Chapter 9
Optimization: One Choice Variable

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1

9.1 Optimum Values and Extreme Values

- Goal vs. non-goal equilibrium
- In the optimization process, we need to identify the objective function to optimize.
- In the objective function the dependent variable represents the object of maximization or minimization

Example:

- Define profit function: $\pi = PQ - C(Q)$
- Objective: Maximize π
- Tool: Q

2

9.2 Relative Maximum and Minimum: First-Derivative Test

Critical Value

The critical value of x is the value x_0 if $f'(x_0) = 0$.

- A stationary value of y is $f(x_0)$.
- A stationary point is the point with coordinates x_0 and $f(x_0)$.
- A stationary point is coordinate of the extremum.

- **Theorem** (Weierstrass)

  Let $f : S \rightarrow R$ be a real-valued function defined on a compact (bounded and closed) set $S \in R^n$. If f is continuous on S, then f attains its maximum and minimum values on S. That is, there exists a point c_1 and c_2 such that

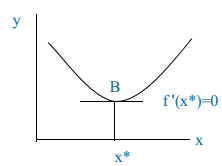
$$f(c_1) \leq f(x) \leq f(c_2) \qquad \forall x \in S.$$

3

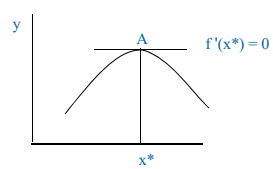
9.2 First-derivative test ☺ ☹

- The *first-order condition* (f.o.c.) or necessary condition for extrema is that $f'(x^*) = 0$ and the value of $f(x^*)$ is:

- A relative minimum if $f'(x^*)$ changes its sign from negative to positive from the immediate left of x_0 to its immediate right. (first derivative test of min.) ☺



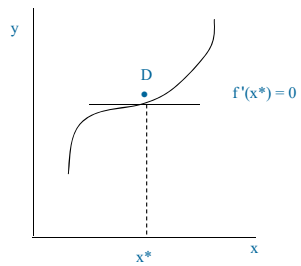
- A relative maximum if the derivative $f'(x)$ changes its sign from positive to negative from the immediate left of the point x^* to its immediate right. (first derivative test for a max.) ☹



4

9.2 First-derivative test ☹

- The first-order condition or necessary condition for extrema is that $f'(x^*) = 0$ and the value of $f(x^*)$ is:
- Neither a relative maxima nor a relative minima if $f'(x)$ has the same sign on both the immediate left and right of point x_0 (first derivative test for *point of inflection*).



5

9.2 Example: Average Cost Function

$AC = Q^2 - 5Q + 8$	Objective function
$f'(Q) = 2Q - 5$	1st derivative function
$f'(Q) = 2Q - 5 = 0$	f.o.c.
$Q^* = 5/2 = 2.5$	extrema

left	$AC = f(Q^*)$	right	
$f(2.4) = 1.76$	$f(2.5) = 1.75$	$f(2.6) = 1.76$	relative min
$f'(2.4) = -0.2$	$f'(2.5) = 0$	$f'(2.6) = 0.2$	$(-,+)$

6

9.3 The smile test ☹ ☺

- Convexity and Concavity: The smile test for aximum/minimum

- If $f''(x) < 0$ for all x, then strictly concave.
  $\Rightarrow$ critical points are global maxima

- If $f''(x) > 0$ for all x, then strictly convex.
  $\Rightarrow$ critical points are global minima

- If a concave utility function (typical for risk aversion) is assumed for a utility maximizing representative agent, there is no need to check for s.o.c. Similar situation for a concave production function

7

9.3 The smile test: Examples: ☹

Example 1: *Revenue Function*

1) $TR = 1200Q - 2Q^2$ Revenue function
2) $MR = 1200 - 4Q^* = 0$ f.o.c.
3) $Q^* = 300$ extrema
4) $MR' = -4$ 2nd derivative
5) $MR' < 0$ $\Rightarrow Q^* = 300$ is a maximum

Example 2: *Average Cost Function*

$AC = Q^2 - 5Q + 8$ Objective function

$f'(Q) = 2Q - 5 = 0$ f.o.c.

$Q^* = 5/2 = 2.5$ extrema

$f''(Q) = 2 > 0$ $\Rightarrow Q^* = 5/2 = 2.5$ is a minimum

8

9.3 Inflection Point

- Definition

  A twice differentiable function $f(x)$ has an *inflection point* at x iff the second derivative of $f(.)$ changes from negative (positive) in some interval $(m, \tilde{x})$ to positive (negative) in some interval $(\tilde{x}, n)$, where $\tilde{x} \in (m, n)$.

- Alternative Definition

  An inflection point is a point (x, y) on a function, $f(x)$, at which the first derivative, $f'(x)$, is at an extremum, -i.e. a minimum or maximum. (Note: $f''(x)=0$ is necessary, but not sufficient condition.)

  Example: $U(w) = w - 2w^2 + w^3$

  $U'(w) = 4w + 3w^2$

  $U''(w) = 4 + 6w \quad \Rightarrow 4 + 6\tilde{w} = 0$

  $\tilde{w} = 2/3$ is an inflection point

9

9.4 Formal Second-Derivative Test: Necessary and Sufficient Conditions

- The zero slope condition is a necessary condition and since it is found with the first derivative, we refer to it as a 1st order condition.

- The sign of the second derivative is *sufficient* to establish the stationary value in question as a relative minimum if $f''(x_0) > 0$, the 2nd order condition or relative maximum if $f''(x_0) < 0$.

10

9.4 Example 1: Optimal Seignorage

$$\frac{M}{P} = e^{-\lambda(\pi^e + r) + \alpha Y}$$ Demand for money

$$S = \pi \frac{M}{P} = \pi e^{-\lambda(\pi^e + r) + \alpha Y}$$ Seignorage

F.o.c. (assume $\pi = \pi^e$):

$$\frac{dS}{d\pi} = e^{-\lambda(\pi + r) + \alpha Y} + \pi(-\lambda)e^{-\lambda(\pi + r) + \alpha Y}$$

$$= e^{-\lambda(\pi + r) + \alpha Y} + \pi(-\lambda)e^{-\lambda(\pi + r) + \alpha Y} = e^{-\lambda(\pi + r) + \alpha Y}(1 - \pi\lambda)$$

$$(1 - \pi^* \lambda) = 0 \quad => \pi^* = \frac{1}{\lambda}$$ (Critical point)

S.o.c. :

$$\frac{d^2 S}{d\pi^2} = -\lambda e^{-\lambda(\pi + r) + \alpha Y}(1 - \pi\lambda) + (-\lambda)e^{-\lambda(\pi + r) + \alpha Y} = -\lambda e^{-\lambda(\pi + r) + \alpha Y}(2 - \pi\lambda)$$

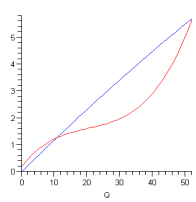
$$\frac{d^2 S}{d\pi^2}(\pi^* = \frac{1}{\lambda}) = -\lambda e^{-\lambda(\pi^* + r) + \alpha Y} < 0 \quad => \pi^* = \frac{1}{\lambda}$$ is a maximum

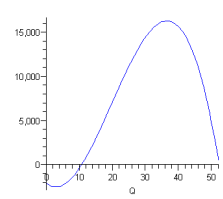
9.4 Example 2: Profit function (Two solutions)

Revenue and Cost functions

1) $TR = 1200Q - 2Q^2$

2) $TC = Q^3 - 61.25Q^2 + 1528.5Q + 2000$





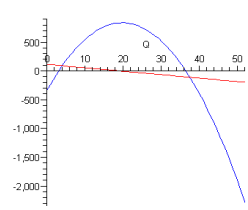
Profit function

3) $\pi = TR\text{-}TC = -Q^3 + 59.25Q^2 - 328.5Q - 2000$

1st derivative of profit function

4) $\pi' = -3Q^2 + 118.5Q - 328.5 = 0$

5) $Q_1^* = 3 \qquad Q_2^* = 36.5$



2nd derivative of profit function

6) $\pi'' = -6Q + 118.5$

7) $\pi''(3) = 100.5 \qquad \pi''(36.5) = -100.5$

applying the smile test

8) $\pi''(Q_1^*) > 0 \rightarrow \min \quad \pi''(Q_2^*) < 0 \rightarrow \max$

12

9.4 Example 3: Optimal Timing - wine storage

$A(t) = Ve^{-rt}$ Present value

$V = ke^{\sqrt{t}}$ Growth in value

$A(t) = ke^{t^{1/2}} e^{-rt} = ke^{t^{1/2} - rt}$

$\ln A(t) = \ln k + \ln e^{t^{1/2} - rt}$ *Monotonic* transformation of objective function

$= \ln k + \left(t^{1/2} - rt\right)\ln e$

$= \ln k + \left(t^{1/2} - rt\right)$

$\frac{dA}{dt}\frac{1}{A} = \frac{1}{2}t^{-1/2} - r$

F.o.c.: $\frac{dA}{dt} = A\left(\frac{1}{2}t^{*-1/2} - r\right) = 0 \quad => \frac{1}{2}t^{*-1/2} - r = 0$

$=> \frac{1}{2\sqrt{t^*}} = r \quad => t^* = \frac{1}{4r^2}$

9.4 Example 3: Optimal Timing - wine storage

Optimal time : $t^* = \frac{1}{4r^2}$

Let $(r) = 10\%$

$t^* = \frac{1}{4(0.10)^2} = 25$ years

Determine optimal values for A(t) and V :

$A(t) = ke^{t^{1/2} - rt}$

let $(k) = \$1$ / bottle

$A(t) = e^{5-(.1)((25)} = e^{2.5} = \$12.18$ / bottle

$V = Ae^{rt}$

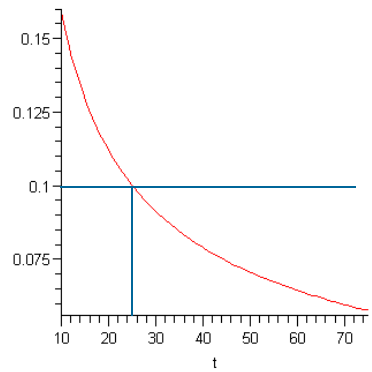
$V = (\$12.18/\text{bottle})\ e^{(.1)(25)} = \$148.38$ / bottle

$V = \$148.38$ / bottle

14

9.4 Example 3: Optimal Timing - wine storage

Plot of Optimal Time with $r = 0.10$ $(\Longrightarrow t = 25)$



15

9.4 Example 4: Least Squares

- In the CLM, we assume a linear model, relating **y** and **X**, which we call the DGP: $\mathbf{y} = \mathbf{X}\beta + \boldsymbol{\varepsilon}$

- The relation is not exact, there is an error term, $\boldsymbol{\varepsilon}$. We want to find the β that minimizes the sum of square errors, $\boldsymbol{\varepsilon}'\boldsymbol{\varepsilon}$.

- Assume there is only one explanatory variable, **x**. Then,

$$Min_{\beta}\ S(\beta \mid y, x) = \sum_{t=1}^{T} (y_t - x_t\beta)^2$$

Then, we write the first order condition as:

$$\frac{\partial S}{\partial \beta} = \sum_{t=1}^{T} 2(y_t - x_t b)(-x_t) = -\sum_{t=1}^{T} (y_t x_t - x_t^2 b) = 0$$

In the general, multivariate case, these f.o.c. are called *normal equations.*

9.4 Example 4: Least Squares

- The LS solution, **b**, is

$$b = \frac{\sum_{t=1}^{T} y_t x_t}{\sum_{t=1}^{T} x_t^2} = (x'x)^{-1} x'y$$

- Second order condition ☺

$$\frac{\partial^2 S}{\partial \beta^2} = -\sum_{t=1}^{T} (-x_t^2) > 0 \qquad \Rightarrow b \text{ a minimum}$$

(It's a globally convex function.)

9.4 Example 5: Maximum Likelihood

- Now, in the CLM, we assume **ε** follow a normal distribution:

$$\boldsymbol{\varepsilon} \,|\, \mathbf{X} \sim N(\mathbf{0}, \sigma^2 \mathbf{I}_T)$$

Then, we write the likelihood function, L, as:

$$L = f(y_1, y_2, \ldots, y_T \mid \beta, \sigma^2) = \Pi_{t=1}^{T} \left(\frac{1}{2\pi\sigma^2}\right)^{1/2} \exp\left[-\frac{1}{2\sigma^2}(y_t - x_t'\beta)^2\right]$$

- We want to find the β that maximizes the likelihood of the occurrence of the data. It is easier to do the maximization after taking logs –i.e, a monotonic increasing transformation. That is, we maximize the log likelihood function w.r.t. β :

$$\ln L = \sum_{t=1}^{T} -\frac{\ln(2\pi\sigma^2)}{2} + \sum_{t=1}^{T} -\frac{1}{2\sigma^2}(y_t - x_t'\beta)^2$$

$$= -\frac{T}{2}\ln 2\pi - \frac{T}{2}\ln \sigma^2 - \frac{1}{2\sigma^2}\varepsilon'\varepsilon$$

9.4 Example 5: Maximum Likelihood

- Assume σ known and only one explanatory variable, **x**. Then,

$$Max_{\beta} \ln L(\beta \mid y, x) = -\frac{T}{2}\ln 2\pi - \frac{T}{2}\sigma^2 - \frac{1}{2\sigma^2}\sum_{t=1}^{T}(y_t - x_t\beta)^2$$

- Then, the f.o.c.:

$$\frac{\partial \ln L}{\partial \beta} = -\frac{1}{2\sigma^2}\sum_{t=1}^{T} 2(y_t - x_t\hat{\beta}_{MLE})(-x_t) = \frac{1}{\sigma^2}\sum_{t=1}^{T}(y_t x_t - x_t^2\hat{\beta}_{MLE}) = 0$$

<u>Note</u>: The f.o.c. delivers the normal equations for β (same condition as in Least Squares) $\Longrightarrow$ *MLE* solution = LS estimator, **b**. That is,

$$\hat{\beta}_{MLE} = b = \frac{\sum_{t=1}^{T} y_t x_t}{\sum_{t=1}^{T} x_t^2} = (x'x)^{-1}x'y$$

- Nice result for OLS **b**: ML estimators have very good properties!

9.4 Example 5: Maximum Likelihood ☹

- Second order condition

$$\frac{\partial \ln^2 L}{\partial \beta^2} = \sum_{t=1}^{T} -x_t^2 < 0 \qquad \Rightarrow \hat{\beta}_{MLE} \text{ a maximum}$$

(It's a globally concave function.) ⋯

9.5 Taylor Series of a polynomial function: Revisited

Taylor series for an arbitrary function : Any function can be approximated by the weighted sum of its derivatives. Then the change is given by

$$f(x)-f(x_0)=\frac{f^{/}(x_0)}{1!}(x-x_0)^1+\frac{f^{//}(x_0)}{2!}(x-x_0)^2+\ldots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+R_{n+1}$$

Where $R_{n+1}=\frac{f^{(n+1)}(p)}{(n+1)!}(x-x_0)^{n+1}$

If n = 2, then

$$f(x)-f(x_0)=\frac{f^{/}(x_0)}{1!}(x-x_0)^1+\frac{f^{//}(x_0)}{2!}(x-x_0)^2+R_3$$

At x_0, $f^{/}(x_0)=0$ - i.e., x_0 is a max., min., or inflection.

Then,

$$f(x)-f(x_0)\approx\frac{f^{//}(x_0)}{2!}(x-x_0)^2$$ => the sign of $f^{//}(x_0)$ determines what x_0 is.

21

9.5 Taylor expansion and relative extremum

A function $f(x)$ attains a relative max (min) value at x_0 if $f(x)-f(x_0)$ is neg. (pos.) for values of x in the immediate neighborhood of x_0 (the critical value) both to its left and right Taylor series approximation for a small change in x :

$$f(x)-f(x_0)=\frac{f^{/}(x_0)}{1!}(x-x_0)^1+\frac{f^{//}(x_0)}{2!}(x-x_0)^2+\ldots+R_{n+1}$$

At the max., min., or inflection, $f^{/}(x_0)=0$

and if $f^{//}(x_0)=0$, and if $f^{(3)}(x_0)=0$, $\cdots$

then $f(x)-f(x_0)=R_{n+1}$

What is the sign of R for the first nonzero derivative?

22

9.5 Nth-derivative test

If the first derivative of a function $f(x)$ at x_0 is $f'(x_0) = 0$ and if the first nonzero derivative value at x_0 encountered in successive derivation is that of the Nth derivative, $f^{(n)}(x_0) \neq 0$, then the stationary value $f(x_0)$ will be:

a relative max if N is even and $f^{(n)}(x_0) < 0$

a relative min if N is even and $f^{(n)}(x_0) > 0$

an inflection point if N is odd

23

9.5 Nth-derivative test

Example:

$Y = (7-x)^4$	primitive function
$Y' = -4(7-x)^3$	1st deriative
$Y' = 0$ at $x^* = 7$	the critical value
$Y''(7) = 12(7-x)^2 = 0$	2nd deriviativ e
$Y^{(3)}(7) = -24(7-x) = 0$	3rd derivative
$Y^{(4)}(7) = 24$	4th derivative

Because first nonzero derivative $Y^{(n)}$ is even (4) and $Y^{(4)} > 0$ (24), critical value is a min.

24

9.5 Nth-derivative test

Example (continuation):

$Y = (7-x)^4$ primitive function

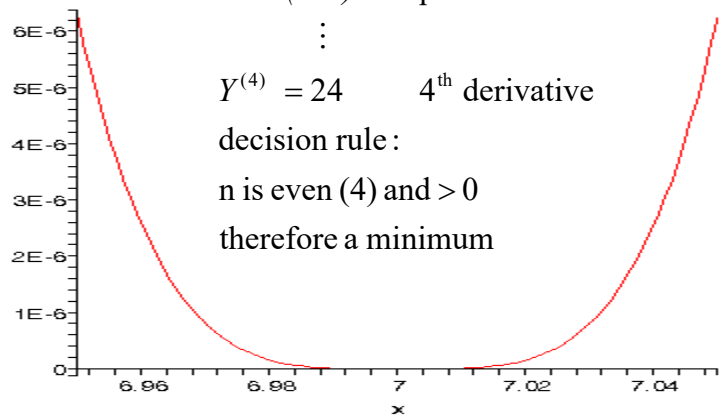
$\vdots$

$Y^{(4)} = 24$ 4th derivative

decision rule:

n is even (4) and > 0

therefore a minimum



25