection is likely to be correlated with ε_i.

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Truncation and OLS: When does (A2) hold?

• Consider the previous regression:

$$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$$

- All CLM assumptions are satisfied.

- Instead of using all the N observations, we use a subsample. Let s_i be a selection indicator: If $s_i = 1$, then person i is included in the regression. If $s_i = 0$, then person i is dropped from the data.

• If we run OLS using the selected subsample, we use only the observation with $s_i = 1$. That is, we run the following regression:

$$s_i y_i = \beta_0 s_i + \beta_1 s_i x_i + s_i \varepsilon_i$$

Now, $s_i x_i$ is the explanatory variable, and $u_i = s_i \varepsilon_i$ is the error term.

• OLS is unbiased if $E[u_i = s_i \varepsilon_i | s_i x_i] = 0$.

⇒ under what conditions is this new (**A2**) satisfied?

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Truncation and OLS: When does (A2) hold?

Q: When does $E[u_i = s_i \varepsilon_i | s_i x_i]$ hold?

It is sufficient to check: $E[u_i | s_i x_i]$. (If this is zero, then new (**A2**) is also zero.)

• $E[u_i | s_i x_i] = s_i E[\varepsilon_i | s_i x_i]$ - s_i is in the conditional set.

• It is sufficient to check the condition which ensures $E[\varepsilon_i | x_i, s_i] = 0$.

• CASES:

(A-1) Sample selection is done randomly.

s_i is independent of ε_i and x_i ⇒ $E[\varepsilon_i | x_i, s_i] = E[\varepsilon_i | x_i]$

Since CLM assumptions are satisfied ⇒ we have $E[\varepsilon_i | x_i] = 0$.

⇒ OLS is unbiased.

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Truncation and OLS: When does (A2) hold?

(A-2) Sample is selected based solely on the value of x-variable.

Example: We study trading in stocks, y_i. One of the dependent variables, x_i, is wealth, and we select person i if wealth is greater than 50K. Then,

$s_i = 1$ if $x_i \geq$ 50K,

$s_i = 0$ if $x_i <$ 50K.

-Now, s_i is a deterministic function of x_i.

• Since s_i is a deterministic function of x_i, $s_i(x_i)$, it drops out from the conditioning set. Then,

$$\text{E}[\varepsilon_i | x_i, s_i] = \text{E}[\varepsilon_i | x_i, s_i(x_i)]$$

$$= \text{E}[\varepsilon_i | x_i] = 0$$ - CLM assumptions satisfied.

⇒ OLS is unbiased.

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Truncation and OLS: When does (A2) hold?

(B-1) *Sample selection is based on the value of y-variable.*

Example: We study determinants of wealth, y. We select individuals whose wealth is smaller than 150K. Then, $s_i = 1$ if $y_i <$150K.

- Now, s_i depends on y_i (and ε_i). It cannot be dropped out from the conditioning set like we did before. Then,

$$\text{E}[\varepsilon_i | x_i, \ s_i] \neq \text{E}[\varepsilon_i | x_i] = 0.$$

• For example, $\text{E}[\varepsilon_i | x_i, s_i] = \text{E}[\varepsilon_i | x_i, s_i(x_i)]$

$$\text{E}[\varepsilon_i | x_i, s_i = 1] = \text{E}[\varepsilon_i | x_i, y_i \leq 150\text{K}]$$

$$= \text{E}[\varepsilon_i | x_i, \beta_0 + \beta_1 x_i + \varepsilon_i \leq 150\text{K}]$$

$$= [\varepsilon_i | x_i, \varepsilon_i \leq 150\text{K} - (\beta_0 + \beta_1 x_i)]$$

$$\neq \text{E}[\varepsilon_i | x_i] = 0$$ ⇒ OLS is biased.

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Truncation and OLS: When does (A2) hold?

(B-2) *Sample selection is correlated with* u_i.
The inclusion of a person in the sample depends on the person's decision, not the surveyor's decision. This type of truncation is called the *incidental truncation*. The bias that arises from this type of sample selection is called the **Sample Selection Bias**.

Example: Dividend payments model:

$$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i.$$

Since it is a company's decision to pay dividends –i.e., to participate–, this sample selection is likely to be based on some unobservable factors which are contained in ε_i. Like in **(B-1)**, s_i cannot be dropped out from the conditioning set:

$$\text{E}[\varepsilon_i | x_i, s_i] \neq \text{E}[\varepsilon_i | x_i] = 0 \qquad \Rightarrow \text{OLS is biased.}$$

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Truncation and OLS: When does (A2) hold?

• CASE (A-2) can be more complicated, when the selection rule based on the ***x*-variable** may be correlated with ε_i.

Example: x is IQ. A survey participant responds if IQ > v.
Now, the sample selection is based on x-variable *and* a random error v.

Q: If we run OLS using only the truncated data, will it cause a bias?
Two cases:
- (1) If v is independent of ε, then it does not cause a bias.
- (2) If v is correlated with ε, then this is the same case as **(B-2)**. Then, OLS will be biased.

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Estimation with Truncated Data.

• CASES

- Under cases (A-1) and (A-2), OLS is appropriate.
- Under case **(B-1)**, we use **Truncated regression**.
- Under case **(B-2)** –i.e., incidental truncation-, we use the ***Heckman Sample Selection Correction*** method. This is also called the **Heckit model**.

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Truncated Regression

• Data truncation is **(B-1)**: the truncation is based on the ***y*-variable**.

• We have the following regression satisfies all CLM assumptions:

$$y_i = \boldsymbol{x}_i'\beta + \varepsilon_i, \qquad \varepsilon_i \sim \text{N}(0, \sigma^2)$$

- We sample only if $y_i < c_i$

$\Rightarrow$ Observations are dropped if $y_i \geq c_i$ by design.

- We know the exact value of c_i for each person.

• We know that OLS on the truncated data will be biased. The model that produces unbiased estimate is based on ML Estimation.

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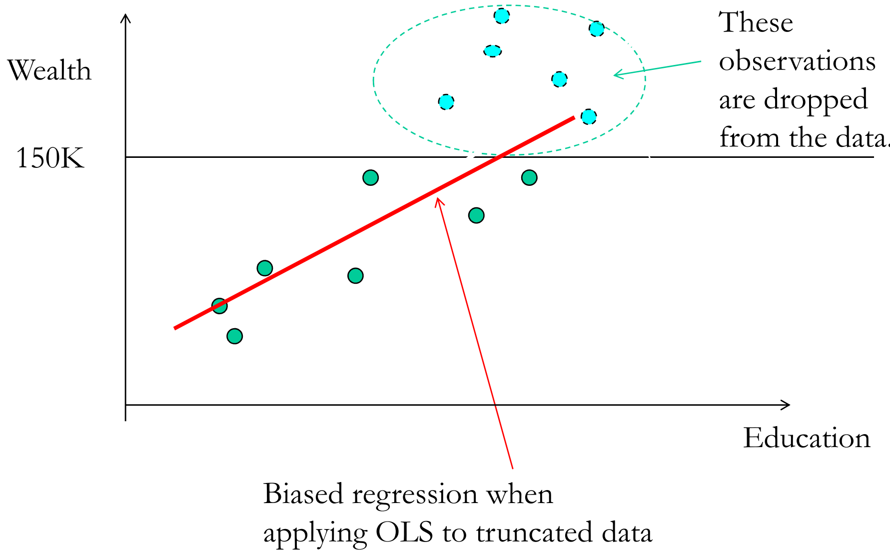
Truncated Regression



Truncated Regression: Conditional Distribution

• Given the normality assumption for ε_i, ML is easy to apply.

- For each, $\varepsilon_i = y_i - \boldsymbol{x}_i'\beta$, the likelihood contribution is $f(\varepsilon_i)$.

- But, we select sample only if $y_i < c_i$

⇒ we have to use the density function of ε_i conditional on $y_i < c_i$:

$$f(\varepsilon_i|\, y_i < c_i) = f(\varepsilon_i|\, \varepsilon_i < c_i - \boldsymbol{x}_i'\beta) = \frac{f(\varepsilon_i)}{P(\varepsilon_i < c_i - \boldsymbol{x}_i'\beta)}$$

$$= \frac{f(\varepsilon_i)}{P\left(\frac{\varepsilon_i}{\sigma} < \frac{c_i - \boldsymbol{x}_i'\beta}{\sigma}\right)} = \frac{f(\varepsilon_i)}{\Phi\left(\frac{c_i - \boldsymbol{x}_i'\beta}{\sigma}\right)}$$

$$= \frac{\frac{1}{\sigma}\phi\left(\frac{\varepsilon_i}{\sigma}\right)}{\Phi\left(\frac{c_i - \boldsymbol{x}_i'\beta}{\sigma}\right)}$$

Truncated Normal (Again)

• Moments:

Let $y^* \sim$ N(μ^*, σ²) and $\alpha = \frac{(c - \mu^*)}{\sigma}$.

- **First moment**:

E[y^* | $y > c$] = μ^* + σ λ(α) <= This is the truncated regression.

⇒ If μ >0 and the truncation is from below –i.e., λ(α)>0–, the mean of the truncated variable is greater than the original mean

Note: For the standard normal distribution λ(α) is the mean of the truncated distribution.

- **Second moment**:

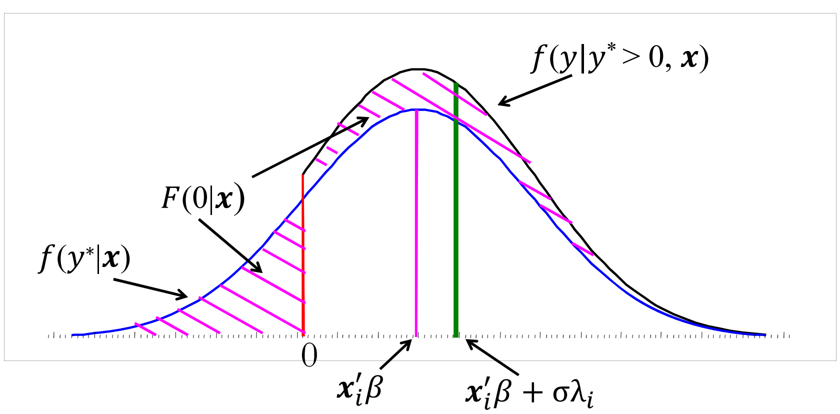
- Var[y^* | $y > c$] = σ²[1 − $\delta(\alpha)$] where $\delta(\alpha)$= λ(α) [λ(α) − α]

⇒ Truncation reduces variance! This result is general, it applies to upper or lower truncation given that 0 ≤ $\delta(\alpha)$ ≤ 1

Truncated Normal (Again)

Model: $y_i^* = \boldsymbol{x}_i'\beta + \varepsilon_i$

Observed Data: $y_i = y_i^* \,|\, y_i^* > 0$



• Truncated (from below, $y_i^* > 0$) regression model:

$$\text{E}[y_i|\, y_i^* > 0, \boldsymbol{x}_i] = \boldsymbol{x}_i'\beta + \sigma\lambda_i > \text{E}[y_i|\boldsymbol{x}_i]$$

Truncated Regression: ML Estimation

• The likelihood contribution for i^{th} observation is given by

$$L_i(\boldsymbol{\beta}, \sigma) = \frac{\frac{1}{\sigma}\phi(\frac{y_i - x_i'\beta}{\sigma})}{\Phi\left(\frac{c_i - x_i'\beta}{\sigma}\right)}$$

ln(joint density of N values of y_i^*)

• The likelihood function is given by (with $c_i = 0$):

$$Log\ L(\beta, \sigma) = \sum_{i=1}^{N} \log L_i = -\frac{N}{2}[\log(2\pi) + \log(\sigma^2)] - \frac{1}{2\sigma^2}\sum_{i=1}^{N}\varepsilon_i^2$$

$$-\sum_{i=1}^{N}\log\left[\Phi(\frac{x_i'\beta}{\sigma})\right]$$

log(joint probability of $y_i^* > 0$)

• The values of $(\boldsymbol{\beta}, \sigma)$ that maximizes Log L are the ML estimators of the *Truncated Regression.*

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The partial effects

• The estimated parameters β_k measures the effect of x_k on y for participating individual. Thus,

$$\frac{\delta E[y_i | y_i > 0, \boldsymbol{x_i}'\boldsymbol{\beta}]}{\delta x} = \beta_k + \sigma\,\frac{\delta\lambda(\boldsymbol{x}_i'\boldsymbol{\beta})}{\delta x} = \beta_k + \sigma\,\frac{\delta\lambda(\boldsymbol{x}_i'\boldsymbol{\beta})}{\delta x} =$$
$$= \beta_k * (1 - d_i)$$

with $d_i = \lambda(\boldsymbol{x}_i'\boldsymbol{\beta}) * [\lambda(\boldsymbol{x}_i'\boldsymbol{\beta}) + \boldsymbol{x}_i'\boldsymbol{\beta}]$.

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Truncated Regression: MLE – Example

• DATA: From a survey of family income in Japan (JPSC_familyinc.dta). The data is originally not truncated.

Model: $y_i = \beta_0 + \beta_1 x_i + \upsilon_i$

y_i = family income in JPY 10,000

x_i: husband's education

• Three cases:

EX1. Use all observations to estimate model

EX2. Truncate sample from above (y_i < 800). Then run the OLS using on the truncated sample.

EXe. Run the truncated regression model for the data truncated from above.


```
. reg familyinc huseduc

      Source |       SS       df       MS              Number of obs =    7695
-------------+------------------------------           F(  1,  7693) =  924.22
       Model |  38305900.9     1  38305900.9           Prob > F      =  0.0000
    Residual |   318850122  7693  41446.7856           R-squared     =  0.1073
-------------+------------------------------           Adj R-squared =  0.1071
       Total |   357156023  7694  46420.0705           Root MSE      =  203.58

------------------------------------------------------------------------------
   familyinc |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
     huseduc |   32.93413   1.083325    30.40   0.000     30.81052    35.05775
       _cons |    143.895   15.09181     9.53   0.000     114.3109     173.479
------------------------------------------------------------------------------
```

OLS using all the observations, unbiased estimated β_1= 32.93413.

```
. reg familyinc huseduc if familyinc<800

      Source |       SS       df       MS              Number of obs =    6274
-------------+------------------------------           F(  1,  6272) =  602.70
       Model |  11593241.1     1  11593241.1           Prob > F      =  0.0000
    Residual |   120645494  6272  19235.5699           R-squared     =  0.0877
-------------+------------------------------           Adj R-squared =  0.0875
       Total |   132238735  6273  21080.621            Root MSE      =  138.69

------------------------------------------------------------------------------
   familyinc |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
     huseduc |   20.27929   .8260432    24.55   0.000     18.65996    21.89861
       _cons |   244.5233   11.33218    21.58   0.000     222.3084    266.7383
------------------------------------------------------------------------------
```

OLS on truncated sample.

The parameter on husband's education is biased towards zero.

```
. truncreg familyinc huseduc, ul(800)
(note: 1421 obs. truncated)

Fitting full model:

Iteration 0:   log likelihood = -39676.782
Iteration 1:   log likelihood = -39618.757
Iteration 2:   log likelihood = -39618.629
Iteration 3:   log likelihood = -39618.629

Truncated regression
Limit:   lower =       -inf                    Number of obs =    6274
         upper =        800                    Wald chi2(1)  =  569.90
Log likelihood = -39618.629                    Prob > chi2   =  0.0000
```

familyinc	Coef.	Std. Err.	z	P>|z|	[95% Conf.	Interval]
huseduc	24.50276	1.0264	23.87	0.000	22.49105	26.51446
_cons	203.6856	13.75721	14.81	0.000	176.7219	230.6492
/sigma	153.1291	1.805717	84.80	0.000	149.59	156.6683

Truncated regression model on the truncated sample

Note: Bias seems to be corrected, but not perfect in this example.

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Sample Selection Bias Correction Model

• The most common case of truncation is **(B-2)**: Incidental truncation.

• This data truncation usually occurs because sample selection is determined by the people's decision, not the surveyor's decision.

• Back to the wage regression example. If person i has chosen to participate (work), person i has *self-selected into the sample.* If person i has decided not to participate, person i has self-selected out of the sample.

• The bias caused by this type of truncation is called *sample selection bias.*

• This model involves two decisions: (1) participation and (2) amount. It is a generalization of the Tobit Model.

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Tobit Model: Type II

• Different ways of thinking about how the latent variable and the observed variable interact produce different Tobit Models.

• The Type I Tobit Model presents a simple relation:

- $y_i = 0$ if $y_i^* = \boldsymbol{x}_i'\boldsymbol{\beta} + \varepsilon_i \leq 0$
  $= y_i^* = \boldsymbol{x}_i'\boldsymbol{\beta} + \varepsilon_i,$ if $y_i^* = \boldsymbol{x}_i'\boldsymbol{\beta} + \varepsilon_i > 0$

The effect of the X's on the probability that an observation is censored and the effect on the conditional mean of the non-censored observations are the same: $\boldsymbol{\beta}$.

• The Type II Tobit Model presents a more complex relation:

- $y_i = 0$ if $y_i^* = \boldsymbol{x}_i'\alpha + \varepsilon_{1,i} \leq 0$, $\varepsilon_{1,i} \sim N(0, 1)$
  $= \boldsymbol{x}_i'\boldsymbol{\beta} + \varepsilon_{2,i}$ if $y_i^* = \boldsymbol{x}_i'\alpha + \varepsilon_{2,i} > 0$, $\varepsilon_{2,i} \sim N(0, \sigma_2^2)$

Now, we have different effects of the $\boldsymbol{x}$'s.

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Tobit Model: Type II

• The Type II Tobit Model:

- $y_i = 0$ if $y_i^* = \boldsymbol{x}_i'\boldsymbol{\alpha} + \varepsilon_{1,i} \leq 0$, $\varepsilon_{1,i} \sim N(0, \sigma_1^2=1)$
  $= \boldsymbol{x}_i'\boldsymbol{\beta} + \varepsilon_{2,i}$ if $y_i^* = \boldsymbol{x}_i'\boldsymbol{\alpha} + \varepsilon_{2,i} > 0$, $\varepsilon_{2,i} \sim N(0, \sigma_2^2)$

- A more flexible model. $\boldsymbol{x}$ can have an effect on the decision to participate (Probit part) and a different effect on the amount decision (truncated regression).

• Type I is a special case: $\varepsilon_{2,i} = \varepsilon_{1,i}$ and $\boldsymbol{\alpha} = \boldsymbol{\beta}$.

Example: Age affects the decision to donate to charity. But it can have a different effect on the amount donated. We may find that age has a positive effect on the decision to donate, but given a positive donation, younger individuals donate more than older individuals.

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Tobit Model: Type II

• The model assumes a **bivariate normal distribution** for ($\varepsilon_{1,i}$, $\varepsilon_{2,i}$); with covariance given by $\sigma_{12}(= \rho\ \sigma_1 \sigma_2)$.

- Conditional expectation:

$$E[y_i \mid y_i > 0, \boldsymbol{x}_i] = \boldsymbol{x}_i{}'\boldsymbol{\beta} + \sigma_{12}\ \lambda(\boldsymbol{x}_i{}'\boldsymbol{\alpha}) \qquad (\sigma_{12}(= \rho\ \sigma_2))$$

- Unconditional Expectation

$$E[y_i \mid \boldsymbol{x}_i] = \text{Prob}(y_i{>}0 \mid \boldsymbol{x}_i) * E[y_i \mid y_i{>}0, \boldsymbol{x}_i] + \text{Prob}(y_i{=}0 \mid \boldsymbol{x}_i) * 0$$
$$= \text{Prob}(y_i > 0 \mid \boldsymbol{x}_i) * E[y_i \mid y_i > 0, \boldsymbol{x}_i]$$
$$= \Phi(\boldsymbol{x}_i{}'\boldsymbol{\alpha}) * [\boldsymbol{x}_i{}'\boldsymbol{\beta} + \sigma_{12}\ \lambda(\boldsymbol{x}_i{}'\boldsymbol{\alpha})]$$

Note: This model is known as the Heckman selection model, or the Type II Tobit model (Amemiya), or the probit selection model (Wooldridge). 31

Tobit Model: Type II – Sample selection

• Now, we generalize the model presented, making the decision to participate dependent on a different variable, z. Then,

- $y_i = 0$ if $y_i^* = z_i{}'\boldsymbol{\alpha} + \varepsilon_{1,i} \leq 0$, $\varepsilon_{1,i} \sim N(0, \sigma_2{}^2{=}1)$
 $= \boldsymbol{x}_i{}'\boldsymbol{\beta} + \varepsilon_{2,i}$ if $y_i^* = z_i{}'\boldsymbol{\alpha} + \varepsilon_{2,i} > 0$, $\varepsilon_{2,i} \sim N(0, \sigma_2{}^2)$

• This model is called the **Sample selection model**, due to Heckman.

Example (from Heckman (*Econometrica*, 1979): Structural Labor model:

• Labor Supply equation:

$$h_i^* = \delta_0 + \delta_1\ w_i + \boldsymbol{Z}_i{}'\boldsymbol{\delta}_2 + \varepsilon_i \qquad (1)$$

- h_i^*: desired hours by i^{th} person (latent variable)
- w_i: wage that could be earned
- $\boldsymbol{Z}_i$: non-labor income, taste variables (married, kids, etc.)
- ε_i (error term): unobserved taste for work. 32

Tobit Model: Type II – Sample selection

Example (from Heckman) (continuation)
• Market wage equation (equation of interest):

$$w_i = \boldsymbol{x}_i{}'\boldsymbol{\beta} + u_i \qquad (2)$$

- $\boldsymbol{x}_i$: productivity, age, education, previous experience, etc.
- u_i (error term): unobserved wage earning ability.
- u_i & ε_i are assumed to follow a bivariate distribution (usually, a normal)

We observe w_i for only those who work –i.e., $h_i^* > 0$.

Goal: Estimation of wage offer equation for people of working age

Q: The sample is non longer random. How can we estimate (2) if we only observe w_i (wages) for those who work?

• Problem: Selection bias. Non-participation is rarely random

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Tobit Model: Type II – Selection Bias

• Selection bias: Non-participation is rarely random
- Not distributed equally across subgroups
- Agents decide to participate or not –i.e., self-select into a group.

Q: Can we test for selection bias?

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Tobit Model: Type II – Terminology

• Terminology:

- **Selection equation**:

$y_i^* = \boldsymbol{z}_i{}'\boldsymbol{\alpha} + \varepsilon_{1,i}$ (often, a latent variable equation, say market wage vs. value of home production)

- **Selection Rule**:

- $y_i = 0$ if $y_i^* \leq 0$ $\Rightarrow D_i = 0,$
  $= \boldsymbol{x}_i{}'\boldsymbol{\beta} + \varepsilon_{2,i}$ if $y_i^* > 0$ $\Rightarrow D_i = 1,$

- **Outcome equation:**

$y_i = \boldsymbol{x}_i{}'\boldsymbol{\beta} + \varepsilon_{2,i}$ (the primary equation of interest)

• To derive moments, we need to make an assumption about the distribution of the errors, $\varepsilon_{1,i}$ & $\varepsilon_{2,i}$. In the Heckman model, we assume a bivariate normal joint distribution.

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Tobit Model: Type II – Expectations

• **Expectations**: Under incidental truncation with a bivariate normal distribution we have:

- Conditional expectation (when is y_i observed) :

$$E[y_i \,|\, y_i > 0, \boldsymbol{x}_i] = \boldsymbol{x}_i{}'\boldsymbol{\beta} + \sigma_{12}\, \lambda(\frac{\boldsymbol{z}_i'\boldsymbol{\alpha}}{\sigma_1})$$

- Unconditional Expectation:

$$E[y_i \,|\, \boldsymbol{x}_i] = \Phi(\frac{\boldsymbol{z}_i'\boldsymbol{\alpha}}{\sigma_1}) * [\boldsymbol{x}_i{}'\boldsymbol{\beta} + \sigma_{12}\, \lambda(\frac{\boldsymbol{z}_i'\boldsymbol{\alpha}}{\sigma_1})]$$

Note: The results look very similar to the results obtained under truncation, but now we have a different variable, $\boldsymbol{z}_i$, determining truncation.

• Again, OLS estimation on the observed part produces a biased and inconsistent estimator. The size of the bias depends on σ_{12} (or ρ).

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Tobit Model: Type II – Conditional Expectation

• From the conditional expectation:

$$E[y_i \mid y_i > 0, \boldsymbol{x}_i] = \boldsymbol{x}_i{'}\boldsymbol{\beta} + \rho\, \sigma_1 \sigma_2\, \lambda(\frac{\boldsymbol{z}_i'\boldsymbol{\alpha}}{\sigma_1}) \qquad (\sigma_1 = 1)$$

• Above we see that applying OLS to observed sample will produce biased (and inconsistent) estimators. This is called *sample selection bias* (an omitted variable problem). It depends on σ_{12} (or ρ) and $\boldsymbol{z}$.

• But regressing y on $\boldsymbol{x}$ and λ on the sub-sample with $y_i^* > 0$ produces consistent estimates (though SE need correction). But, we need an estimator for λ. This idea is the basis of Heckman's two-step estimation.

• Estimation
- ML –complicated, but efficient
- Two-step –easier, but not efficient. Not the usual standard errors

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Tobit Model: Type II – Partial Effects

• Marginal effects of changes in exogenous variables have two components:
- Direct effect on mean of y_i, β_i via (2)
- If a variable affects the Prob[$y_i^* > 0$], then it will affect y_i via (1).

• Marginal effect if regressor appears in both z_i and x_i:

$$\frac{\delta E[y_i \mid y_i > 0, x_i{'}\boldsymbol{\beta}]}{\delta x} = \beta_k - \alpha_k * \rho\sigma_2 * \{\lambda(\boldsymbol{z}_i'\boldsymbol{\alpha})^2 - [\,\underbrace{(-\frac{z_i'\alpha}{\sigma_1})\,\lambda(\boldsymbol{z}_i'\boldsymbol{\alpha})}_{\text{between 0 and 1}}]\}$$

• Suppose $\rho > 0$ and $E(y_i)$ is greater when $y_i^* > 0$ and given that the last term above is between 0 and 1, then, the additional term reduces the marginal effects (it controls for increased mean due to probability impacts). That is, β_k overstates partial effects.

Note: If $\rho = 0$, the partial effect is exactly given by β_k.

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Review: Conditional Bivariate Normal

• To derive the likelihood function for the Sample selection model, we will use results from the conditional distribution of two bivariate normal RVs.

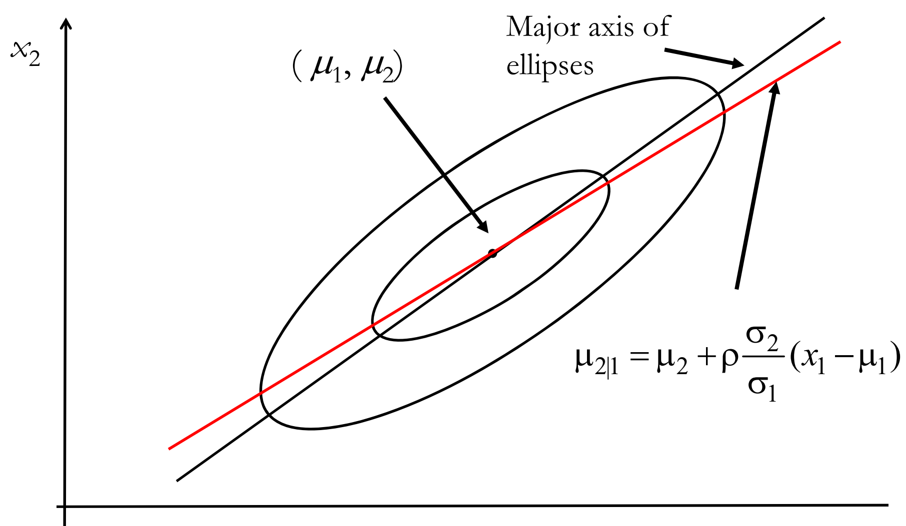
• Recall the definition of conditional distributions for continuous RVs:

$$f_{1|2}\left(x_1|x_2\right)=\frac{f\left(x_1,x_2\right)}{f_2\left(x_2\right)} \quad \text{and} \quad f_{2|1}\left(x_2|x_1\right)=\frac{f\left(x_1,x_2\right)}{f_1\left(x_1\right)}$$

• In the case of the bivariate normal distribution the conditional distribution of x_i given x_j is Normal with mean and standard deviation (using the standard notation):

$$\mu_{i|j}=\mu_i+\rho\frac{\sigma_i}{\sigma_j}\left(x_j-\mu_j\right) \quad \text{and} \quad \sigma_{i|j}=\sigma_i\sqrt{1-\rho^2}$$

Review: Conditional Bivariate Normal



Note: $\mu_{2|1}=\mu_2+(\frac{\sigma_{12}}{\sigma_1\sigma_2})\frac{\sigma_2}{\sigma_1}(x_1-\mu_1)=\mu_2+\frac{\sigma_{12}}{\sigma_1^{\ 2}}(x_1-\mu_1)$

Tobit Model: Type II – ML Estimation

• The model assumes a bivariate normal distribution for $(\varepsilon_{1,i}; \varepsilon_{2,i})$, with covariance given by $\sigma_{12} (= \rho\, \sigma_1 \sigma_2)$. We use a participation dummy variable: $D_i = 0$ (No), $D_i = 1$ (Yes).

• The likelihood reflects two contributions:

(1) Observations with $y_i = 0$ –i.e., $y_i^* = \boldsymbol{z}_i'\boldsymbol{\alpha} + \varepsilon_{1,i} \leq 0 \;\Rightarrow D_i = 0$.

- Prob$(D_i = 0\,|\,\boldsymbol{x}_i)$ = P$(y_i^* = \boldsymbol{z}_i'\boldsymbol{\alpha} + \varepsilon_{1,i} \leq 0\,|\,\boldsymbol{x}_i)$ = P$(\varepsilon_{1,i} \leq -\boldsymbol{z}_i'\boldsymbol{\alpha}\,|\,\boldsymbol{x}_i)$

$= 1 - \Phi(\boldsymbol{z}_i'\boldsymbol{\alpha})$

(2) Observations with $y_i > 0$ –i.e., $y_i^* = \boldsymbol{z}_i'\boldsymbol{\alpha} + \varepsilon_{1,i} > 0 \;\Rightarrow D_i = 1$.

- $f(y_i|D_i = 1, \boldsymbol{x}_i, \boldsymbol{z}_i)$ * Prob$(D_i = 1\,|\,\boldsymbol{x}_i, \boldsymbol{z}_i, y_i)$

(2.a) $f(y_i|D_i = 1, \boldsymbol{x}_i) = \dfrac{P(D_i = 1\,|\,\boldsymbol{x}_i, y_i) * f(y_i|\boldsymbol{x}_i)}{P(D_i = 1, \boldsymbol{x}_i)}$ (Bayes' Rule)

where $f(y_i|\boldsymbol{x}_i) = (1/\sigma_2)\, \phi((y_i - \boldsymbol{x}_i'\boldsymbol{\beta})/\sigma_2)$

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Tobit Model: Type II – ML Estimation

(2.b) $P(y_i|D_i = 1\,|\,\boldsymbol{x}_i, \boldsymbol{z}_i, y_i) = P(\varepsilon_{1,i} > -\boldsymbol{z}_i'\boldsymbol{\alpha}|\boldsymbol{x}_i, y_i)$

$$= P\left[\frac{\varepsilon_{1,i} - (\rho/\sigma_2) * (y_i - \boldsymbol{x}_i'\boldsymbol{\beta})}{\sqrt{\sigma_1^2(1-\rho)^2}} > \frac{-\boldsymbol{z}_i'\boldsymbol{\alpha} - (\rho/\sigma_2) * (y_i - \boldsymbol{x}_i'\boldsymbol{\beta})}{\sqrt{\sigma_1^2(1-\rho)^2}}\right]$$

$$= 1 - \Phi\left(-\frac{\boldsymbol{z}_i'\boldsymbol{\alpha} + (\rho/\sigma_2) * (y_i - \boldsymbol{x}_i'\boldsymbol{\beta})}{\sqrt{\sigma_1^2(1-\rho)^2}}\right)$$

$$= \Phi\left(\frac{\boldsymbol{z}_i'\boldsymbol{\alpha} + (\rho/\sigma_2) * (y_i - \boldsymbol{x}_i'\boldsymbol{\beta})}{\sqrt{\sigma_1^2(1-\rho)^2}}\right)$$

- Moments of the conditional distribution $(y_1|y_2)$ of a normal RV:
  - Mean for RV 1: $\mu_1 + (\sigma_{12}/\sigma_2^2)\,(y_2 - \mu_2) = (\rho/\sigma_2) * (y_i - \boldsymbol{x}_i'\boldsymbol{\beta})$
  - Variance for RV 1: $\sigma_1^2\,(1 - \rho^2) = 1 - \rho^2$ (Recall: $\sigma_1 = 1$)

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Tobit Model: Type II – ML Estimation

• Now, we can put all the contributions together:

$L(\boldsymbol{\beta}) = \prod_{i,y_i=0} P(y_i = 0) * \prod_{i,y_i>0} \{P(y_i > 0) * f(y_i|\boldsymbol{x}_i, \boldsymbol{z}_i)\}$

• Taking logs:

$\log L(\beta,\boldsymbol{\alpha},\sigma,\rho) = \sum_{i=1}^{N}(1 - D_i) * \ln(1 - \Phi(\boldsymbol{z}_i'\boldsymbol{\alpha})) +$

$$+ \sum_{i=1}^{N} D_i * \ln\{\Phi\left(\frac{\boldsymbol{z}_i'\boldsymbol{\alpha} + (\rho/\sigma_2) * (y_i - \boldsymbol{x}_i'\boldsymbol{\beta})}{\sqrt{\sigma_1{}^2(1-\rho)^2}}\right)\}$$

$$+ \sum_{i=1}^{N} D_i * \ln\{\frac{1}{\sigma_2}\phi(\frac{y_i - \boldsymbol{x}_i'\boldsymbol{\beta}}{\sigma_2})\}$$

• Complicated likelihood. The algorithm tends to be badly behaved:
 ⇒ Iterative methods do not always converge to the MLE.

Note: If $\rho = 0$ this log likelihood is just the sum a Gaussian linear regression log likelihood and a probit log likelihood.

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Tobit Model: Type II – Two-step estimator

• It is much easier two use Heckman's two-step (**Heckit**) estimator:

(1) Probit part: Estimate $\boldsymbol{\alpha}$ using ML ⇒ get $\hat{\boldsymbol{\alpha}}$

(2) Truncated regression:
- For each $D_i = 1$ (participation), calculate $\lambda_i = \lambda(\boldsymbol{z}_i'\hat{\boldsymbol{\alpha}})$.
- Regress y_i against $\boldsymbol{x}_i$ & $\lambda(\boldsymbol{z}_i'\hat{\boldsymbol{\alpha}})$ ⇒ get **b** & b_λ ($= \rho\sigma_2$).

• Problems:
- Consistent, but not efficient (relative to MLE)
- Getting Var[**b**] is not easy (we are estimating $\boldsymbol{\alpha}$ too).

• We can get consistent estimators of ρ & σ_2, individually. For each observation, the true conditional variance of the disturbance would be

$$\sigma_i{}^2 = \sigma_2{}^2 (1 - \rho^2 \delta_i) \qquad (\delta(\alpha) = \lambda(\alpha)\,[\lambda(\alpha) - \alpha])$$

where we can estimate

$$\hat{\sigma}_2{}^2 = \frac{e'e}{N} + (\textstyle\sum_{i=1}^{N} \delta_i/N)\, b_\lambda \quad \& \quad \hat{\rho} = \frac{b_\lambda{}^2}{\sigma_2{}^2}.$$

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Tobit Model: Type II – Two-step estimator

• In theory, we can use the delta method to get SE for ρ & σ_2. But, we have heteroscedasticity and the usual 2-step SE estimation problem.

• Heckman (1979) shows the correct asymptotic covariance matrix for $\boldsymbol{\beta}$ & $\boldsymbol{\beta}_\lambda$ is given by: the following:

$$\text{Est.Asy.Var}[\beta,\beta_\lambda] = \hat{\sigma}_\varepsilon^2 [X_*'X_*]^{-1} \left[X_*' \left(I - \hat{\rho}\hat{\Delta} \right) X_* + Q \right] [X_*'X_*]^{-1}$$

where $\left(I - \hat{\rho}\hat{\Delta} \right)$ is a diagonal matrix with

$(1 - \rho^2 \delta_i)$ on the diagonal

$X_{i*} = [X_i, \lambda_i]$

$Q = \hat{\rho}^{\,2} (\boldsymbol{z}' \hat{\Delta} \boldsymbol{X}_*) Var[\hat{\boldsymbol{\alpha}}] (\boldsymbol{z}' \hat{\Delta} \boldsymbol{X}_*)$

Note: Murphy and Topel (1985) SE for 2-step estimators can be used. 45

Tobit Model: Type II – Identification

• In general, it is difficult to justify different variables for $\boldsymbol{z}_i$ and $\boldsymbol{x}_i$. This is a problem for the estimates. It creates an identification problem.

• Technically, the parameters of the model are identified, even when $\boldsymbol{z}_i = \boldsymbol{x}_i$. But, identification is based on the distributional assumptions.

• Estimates are very sensitive to assumption of bivariate normality - Winship and Mare (1992) and $\boldsymbol{z}_i = \boldsymbol{x}_i$.

• ρ parameter very sensitive in some common applications. Sartori (2003) comes with 95% C.I. for ρ = -.999999 to +0.99255!

• Identification is driven by the non-linearity in the selection equation, through λ_i (and, thus, we need variation in the $\boldsymbol{z}_i$'s too!).

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Tobit Model: Type II – Identification

• In general, it is difficult to justify different variables for $\boldsymbol{z}_i$ and $\boldsymbol{x}_i$. This is a problem for the estimates. It creates an identification problem.

• We find that when $\boldsymbol{z}_i = \boldsymbol{x}_i$, identification tends to be tenuous unless there are many observations in the tails, where there is substantial nonlinearity in the λ_i. We need exclusion restrictions.

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Tobit Model: Type II – Testing the model

• Q: Do we have a sample selection problem?
Based on the conditional expectation, a test is very simple. We need to test if there is an omitted variable. That is, we need to test if λ_i belongs in the conditional expectation E$[y_i \,|\, y_i > 0]$.

• Easy test: H_0: $\beta_\lambda = 0$.

We can do this test using the estimator for β_λ, b_λ, from the second step of Heckman's two-step procedure.

• Usual problems with testing.
- The test assumes correct specification. If the selection equation is incorrect, we may be unable to reject H_0.
- Rejection of H_0 does not imply accepting the alternative –i.e., sample selection problem. We may have non-linearities in the data!

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Tobit Model: Type II – Testing the model

• Rejection of H_0 does not imply accepting the alternative –i.e., sample selection problem. We may have non-linearities in the data!

Identification issue II

We are not sure about the functional form. We may not be comfortable interpreting nonlinearities as evidence for endogeneity of the covariates.

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Tobit Model: Type II – Application

Estimating Heckit Manually. (note: you will not get the correct standard errors.

```
. ***********************************************
. * Estimating heckit model manually            *
. ***********************************************
. **************************
. * First create selection  *
. * Variable                *
. **************************
.     gen s=0 if wage==.
(428 missing values generated)

.     replace s=1 if wage~=.
(428 real changes made)

. ******************************
. *Next, estimate the probit    *
. *selection equation           *
. ******************************
. probit s educ exper expersq nwifeinc age kidslt6 kidsge6

Iteration 0:   log likelihood =  -514.8732
Iteration 1:   log likelihood = -405.78215
Iteration 2:   log likelihood = -401.32924
Iteration 3:   log likelihood = -401.30219
Iteration 4:   log likelihood = -401.30219

Probit regression                                 Number of obs   =        753
                                                  LR chi2(7)      =     227.14
                                                  Prob > chi2     =     0.0000
Log likelihood = -401.30219                       Pseudo R2       =     0.2206
```

First step:
Probit selection equation

s	Coef.	Std. Err.	z	P>|z|	[95% Conf.	Interval]
educ	.1309047	.0252542	5.18	0.000	.0814074	.180402
exper	.1233476	.0187164	6.59	0.000	.0866641	.1600311
expersq	-.0018871	.0006	-3.15	0.002	-.003063	-.0007111
nwifeinc	-.0120237	.0048398	-2.48	0.013	-.0215096	-.0025378
age	-.0528527	.0084772	-6.23	0.000	-.0694678	-.0362376
kidslt6	-.8683285	.1185223	-7.33	0.000	-1.100628	-.636029
kidsge6	.036005	.0434768	0.83	0.408	-.049208	.1212179
_cons	.2700768	.508593	0.53	0.595	-.7267473	1.266901

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Tobit Model: Type II – Application

```
. ******************************
. *Then create inverse lambda   *
. ******************************
.  predict xdelta, xb

.  gen lambda =normalden(xdelta)/normal(xdelta)

. *************************************
. *Finally, estimate the Heckit model *
. *************************************
. reg lwage educ exper expersq lambda

      Source |       SS       df       MS              Number of obs =     428
-------------+------------------------------           F(  4,   423) =   19.69
       Model |  35.0479487     4  8.76198719           Prob > F      =  0.0000
    Residual |  188.279492   423  .445105182           R-squared     =  0.1569
-------------+------------------------------           Adj R-squared =  0.1490
       Total |  223.327441   427  .523015084           Root MSE      =  .66716

------------------------------------------------------------------------------
       lwage |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        educ |   .1090655   .0156096     6.99   0.000     .0783835    .1397476
       exper |   .0438873   .0163534     2.68   0.008     .0117434    .0760313
     expersq |  -.0008591   .0004414    -1.95   0.052    -.0017267    8.49e-06
      lambda |   .0322619   .1343877     0.24   0.810    -.2318889    .2964126
       _cons |  -.5781032    .306723    -1.88   0.060    -1.180994     .024788
------------------------------------------------------------------------------
```

Second step: Truncated regression

<u>Note</u>: The standard errors are not correct.

Tobit Model: Type II – Application

```
. heckman lwage educ exper expersq, select(s=educ exper expersq nwifeinc age kidslt6 kidsge6) twostep

Heckman selection model -- two-step estimates   Number of obs      =       753
(regression model with sample selection)        Censored obs       =       325
                                                Uncensored obs     =       428

                                                Wald chi2(3)       =     51.53
                                                Prob > chi2        =    0.0000

------------------------------------------------------------------------------
             |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
lwage        |
        educ |   .1090655    .015523     7.03   0.000     .0786411      .13949
       exper |   .0438873   .0162611     2.70   0.007     .0120163    .0757584
     expersq |  -.0008591   .0004389    -1.96   0.050    -.0017194    1.15e-06
       _cons |  -.5781032   .3050062    -1.90   0.058    -1.175904     .019698
-------------+----------------------------------------------------------------
s            |
        educ |   .1309047   .0252542     5.18   0.000     .0814074     .180402
       exper |   .1233476   .0187164     6.59   0.000     .0866641    .1600311
     expersq |  -.0018871      .0006    -3.15   0.002     -.003063   -.0007111
    nwifeinc |  -.0120237   .0048398    -2.48   0.013    -.0215096   -.0025378
         age |  -.0528527   .0084772    -6.23   0.000    -.0694678   -.0362376
     kidslt6 |  -.8683285   .1185223    -7.33   0.000    -1.100628    -.636029
     kidsge6 |    .036005   .0434768     0.83   0.408     -.049208    .1212179
       _cons |   .2700768    .508593     0.53   0.595    -.7267473    1.266901
-------------+----------------------------------------------------------------
mills        |
      lambda |   .0322619   .1336246     0.24   0.809    -.2296376    .2941613
-------------+----------------------------------------------------------------
         rho |    0.04861
       sigma |  .66362875
      lambda |  .03226186   .1336246
------------------------------------------------------------------------------
```

Heckit Model estimated automatically.

Note H_0: ρ =0 cannot be rejected. There is little evidence that sample selection bias is present.