Chapter 7
Rules of Differentiation
& Taylor Series

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7.1 Review: Derivative and Derivative Rules

- Review: Definition of derivative.

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \frac{dy}{dx}$$

- Applying this definition, we review the 9 rules of differentiation:

1) Constant: $\frac{d[f(x)=c]}{dx} = 0$

2) Power: $\frac{d[x^n]}{dx} = nx^{n-1}$

3) Sum/Difference $\frac{d[f(x)+g(x)]}{dx} = \frac{df}{dx} + \frac{dg}{dx}$

4) Product $\frac{d[f(x)*g(x)]}{dx} = \frac{df}{dx} g(x) + \frac{dg}{dx} f(x)$

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7.1 Review: Derivative and Derivative Rules

- (continuation) 9 rules of differentiation:

5) Quotient (from 4) $\frac{d[f(x)/g(x)]}{dx} = [\frac{df}{dx} g(x) - \frac{dg}{dx} f(x)]/g(x)^2$

6) Exponential $\frac{d[e^{kx}c]}{dx} = ke^{kx}$

7) Chain Rule $\frac{d[z=f(g(x))]}{dx} = \frac{dz}{dy} * \frac{dy}{dx}$ (with $y = g(x)$)

8) Inverse function. Let $y = f(x)$ be a strictly monotonic function.

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}.$$

9) Constant, Product and Power (from 1, 2 & 4)

$$\frac{d[cx^n]}{dx} = cnx^{n-1}$$

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7.1.1 Constant Rule

- Recall the definition of derivative.

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \frac{dy}{dx}$$

- Applying this definition, we derive the constant rule:

The derivative of a constant function is zero for all values of *x*.

$$y = f(x) = k \quad \Rightarrow \quad \frac{dy}{dx} = \frac{d}{dx} k = 0$$

$$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

If $f(x) = k$ then $f(x + \Delta x) = k$

$$\lim_{\Delta x \to 0} \frac{k - k}{\Delta x} = \lim_{\Delta x \to 0} 0 = 0$$

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7.1.2 Power-Function Rule

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x^n + nx^{n-1}\Delta x + (n-1)x^{n-2}\Delta x^2 + \ldots + nx\Delta x^{n-1} + \Delta x^n) - x^n}{\Delta x}$$

$$= \lim_{\Delta x \to 0} x^n / \Delta x + nx^{n-1} + (n-1)x^{n-2}\Delta x + \ldots + nx\Delta x^{n-2} + \Delta x^{n-1} - x^n / \Delta x$$

$$= nx^{n-1}$$

$$\lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x} = nx^{n-1}$$

Example: Let Total Revenue (R) be:

R = 15 Q – Q² $\Rightarrow \frac{dR}{dQ} = MR = 15 - 2Q.$

As Q increases R increases (as long as Q > 7.5).

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7.1.3 Sum or Difference Rule

3) $$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$$

• The derivative of a sum (or difference) of two functions is the sum (or difference) of the derivatives of the two functions

Example : $C = Q^3 - 4Q^2 + 10Q + 75$

$$\frac{dC}{dQ} = \frac{d}{dQ}Q^3 - \frac{d}{dQ}4Q^2 + \frac{d}{dQ}10Q + \frac{d}{dQ}75$$

$$\frac{dC}{dQ} = 3Q^2 - 8Q + 10 + 0$$

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7.1.4 Product Rule

4) $$\frac{d[f(x)*g(x)]}{dx} = \frac{df}{dx}g(x) + \frac{dg}{dx}f(x)$$

The derivative of the product of two functions is equal to the second function times the derivative of the first *plus* the first function times the derivative of the second.

Example: Marginal Revenue (MR)

Total Revenue: R = P Q

Given $P = 15 - Q \quad \Rightarrow$ R $= (15 - Q)Q$

$$\Rightarrow \frac{dR}{dQ} = \frac{dP}{dQ}Q + \frac{dQ}{dQ}P = -Q + 1 * (15 - Q) = 15 - 2Q$$

Same as in previous example.

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7.1.5 Quotient Rule

5) $$\frac{d[f(x)/g(x)]}{dx} = [\frac{df}{dx}g(x) - \frac{dg}{dx}f(x)]/g(x)^2$$

Example :

TC = C(Q) Total cost

AC = C(Q)/Q Average cost

$$\frac{d}{dQ}\frac{C(Q)}{Q} = \frac{Q \cdot C'(Q) - C(Q) \cdot 1}{Q^2} = \frac{1}{Q}\left[C'(Q) - \frac{C(Q)}{Q}\right] = \frac{1}{Q}[MC - AC]$$

if $\frac{d}{dQ}\frac{C(Q)}{Q} = 0$, then AC = MC (Average Cost = Marginal Cost)

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7.1.6 Exponential-Function Rule

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{e^{\kappa(x+\Delta x)} - e^{\kappa x}}{\Delta x} = e^{\kappa x} \lim_{\Delta x \to 0} \frac{(e^{\kappa \Delta x} - 1)}{\Delta x}$$

Definition of e : e unique positive number for which $\lim_{h \to 0} \frac{(e^h - 1)}{h} = 1$

Let $h = \kappa \Delta x$. Then, $\lim_{h \to 0} \frac{(e^h - 1)}{h} = \kappa$

Thus, $\lim_{\Delta x \to 0} \frac{e^{\kappa(x+\Delta x)} - e^{\kappa x}}{\Delta x} = \kappa e^{\kappa x}$

- *Example* : Exponential Growth

$y = f(t) = e^{0.5t}$ $\qquad \frac{d}{dt} e^{0.5t} = 0.5e^{0.5t}$

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7.1.6 Exponential-Function Rule: Joke

- A mathematician went insane and believed that he was the differentiation operator. His friends had him placed in a mental hospital until he got better. All day he would go around frightening the other patients by staring at them and saying "*I differentiate you*!"
- One day he met a new patient; and true to form he stared at him and said "I differentiate you!", but for once, his victim's expression didn't change.
- Surprised, the mathematician collected all his energy, stared fiercely at the new patient and said loudly "*I differentiate you!",* but still the other man had no reaction. Finally, in frustration, the mathematician screamed out "I DIFFERENTIATE YOU!"
- The new patient calmly looked up and said, "*You can differentiate me all you like: I'm* e^x."

7.1.7 Chain Rule

This is a case of two or more differentiable functions, in which each has a distinct independent variable, where $z = f(g(x))$. That is,
$z = f(y)$, i.e., z is a function of variable y and
$y = g(x)$, i.e., y is a function of variable x

7) $$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}$$

$$\frac{df(y)}{dx} = \frac{df(y)}{dy}\frac{dg(x)}{dx} = f'(y)g'(x)$$

Example: $R = f(Q)$ (revenue) & $Q = g(L)$ (output)

$$\begin{aligned}\frac{dR}{dL} &= \frac{dR}{dQ}\cdot\frac{dQ}{dL} \\ &= f'(Q)\cdot g'(L) \\ &= MR \cdot MPP_L = MRP_L\end{aligned}$$

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7.1.7 Chain rule: Application – Log rule

- Chain Rule : $\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}$

$$\frac{df(y)}{dx} = \frac{df(y)}{dy}\frac{dg(x)}{dx} = f'(y)g'(x)$$

- Consider $h(x) = e^{\ln(x)} = x$. $\Rightarrow h'(x) = 1$.

Now, apply Chain rule to $h(x)$:

$$h'(x) = e^{\ln(x)}\frac{d\ln(x)}{dx} = x\frac{d\ln(x)}{dx}$$

$$1 = \frac{d\ln(x)}{dx}x \quad \Rightarrow \frac{d\ln(x)}{dx} = \frac{1}{x}$$

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7.1.8 Inverse-function Rule

• Let $y = f(x)$ be a differentiable strictly monotonic function:

$$\frac{dx}{dy} = f^{-1\prime}(y) = \frac{1}{\frac{dy}{dx}}.$$

Note: A monotonic function is one in which a given value of x yields a unique value of y, and given a value of y will yield a unique value of x (a one-to-one mapping). These types of functions have a defined inverse.

Example: Inverse supply function

$Q_s = b_0 + b_1 P \quad \Rightarrow \quad P = -\frac{b_0}{b_1} + \frac{1}{b_1} Q_s$ (where $b_1 > 0$)

$\frac{dQ_s}{dP} = b_1 \quad \Rightarrow \quad \frac{dP}{dQ} = \frac{1}{b_1}$

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7.1.8 Inverse-function Rule

- This property of one-to-one mapping is unique to the class of functions known as monotonic functions:
- Recall the definition of a function:

function:	one y for each x
monotonic function:	one x for each y (inverse function)

- if $x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$ — *monotonically increasing*

$Q_s = b_0 + b_1 P$	supply function (where $b_1 > 0$)
$P = -b_0/b_1 + (1/b_1)Q_s$	inverse supply function

- if $x_1 > x_2 \Rightarrow f(x_1) < f(x_2)$ — *monotonically decreasing*

$Q_d = a_0 - a_1 P$	demand function (where $a_1 > 0$)
$P = a_0/a_1 - (1/a_1)Q_d$	inverse demand function

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7.2 Extension: Partial Differentiation

- In multivariate calculus, y depends on several variables:

$$y = f(x_1, x_2, \ldots, x_n)$$

- The derivative of y with respect to one of the variables –while the other variables are held constant– is called a *partial derivative*.

$$y = f(x_1, x_2, \cdots, x_n)$$

$$\lim_{\Delta x_1 \to 0} \frac{\Delta y}{\Delta x_1} = \lim_{\Delta x_1 \to 0} \frac{f(x_1 + \Delta x_1, x_2, \cdots, x_n) - f(x_1, x_2, \cdots, x_n)}{\Delta x_1}$$

$$\equiv \frac{\partial y}{\partial x_1} \equiv f_1 \qquad \text{(partial derivative w.r.t. } x_1\text{)}$$

In general, $\lim_{\Delta x_i \to 0} \frac{\Delta y}{\Delta x_i} \equiv \frac{\partial y}{\partial x_i} \equiv f_i, \quad i = 1 \ldots n$

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7.2 Application: Black-Scholes – Greeks

- The Black-Scholes (BS) formula prices an European call option on a non-dividend paying stock, as a function of the stock price (S_t), time to maturity (T-t), strike price (K), interest rates (i) and the stock price volatility (σ):

$$C_t = S_t\, N(d1) - K\, e^{-i(T-t)} N(d2)$$

where

$$N(d) = \int_{-\infty}^{d} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \quad \text{(standard normal distribution function)}$$

$$d1 = [\ln(S_t/K) + (i + \sigma^2/2)\,(T - t)]/(\sigma \sqrt{T-t}),$$

$$d2 = [\ln(S_t/K) + (i - \sigma^2/2)\,(T - t)]/(\sigma \sqrt{T-t}) = d1 - \sigma \sqrt{T-t}$$

• The Greeks represent the first derivatives of the BS pricing formulas (*ceteris paribus*) with respect to the driver variables: S_t, (T-t), i, σ. For example, the first derivative with respect to S_t is called Δ (or *BS Delta*).

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7.2 Application: Black-Scholes – Greeks (Delta)

• Δ = BS Delta

$$\Delta = \frac{dC_t}{dS_t} = N(d1) + S_t \frac{dN}{d(d1)} * \frac{d(d1)}{dS_t} - K\, e^{-i(T-t)} \frac{dN}{d(d2)} * \frac{d(d2)}{dS_t}$$

• Taking derivatives and using the FTC to get N′(d):

$$\frac{N(d1)}{d(d)} = N'(d1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{d1^2}{2}}$$

$$\frac{N(d2)}{d(d)} = N'(d2) = \frac{1}{\sqrt{2\pi}} e^{-\frac{d2^2}{2}} = \frac{1}{\sqrt{2\pi}} e^{-\{\frac{d1^2}{2} + \frac{\sigma^2}{2}(T-t) - d1\sigma\sqrt{T-t}\}}$$

$$\frac{d(d1)}{dS_t} = \frac{d(d2)}{dS_t} = \frac{1}{\sigma S_t \sqrt{T-t}}$$

Then,

$$\Delta = N(d1) + S_t \frac{1}{\sqrt{2\pi}} e^{-\frac{d1^2}{2}} * \frac{d(d1)}{dS_t} - K\, e^{-i(T-t)} \frac{1}{\sqrt{2\pi}} e^{-\frac{d2^2}{2}} * \frac{d(d1)}{dS_t}$$

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7.2 Application: Black-Scholes – Greeks (Delta)

$$\Delta = N(d1) + N'(d1) * \frac{d(d1)}{dS_t} * [S_t - K\, e^{-\{\left(i+\frac{\sigma^2}{2}\right)(T-t)\}+d1\sigma\sqrt{T-t}}] = N(d1)$$

since $\frac{S_t}{K} = e^{-\{\left(i+\frac{\sigma^2}{2}\right)(T-t)\}+d1\sigma\sqrt{T-t}}$ (from definition of d1)

• We can use Δ to establish a portfolio that is not sensitive to changes in S_t: A long position in one call and a short position Δ stocks. The profits from this portfolio are:

$$\Pi = C_t - \Delta\, S_t$$

Then,

$$\frac{d\Pi}{dS_t} = \frac{dC_t}{dS_t} - \Delta = 0.$$

Note: A position with a delta of zero is referred to as being *delta-neutral*.

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7.2 Application: Black-Scholes – Greeks (Vega)

• V = BS Vega: It measures the sensitivity of option prices to changes in volatility. Recall BS formula:

$$C_t = S_t\, N(d1) - K\, e^{-i(T-t)} N(d2)$$

Then,

$$\text{V} = \frac{dC_t}{d\sigma} = S_t N'(d1) * \frac{d(d1)}{d\sigma} - K\, e^{-i(T-t)} N'(d2) * \frac{d(d2)}{d\sigma}$$

Using the result (check it):

$$\frac{d(d1)}{d\sigma} = \frac{d(d2)}{d\sigma} + \sqrt{T-t}$$

and after some algebra, similar to what we did above with the result:

$$S_t N'(d1) = K\, e^{-i(T-t)} N'(d2),$$

we get to:

$$\text{V} = S_t\, N'(d1)\, \sqrt{T-t} > 0$$

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7.2 Application: Black-Scholes – Greeks (Rho)

• P = BS Rho: It measures the sensitivity of option prices to changes in interest rates. For the call option we get:

$$\text{P} = \frac{dC_t}{di}\; S_t N'(d1) * \frac{d(d1)}{di} - K\, e^{-i(T-t)} N'(d2) * \frac{d(d2)}{di} + \\ + (T-t) K\, e^{-i(T-t)} N(d2)$$

Using the result (check it):

$$\frac{d(d1)}{di} = \frac{d(d2)}{di}$$

A bit of algebra, again, mainly using: $S_t N'(d1) = K\, e^{-i(T-t)} N'(d2)$

delivers

$$\text{P} = (T-t) K e^{-i(T-t)} * N(d2)$$

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7.3 Second & Higher Derivatives

Derivative of a derivative

- Given $y = f(x)$
- The first derivative $f'(x)$ or dy/dx is itself a function of x, it should be differentiable with respect to x, provided that it is continuous and smooth.
- The result of this differentiation is known as the second derivative of the function f and is denoted as $f''(x)$ or d^2y/dx^2.
- The second derivative can be differentiated with respect to x again to produce a third derivative:

  $f'''(x)$ and so on to $f^{(n)}(x)$ or d^ny/dx^n
- This process can be continued to produce an n-th derivative.

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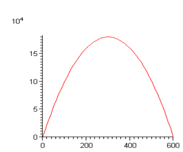
7.3 Higher Derivatives: Example – 1st, 2nd & 3rd

1)	$R = f(Q) = 1200Q - 2Q^2$	primitive function
2)	$f'(Q) = 1200 - 4Q$	1st derivative
3)	$f''(Q) = -4$	2nd derivative
4)	$f'''(Q) = 0$	3rd derivative

Graphically:

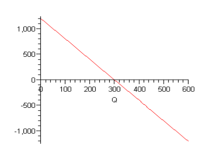
1) $R = 1200Q - 2Q^2$

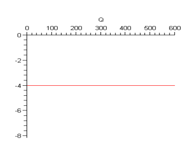


2) $MR = 1200 - 4Q$

$1200 - 4Q = 0$

$Q = 300$





3) $MR' = -4$

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7.3 Higher Derivatives: Example – 1st to 5th

1)	$y = f(x) = 4x^4 - x^3 + 17x^2 + 3x - 1$	primitive function
2)	$f'(x) = 16x^3 - 3x^2 + 34x + 3$	1st derivative
3)	$f''(x) = 48x^2 - 6x + 34$	2nd derivative
4)	$f^{(3)}(x) = 96x - 6$	3rd derivative
5)	$f^{(4)}(x) = 96$	4th derivative
6)	$f^{(5)}(x) \equiv 0$	5th derivative

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7.3 Example: Black-Scholes – Greeks (Gamma)

• The BS Gamma of a derivative security, Γ, represents the rate of change of Δ with respect to the price of the underlying asset. That is, Γ is the second derivative of the call option with respect to S_t. Recall:

$$\Delta = \frac{dC_t}{dS_t} = N(d1)$$

Then,

$$\Gamma = \frac{d^2C_t}{dS_t{}^2} = \frac{d\Delta}{dS_t} = \frac{dN(d1)}{dS_t} = N'(d1) * \frac{d(d1)}{dS_t}$$

Using $N'(d1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{d1^2}{2}}$ & $\frac{d(d1)}{dS_t} = \frac{d(d2)}{dS_t} = \frac{1}{\sigma S_t \sqrt{T-t}}$

we get:

$$\Gamma = \frac{1}{\sqrt{2\pi}} e^{-\frac{d1^2}{2}} * \frac{1}{\sigma S_t \sqrt{T-t}} \qquad \Rightarrow \Gamma > 0$$

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7.3 Interpretation of the second derivative

- $f'(x)$ measures the rate of change of a function
  - e.g., whether the slope is increasing or decreasing
- $f''(x)$ measures the rate of change in the rate of change of a function
  - e.g., whether the slope is increasing or decreasing at an increasing or decreasing rate
  - how the curve tends to bend itself

- Utility functions are increasing in consumption $f'(x)>0$. But they differ by the rate of change in $f'(x)>0$; that is, they differ on $f''(x)$.
  - $f''(x) > 0$, increasing $f'(x)>0$
  - $f''(x) = 0$, constant $f'(x)>0$
  - $f''(x) < 0$, decreasing $f'(x)>0$ (usual assumption)

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7.4 Strict concavity and convexity

- *Strictly concave*: if we pick any pair of points M and N on its curve and joint them by a straight line, the line segment MN must lie entirely *below the curve*, except at points MN.
- A strictly concave curve can never contain a linear segment anywhere (if it does it is just concave, not strictly concave).
- Test: if $f''(x)$ is negative for all x, then it is strictly concave.

- *Strictly convexity*: if we pick any pair of points M and N on its curve and joint them by a straight line, the line segment MN must lie entirely *above the curve*, except at points MN.
- A strictly convex curve can never contain a linear segment anywhere (if it does it is just convex, not strictly convex)
- Test: if $f''(x)$ is positive for all x, then it is strictly convex

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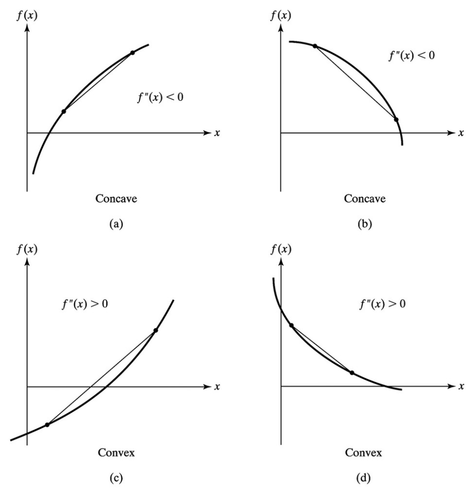
Figure 7.6 Concave and Convex Functions



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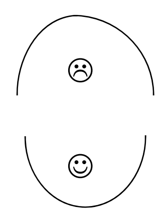
7.4 Concavity and Convexity: ☹ & ☺



- If $f''(x) < 0$ for all x ⇒ strictly concave.
  ⇒ There is a global maxima
- If $f''(x) > 0$ for all x ⇒ strictly convex.
  ⇒ There is a global minima

- Concave functions have valuable properties: critical points are global maxima, & the weighted sum of concave functions is also concave. A popular choice to describe an average utility and production functions.

Example: AP = Arrow-Pratt risk aversion measure = $-U''(w)/U'(w)$
Let $U(w) = \beta \ln(w)$ $(\beta > 0)$
$U'(w) = \beta/w > 0$
$U''(w) = -\beta/w^2 < 0$
$AP = 1/w$ ⇒ As w (wealth) increases, risk aversion decreases.

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Figure 7.5 Logarithmic Utility Function

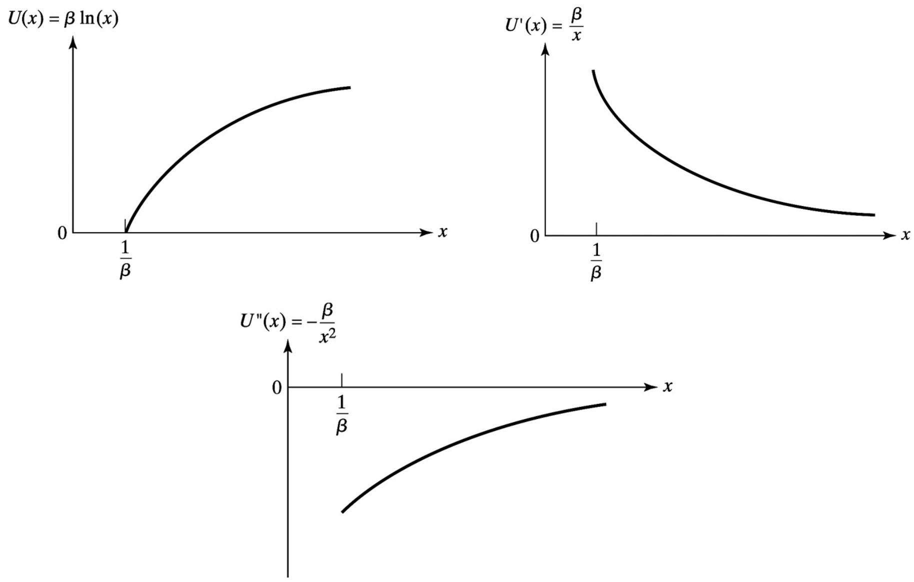

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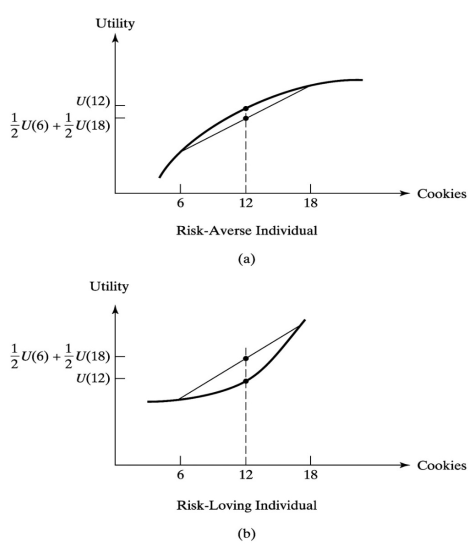
Figure 7.7 Utility Functions for Risk-Averse & Risk-Loving Individuals



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7.5 Series

• Definition: Series, Partial Sums and Convergence

Let $\{a_n\}$ be an infinite sequence.

1. The formal expression $\Sigma_n a_n$ is called an (infinite) *series*.
2. For $N = 1, 2, 3, ...$ the expression $S_n = \Sigma_n a_n$ is called the *N-th partial sum* of the series.
3. If $lim\ S_n$ exists and is finite, the series is said to *converge*.
4. If $lim\ S_n$ does not exist or is infinite, the series is said to *diverge*.

Example: $\Sigma_n (1/2)^n = 1/2 + 1/4 + 1/8 + 1/16 + ...$ (an infinite series). The 3rd, and 4th partial sums are, respectively: 0.875, & 0.9375.

The n-th partial sum for this series is defined as

$$S_n = 1/2 + 1/2^2 + 1/2^3 + ... + 1/2^n$$

Divide S_n by 2 and subtract it from the original one, we get:

$$S_n - 1/2\, S_n = 1/2 - 1/2^{n+1} \quad \Rightarrow S_n = 2\,(1/2 - 1/2^{n+1})$$

Then, $lim\ S_n = 1$ (the infinite series converges to 1)

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7.5 Series: Convergence

• A series may contain positive and negative terms, many of them may cancel out when added together. Hence, there are different modes of convergence: one mode for series with positive terms, and another mode for series whose terms may be negative and positive.

• Definition: Absolute and Conditional Convergence

A series $\Sigma_n a_n$ *converges absolutely* if the sum of the absolute values $\Sigma_n |a_n|$ converges.

A series *converges conditionally*, if it converges, but not absolutely.

Example: $\Sigma_n (-1)^n = -1 + 1 - 1 + 1 ... \Rightarrow$ no absolute convergence

Conditional convergence? Consider the sequence of partial sums:

$S_n = -1 + 1 - 1 + 1 ... - 1 = -1$ if n is odd, and

$S_n = -1 + 1 - 1 + 1 ... - 1 + 1 = 0$ if n is even.

Then, $S_n = -1$ if n is odd and 0 if n is even. The series is divergent.

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7.5 Series: Rearrangement

• Conditionally convergent sequences are rather difficult to work with. Several operations do not work for such series. For example, the commutative law. Since $a + b = b + a$ for any two real numbers a and b, positive or negative, one would expect also that changing the order of summation in a series should have little effect on the outcome

• **Theorem**: Convergence and Rearrangement
A series $\Sigma_n a_n$ be an absolutely convergent series. Then, any rearrangement of terms in that series results in a new series that is also absolutely convergent to the same limit.

Let $\Sigma_n a_n$ be a conditionally convergent series. Then, for any real number c there is a rearrangement of the series such that the new resulting series will converge to c.

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7.5 Series: Absolute Convergent Series

• Absolutely convergent series behave just as expected.

• **Theorem**: Algebra of Absolute Convergent Series
Let $\Sigma_n a_n$ and $\Sigma_n b_n$ be two absolutely convergent series. Then:

1. The sum of the two series is again absolutely convergent. Its limit is the sum of the limit of the two series.

2. The difference of the two series is again absolutely convergent. Its limit is the difference of the limit of the two series.

3. The product of the two series is again absolutely convergent. Its limit is the product of the limit of the two series (*Cauchy Product*).

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7.5 Series: Convergence Tests

• There are many tests for convergence or divergence of series. Here are the most popular in economics.

• *Divergence Test*
If the series $\Sigma_n a_n$ converges, then $\{a_n\}$ converges to 0. Equivalently:
If $\{a_n\}$ does *not* converge to 0, then the series $\Sigma_n a_n$ can not converge.

• *Limit Comparison Test*
Suppose $\Sigma_n a_n$ and $\Sigma_n b_n$ are two infinite series. Suppose also that

$r = lim\ |a_n/b_n|$ exists and $0 < r < \infty$.

Then $\Sigma_n a_n$ converges absolutely iff $\Sigma_n b_n$ converges absolutely.

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7.5 Series: Convergence Tests

• *p Series Test*
The series $\Sigma_n (1/n^p)$ is called a p Series.

if $p > 1$ the p-series converges
if $p \leq 1$ the p-series diverges.

• *Alternating Series Test*
A series of the form $\Sigma_n (-1)^n b_n$, with $b_n \geq 0$ is called *alternating series*. If $\{b_n\}$ is decreasing and converges to 0, then the sum converges.

• *Geometric Series Test*
Let $a \in R$. The series $\Sigma_n a^n$ is called *geometric series*. Then,

if $|a| < 1$ the geometric series converges
if $|a| \geq 1$ the geometric series diverges.

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7.5 Series: Power Series

• Definition: *Power Series*
A function series of the form

$$\Sigma_n a_n (x - c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \dots$$

is called a *power series* centered at c.

That is, a power series is an infinite series of functions where each term consists of a coefficient a_n and a power $(x-c)^n$.

Examples:
- $\Sigma_{n=0} (n+1)\, x^n = 1 + 2x + 3x^2 + 4x^3 + \dots$
- $\Sigma_{n=0} (-1)^n\, 1/2^n\, x^n = 1 - 1/2\, x + 1/4\, x^2 - 1/8\, x^3 + 1/16\, x^4 + \dots$
- $\Sigma_{n=0} (3n/2^n)\, (x-2)^n = 3/2\, (x-2) + 6/4\, (x-2)^2 + 9/8\, (x-2)^3 + \dots$
- Popular example in Finance, DDM (discounted dividend model):

$$P = \sum_{t=1}^{\infty} \frac{d}{(1+r)^t} = d\left(\frac{1}{(1+r)} + \frac{1}{(1+r)^2} + \dots + \frac{1}{(1+r)^t} + \dots\right)$$

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7.5 Series: Power Series

• Properties:
- The power series converges at its center, i.e. for $x = c$
- There exists an r such that the series converges absolutely and uniformly for all $|x - c| \leq p$, where $p<r$, and diverges $\forall\, |x - c| > r$. r is called the *radius of convergence* for the power series and is given by:

$$r = \lim\sup\, |a_n / a_{n+1}|$$

Note: It is possible for r to be zero –i.e., the power series converges *only* for $x = c$– or to be ∞ -i.e., the series converges *for all* x.

Example: $\Sigma_{n=0} (3n/2^n)\, (x - 2)^n$; $a_n = 3n/2^n$

$$r = \lim\sup |a_n/a_{n+1}| = \lim\sup |(3n/2^n)/(3(n+1)/2^{n+1})|$$
$$= \lim\sup |\, n/(n+1) * 2\,| = 2$$

⇒ Series converges absolutely and uniformly on any subinterval of $|x - 2| < 2$.

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7.5 Series: Power Series

• Polynomials are relatively simple functions: they can be added, subtracted, and multiplied (but not divided), and, again, we get a polynomial. Differentiation and integration are particularly simple and yield again polynomials.

• We know a lot about polynomials (e.g. they can have at most *n* zeros) and we feel pretty comfortable with them.

• Power series share many of these properties. Since we can add, subtract, and multiply absolutely convergent series, we can add, subtract, and multiply (think Cauchy product) power series, as long as they have overlapping regions of convergence.

• Differentiating and integrating works as expected. Important result: Power series are infinitely often (lim sup) differentiable.

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7.6 Taylor Series

• The *Taylor series* is a representation of a (infinitely differentiable) function as an infinite sum of terms calculated from the values of its derivatives at a single point, x_0.

Brook Taylor (1685 – 1731, England)

Definition: Taylor Series

Suppose f is an infinitely often differentiable function on a set $\boldsymbol{D}$ and $c \in \boldsymbol{D}$. Then, the series

$$T_f(x,c) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

is called the (formal) *Taylor series* of f centered at, or around, c.

Note: If $c = 0$, the series is also called *MacLaurin Series.*

• The partial sum formed by the first $n + 1$ terms of a Taylor series is a polynomial of degree n. It is called the nth *Taylor polynomial* of f.

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7.6 Taylor Series: Remarks

- A Taylor series is associated with a given function *f*. A power series contains (in principle) arbitrary coefficients a_n. Therefore, *every Taylor series is a power series but not every power series is a Taylor series.*
- $T_f(x, c)$ converges trivially for $x = c$, but it may or may not converge anywhere else. In other words, the "*r*" *of* $T_f(x, c)$ *is not necessarily greater than zero.*
- Even if $T_f(x, c)$ converges, it *may or may not converge to f.*

Example: A Taylor Series that does not converge to its function

$$f(x) = \exp(-1/x^2) \quad \text{if } x \neq 0$$
$$= 0 \quad \text{if } x = 0$$

• The function is infinitely often differentiable, with $f'(0) = 0$. $T_f(x, 0)$ around $c = 0$ has radius of convergence infinity.

• $T_f(x, 0)$ around $c = 0$ does not converge to the original function ($T_f(x, 0) = 0$ for all x).

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7.6 Maclaurin Series: Power Series Derivation

$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n$ primitive function

$f'(x) = a_1 + 2a_2x + 3a_3x^2 + \ldots + na_nx^{n-1}$ 1st derivative

$f''(x) = 2a_2 + 6a_3x + \ldots + n(n-1)a_nx^{n-2}$ 2nd derivative

$f'''(x) = 6a_3 + \ldots + n(n-1)(n-2)a_nx^{n-3}$ 3rd derivative

$\vdots$ $\vdots$

$f^n(x) = n(n-1)(n-2)(n-3)\ldots(3)(2)(1)a_n$ nth derivative

Evaluating each function at c = 0, simplifying & solving for the coefficient

$f(0) = a_0 \rightarrow f(0) = 0!a_0 \rightarrow a_0 = f(0)/0!$

$f'(0) = a_1 \rightarrow f'(0) = 1!a_1 \rightarrow a_1 = f'(0)/1!$

$f''(0) = 2a_2 \rightarrow f''(0) = 2!a_2 \rightarrow a_2 = f''(0)/2!$

$f'''(0) = 6a_3 \rightarrow f'''(0) = 3!a_3 \rightarrow a_3 = f'''(0)/3!$

$\vdots$ $\vdots$

$f^n(0) = n(n-1)(n-2)(n-3)\ldots(3)(2)(1)a_n \rightarrow a_n = f^n(0)/n!$

Substituting the value of the coefficients into the primitive function

$$f(x) \approx \frac{f(0)}{0!}(x)^0 + \frac{f'(0)}{1!}(x)^1 + \frac{f''(0)}{2!}(x)^2 + \ldots + \frac{f^{(n)}(0)}{n!}(x)^n$$

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7.6 Taylor Series: Taylor's Theorem

Suppose $f \in C^{n+1}([a, b])$ –i.e., f is $(n+1)$-times continuously differentiable on $[a, b]$. Then, for $c \in [a,b]$ we have:

$$f(x) = \frac{f(c)}{0!}(x-c)^0 + \frac{f^{/}(c)}{1!}(x-c)^1 + \frac{f^{//}(c)}{2!}(x-c)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + R$$

$$\text{where } R_{n+1}(x) = \frac{1}{n!}\int f^{(n+1)}(p)(x-p)^n\, dp$$

In particular, the $T_f(x, c)$ for an infinitely often differentiable function f converges to f iff the remainder $R_{(n+1)}(x) \rightarrow 0$ as $n \rightarrow \infty$.

• We can show that a function really has a Taylor series by checking that the remainder goes to zero. Lagrange found an easier expression:

$$R_{n+1}(x) = \frac{f^{(n+1)}(p)}{(n+1)!}(x-c)^{n+1}$$

for some p between x and c.

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7.6 Taylor Series: Taylor's Theorem

• Implications:

- A function that is $(n+1)$-times continuously differentiable can be approximated by a polynomial of degree n.
- If f is a function that is $(n+1)$-times continuously differentiable and $f^{(n+1)}(x) = 0$ for all x, then f is necessarily a polynomial of degree n.
- If a function f has a Taylor series centered at c then the series converges in the largest interval $(c - r, c + r)$ where f is differentiable.

• In practice, a function is *approximated* by its Taylor series using a small n, say $n = 2$:

$$f(x) \approx (x-c)^0 + \frac{f^{(1)}(1)}{1!}(x-c)^1 + \frac{f^{(2)}(c)}{2!}(x-c)^2$$

• The error (& the approximation) depends on the curvature of f.

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7.6 Taylor Series: Taylor Polynomial

• The partial sum formed by the first $n + 1$ terms of a Taylor series is a polynomial of degree n that is called the nth *Taylor polynomial* of the function.

• Taylor polynomials are approximations of a function, which become generally better as n increases.

Example: We approximate the following quadratic function with a Taylor polynomial around $c = 1$:

$f(x) = 5 + 2x + x^2$ $\qquad$ $f(c=1) = 8$

$f'((x) = 2 + 2x$ $\qquad$ $f'(c=1) = 4$

For $n = 1$: $\quad T_f(x,c) \approx \frac{f(1)}{0!}(x-1)^0 + \frac{f^{(1)}(1)}{1!}(x-1)^1$

$\approx 8 + 4(x-1) = 4 + 4x$

with $\quad R_2 = [2/2!](x-1)^2 = (x-1)^2$

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7.6 Taylor Series Approximations

Example (continuation): Let's check the approximation error, R_2:

	$f(x) = 5 + 2x + x^2$	$f(x) \approx 4 + 4x$	R_2
$c = 1$	$f(1) = 8$	$f(1) = 8$	0
$c = 1.1$	$f(1.1) = 8.41$	$f(1.1) = 8.4$	0.1^2
$c = 1.2$	$f(1.2) = 8.84$	$f(1.2) = 8.8$	0.2^2

For $n = 2$:

$$T_f(x,c) \approx \frac{f(1)}{0!}(x-1)^0 + \frac{f^{(1)}(1)}{1!}(x-1)^1 + \frac{f^{(2)}(1)}{2!}(x-1)^2$$

$$\approx 8 + 4(x-1) + 2/2\,(x-1)^2 = 4 + 4x + (x-1)^2$$

$$\approx 4 + 4x + x^2 - 2x + 1$$

$$\approx 5 + 2x + x^2$$

with $\quad R_3 = 0$

Note: Polynomials can be approximated with great accuracy.

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7.6 Taylor Series Approximations: BS Example

• We do an expansion of the BS pricing formula (*ceteris paribus*) with respect to S_t –i.e., take (T-t), i, and σ as fixed, usually at current or average values. Recall the BS call option pricing formula:

$$C(S_t) = S_t\, N(d1) - K\, e^{-i(T-t)} N(d2)$$

For n = 1, around $S_t = S^*$, we have:

$$C(S_t) \approx C(S^*) + \Delta(S^*)\,(S_t - S^*) = \text{constant} + \Delta(S^*)\, S_t$$

If we want to approximate $C(S_t + \delta)$ around S_t, we get:

$$C(S_t + \delta)_t \approx C(S_t) + \Delta(S_t)\,\delta$$

For n = 2, around $S_t = S^*$ we have:

$$C(S_t + \delta)_t \approx C(S_t) + \Delta(S_t)\,\delta + \tfrac{1}{2}\,\Gamma(S_t)\,\delta^2$$

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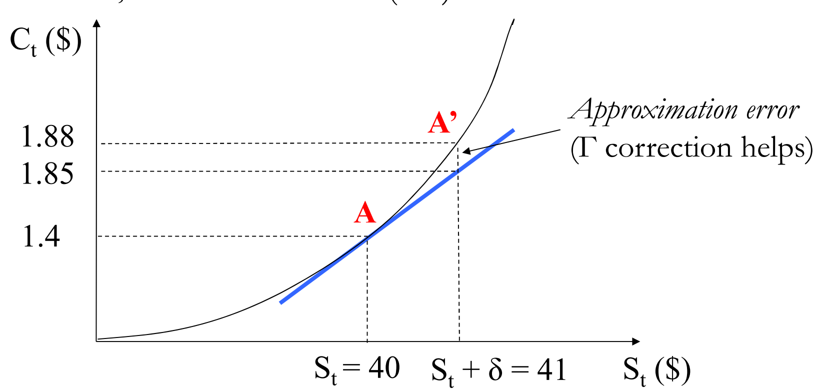
7.6 Taylor Series Approximations: BS Example

• From $C(S_t + \delta)_t \approx C(S_t) + \Delta(S_t)\,\delta$

At **A**, Δ=0.45 (σ = 19% annual, i = 1.5%, K = \$41, T – t = 90/365).

Let δ = \$1, then C(\$41) = \$1.4 + 0.45*\$1 = \$1.85



• At **A'**, C_t = \$1.88. Then, the approximation error is \$0.03.

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7.6 Taylor Series Approximations: BS Example

- At **A'**, the approximation error is: \$1.88 - \$1.85 = \$.03

- To improve the approximation, we can use a 2nd-order Taylor series:

$$C(S_t + \delta)_t \approx C(S_t) + \Delta(S_t)\,\delta + \frac{1}{2}\,\Gamma(S_t)\,\delta^2$$

At **A**, Δ=0.45 & Γ=0.09. Then,

$$C(\$41) \approx 1.4 + 0.45*1 + 0.5*.09*1^2 = \$1.895,$$

which delivers a smaller error (\$-0.015).

- Note: The change, δ (=\$1), is not small. At **A'**, there is a new Δ (=55). Delta-neutral portfolios need to be adjusted!

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7.6 Maclaurin Series of e^x

Let's do a Taylor series around $c = 0$:

$f(x) = e^x$ primitive function $\Rightarrow f(0) = e^0 = 1$

$f'(x) = e^x$ 1^{st} derivative $\Rightarrow f'(0) = e^0 = 1$

$f''(x) = e^x$ 2^{nd} derivative $\Rightarrow f''(0) = e^0 = 1$

$f'''(x) = e^x$ 3^{rd} derivative $\Rightarrow f'''(0) = e^0 = 1$

$\vdots \quad \vdots \qquad \cdots \qquad \cdots$

$f^n(x) = e^x$ n^{th} derivative $\Rightarrow f^{(n)}(0) = e^0 = 1$

Substituting the value of the coefficients into the primitive function

$$e^x = \frac{1}{0!}(x)^0 + \frac{1}{1!}(x)^1 + \frac{1}{2!}(x)^2 + \cdots + \frac{1}{n!}(x) + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

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7.6 Maclaurin Series of $e^{-\frac{x^2}{2}}$

Let's do a Taylor series around $c = 0$:

$f(x) = e^{-\frac{x^2}{2}}$ $\Rightarrow f(0) = 1$

$f'(x) = -xe^{-\frac{x^2}{2}} = -xf(x)$ $\Rightarrow f'(0) = 0$

$f''(x) = (x^2 - 1)e^{-\frac{x^2}{2}} = (x^2 - 1)f(x)$ $\Rightarrow f''(0) = -1$

$f'''(x) = 2xe^{-\frac{x^2}{2}} + (-x^3 + x)e^{-\frac{x^2}{2}} = (-x^3 + 3x)f(x)$ $\Rightarrow f'''(0) = 0$

$f^{IV}(x) = (x^4 - 6x^2 + 3)f(x)$ $\Rightarrow f^{IV}(0) = 3$

$f^{V}(x) = (-x^5 + 10x^3 - 15x)f(x)$ $\Rightarrow f^{V}(0) = 0$

$f^{VI}(x) = (x^6 - 10x^4 + 45x^2 - 15)f(x)$ $\Rightarrow f^{V}(0) = 15$

$\vdots \quad \vdots$

Continue. Then, substituting into a n^{th}-order Taylor series:

$$e^{-\frac{x^2}{2}} = 1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} x^{2n}$$

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7.6 Maclaurin Series of $e^{-\frac{x^2}{2}}$

Apply the Taylor series to the standard normal pdf:

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

Now, we approximate $f(x)$ with a 6^{th}-order Taylor series:

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \approx \frac{1}{\sqrt{2\pi}} \left(1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48}\right)$$

	Normal	**2nd order**	**4th order**	**6th order**
$x = 0$	0.3989423	0.3989423	0.3989423	0.3989423
$x = 0.2$	0.3910427	0.3909634	0.3910432	0.3910427
$x = 0.5$	0.3520653	0.3490745	0.3521912	0.3520614
$x = 1$	0.2419707	0.1994711	0.2493389	0.2410276
$x = 1.5$	0.1295176	-0.0498678	0.2025879	0.107917
$x = 2$	0.05399097	-0.3989423	0.3989423	-0.1329808

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7.6 Maclaurin Series of cos(x)

Let's do a Taylor series around $c = 0$:

$$
\begin{array}{lll}
f(x) = \cos(x) & \text{primitive function} & \Rightarrow f(0) = \cos(0) = 1 \\
f^{/}(x) = -\sin(x) & 1^{\text{st}} \text{ derivative} & \Rightarrow f'(0) = -\sin(0) = 0 \\
f^{//}(x) = -\cos(x) & 2^{\text{nd}} \text{ derivative} & \Rightarrow f''(0) = -\cos(0) = -1 \\
f^{///}(x) = \sin(x) & 3^{\text{rd}} \text{ derivative} & \Rightarrow f'''(0) = -\sin(0) = 0 \\
f^{(4)}(x) = \cos(x) & 4^{\text{th}} \text{ derivative} & \Rightarrow f^{(4)}(0) = \cos(0) = 1 \\
\qquad\ldots & & \qquad\ldots
\end{array}
$$

Substituting the value of the coefficients into the primitive function

$$\cos(x) = 1 - \frac{1}{2!}(x)^2 + \frac{1}{4!}(x)^4 + \ldots + = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n!} x^{2n}$$

• Now, let's check if the remainder $R_{2(n+1)}$ goes to 0 as $n \to \infty$:

$$R_{2n+2}(x) = \left|\frac{f^{(2n+2)}(p)}{(2n+2)!}(x-0)^{2n+2}\right| = \left|\frac{\cos(p)}{(2n+2)!} x^{2n+2}\right| \le \frac{|x|^{2n+2}}{(2n+2)!}$$

and the last term is a converging series to 0, as $n \to \infty$.

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7.6 Maclaurin Series of sin(x) & Euler's formula

Similarly, we can do a Taylor series for sin(x):

$$\sin(x) = x - \frac{1}{3!}(x)^3 + \frac{1}{5!}(x)^5 - \frac{1}{7!}(x)^7 + \cdots + = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

• Now, let's go back to the Taylor series of e^x. Let's look at e^{ix}:

$$e^{ix} = \sum_{n=0}^{\infty} \frac{1}{n!}(ix)^n$$

$$= 1 + ix + \frac{1}{2!}(ix)^2 + \frac{1}{3!}(ix)^3 + \frac{1}{4!}(ix)^4 + \frac{1}{5!}(ix)^5 + \cdots + \frac{1}{n!}(ix)^n + \cdots$$

$$= 1 + ix - \frac{1}{2!}(x)^2 - i\frac{1}{3!}(x)^3 + \frac{1}{4!}(x)^4 + i\frac{1}{5!}(x)^5 + \cdots$$

$$= \cos(x) + i\sin(x)$$

<u>Note</u>: This last result is called *Euler's formula*. (It will re-appear when solving differential equations with complex roots.)

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7.6 Maclaurin Series of log (1+x)

$f(x) = \log(1+x)$ primitive function
$f^{/}(x) = (1+x)^{-1}$ 1^{st} derivative
$f^{//}(x) = -(1+x)^{-2}$ 2^{nd} derivative
$f^{///}(x) = 2(1+x)^{-3}$ 3^{rd} derivative
$\vdots \quad \vdots \quad \ldots \qquad \ldots$
$f^{n}(x) = (-1)^{(n-1)}(n-1)!\,(1+x)^{-n}$ n^{th} derivative

Evaluating each function at $x_0=0$,
$f(0) = 0;\ f'(0) = 1; f''(0) = -1; f''(0) = 2; \ldots; f^{n}\,(0) = (-1)^{(n-1)}(n-1)!$

Substituting the value of the coefficients into the primitive function:

$$\log(1+x) \ = \frac{0}{0!}(x)^0 \ + \frac{1}{1!}(x)^1 \ - \frac{1}{2!}(x)^2 \ + \cdots + \frac{(-1)^{(n-1)}(n-1)!}{n!}(x)^n$$
$$= \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} x^i$$

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7.6 Maclaurin Series of log (1+*x*)

$$\log(1+x) \ = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

• A 1^{st} order series expansion: $\log(1+x) = x + O(x^2)$.
Notation: $O(x^2)$: R is bounded by Ax^2 as $x \to 0$ for some $A < \infty$.

• Q: Why do we care about this approximation $\log(1+x) = x$?
A: Let's define net or simple (total) return, R_t:

$$R_t = \frac{(P_t - P_{t-1}) + D_t}{P_{t-1}} = \text{Capital gain + Dividend yield}$$

where P_t = Stock price or Value of investment at time t
D_t = Dividend or payout of investment at time t

Then, we define the gross (total) return as:

$$R_t + 1 = \frac{P_t + D_t}{P_{t-1}}$$

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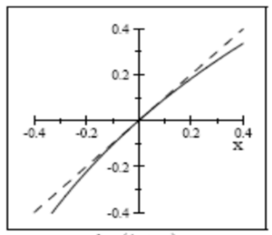
7.6 Maclaurin Series of log (1+x)

• There is another commonly used definition of return, the *log (total) return*, r_t, defined as the log of the gross return:

$$r_t = \log(1 + R_t) = \log(P_t + D_t) - \log(P_{t-1})$$

<u>Note</u>: When the values are small (-0.1 to +0.1), the two returns are *approximately* the same: $r_t = \log(1 + R_t) \approx R_t$.



• In general –i.e., when returns are not small, $r_t < R_t$.

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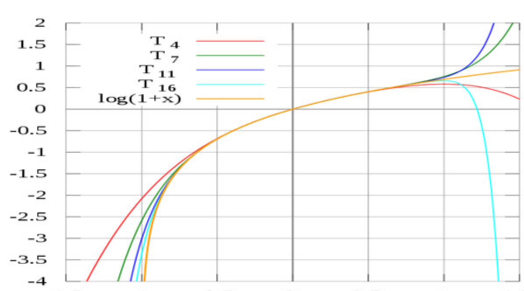
7.6 Taylor series: Approximations

• Taylor series work very well for polynomials; the exponential function e^x and the sine and cosine functions. (They are all examples of *entire functions* –i.e., $f(x)$ equals its Taylor series everywhere).
• Taylor series do not always work well. For example, for the logarithm function, the Taylor series do not converge if x is far from x_0.
• Log approximation around 0:



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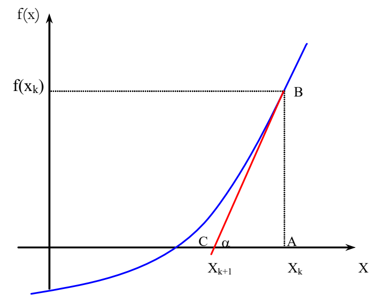
7.6 Taylor Series: Newton Raphson Method

- The Newton Raphson (NR) method is a procedure to iteratively find roots of functions, or a solution to a system of equations, where $f(\mathbf{x})=\mathbf{0}$. Very popular in numerical optimization, where $f'(\mathbf{x})=\mathbf{0}$.

- To find the roots of a function $f(x)$. We start with:

$$\Delta y \cong \frac{\partial f(x)}{\partial x}\Delta x \quad \Rightarrow \Delta x = \frac{\Delta y}{f'(x)}$$
$$\Rightarrow x_{k+1} = x_k + \frac{f(x_{k+1}) - f(x_k)}{f'(x_k)}$$
$$= x_k + \frac{0 - f(x_k)}{f'(x)} = x_k - \frac{f(x_k)}{f'(x_k)}$$



- We derived an iterative process to find the roots of a function $f(x)$.

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7.6 Taylor Series: NR Method

- We can use the NR method to minimize a function.

- Recall that $f\ '(x^*) = 0$ at a minimum or maximum, thus stationary points can be found by applying NR method to the derivative. The iteration becomes:

$$x_{k+1} = x_k - \frac{f'(x_k)}{f''(x_k)}$$

- We need $f\ ''(x_k) \neq 0$; otherwise the iterations are undefined. Usually, we add a step-size, λ_k, in the updating step of x:

$$x_{k+1} = x_k - \lambda_k \frac{f'(x_k)}{f''(x_k)}$$

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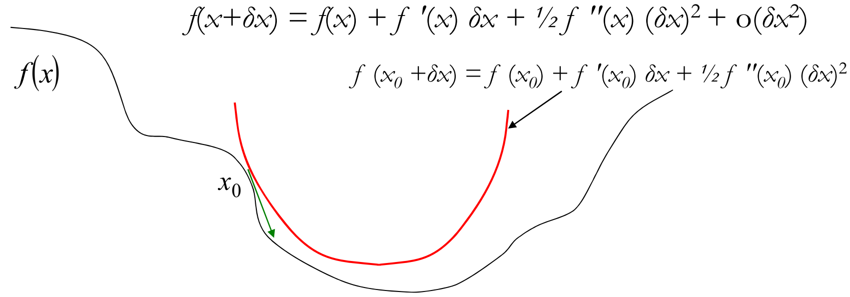
7.6 Taylor Series: NR – Quadratic Approximation

• When used for minimization, the NR method approximates $f(x)$ by its quadratic approximation near x_k.

• Expand $f(x)$ locally using a 2nd-order Taylor series:

$$f(x+\delta x) = f(x) + f'(x)\,\delta x + ½ f''(x)\,(\delta x)^2 + o(\delta x^2)$$



• Find the δx which minimizes this local quadratic approximation:

$$\delta x = -f'(x)/f''(x)$$

• Update x: $\quad x_{k+1} = x_k - f'(x)/f''(x)$.

7.7 Geometric series

• Geometric series: Each term is obtained from the preceding one by multiplying it by x, convergent if $|x|<1$.

Given $f(x) = (1-x)^{-1}$. Find the first five terms of the Maclaurin series (n = 4) around c = 0.

$$f(x) \approx f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f^{(3)}(c)}{3!}(x-c)^3 + \frac{f^{(4)}(c)}{4!}(x-c)^4$$

$$\begin{array}{llll}
f(x) = \frac{1}{1-x} & = (1-x)^{-1} & \rightarrow & f(c) = (1-0)^{-1} = 1 \\
f'(x) = (-1)(-1)(1-x)^{-2} & = (1-x)^{-2} & \rightarrow & f'(c) = (1-0)^{-2} = 1 \\
f''(x) = (-2)(-1)(1-x)^{-3} & = 2(1-x)^{-3} & \rightarrow & f''(c) = 2(1-0)^{-3} = 2 \\
f^{(3)}(x) = (-3)(2)(-1)(1-x)^{-4} & = 6(1-x)^{-4} & \rightarrow & f^{(3)}(c) = 6(1-0)^{-4} = 6 \\
f^{(4)}(x) = (-4)(3)(2)(-1)(1-x)^{-5} & = 24(1-x)^{-5} & \rightarrow & f^{(4)}(c) = 24(1-0)^{-5} = 24
\end{array}$$

$$f(x) = 1 + 1(x-0) + \frac{2}{2}(x-0)^2 + \frac{6}{6}(x-0)^3 + \frac{24}{24}(x-0)^4 + \cdots$$

$$f(x) = 1 + x + x^2 + x^3 + x^4 + \ldots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \rightarrow \sum_{n=0}^{\infty} ax^n = \frac{a}{1-x} \rightarrow \sum_{n=1}^{\infty} ax^n = a\left(\frac{1}{1-x} - 1\right)$$

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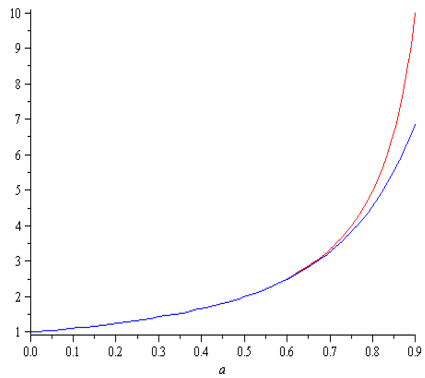
7.7 Geometric series: Approximating $(1\text{-}a)^{-1}$

• It is possible to accurately approximate some ratios, with $(1 - x)$ term in the denominator, with a geometric series. Recall:

$$y = (1 - a)^{-1} \qquad \text{where } a = 0, 0.1, \ldots, 0.9$$
$$y = 1 + a + a^2 + a^3 + a^4 + \ldots$$



• For $n = 4$.
• $a = 0.1 \quad \Rightarrow 1/(1 - a) = 1.1111$
& $1+.1+.1^2+.1^3+.1^4 = 1.11110$

• $a = 0.9 \quad \Rightarrow 1/(1 - a) = 10$
& $1+.9+.9^2+.9^3+.9^4 = 4.09510$

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7.7 Geometric series: Approximating A^{-1}

• Taylor series approximation

$$(1-x)^{-1} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots + x^{n-1} \qquad \text{if } |x| < 1 \text{ (scalar}$$

$$(I-A)^{-1} = \sum_{n=0}^{\infty} A^n = I + A + A^2 + \ldots + A^{n-1} \quad \text{by analogy}$$

• Application to the Leontief Model ($\mathbf{x} = \mathbf{Ax} + \mathbf{d} \Rightarrow (\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{d}$):

$$A = \begin{bmatrix} .15 & 0.25 \\ 0.20 & .05 \end{bmatrix} \quad \Rightarrow (I-A)^{-1} = \begin{bmatrix} 1.2541 & 0.3300 \\ 0.2640 & 1.1221 \end{bmatrix}$$

$$(I-A)^{-1} \approx \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0.15 & 0.25 \\ 0.20 & 0.05 \end{bmatrix} + \begin{bmatrix} 0.15 & 0.25 \\ 0.20 & 0.05 \end{bmatrix}^2$$
$$= \begin{bmatrix} 1.1725 & 0.3125 \\ 0.2400 & 1.0525 \end{bmatrix}$$

with $n = 6 \quad (I-A)^{-1} \approx \begin{bmatrix} 1.2506 & 0.3290 \\ 0.2632 & 1.1214 \end{bmatrix}$

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7.7 Application: Geometric series & PV Models

- A stock price (P) is equal to the discounted some of all futures dividends. Assume dividends are constant (d) and the discount rate is r. Then:

$$P = \sum_{t=1}^{\infty} \frac{d}{(1+r)^t} = d(\frac{1}{(1+r)} + \frac{1}{(1+r)^2} + \cdots + \frac{1}{(1+r)^t} + \cdots)$$

$$P = d(x + x^2 + x^3 \ldots + x^t + \cdots) = d\left(\frac{1}{(1-x)} - 1\right)$$

$$= d\left(\frac{1}{\left(1-\frac{1}{1+r}\right)} - 1\right) = d\left(\frac{1+r}{(1+r-1)} - 1\right) = d\left(\frac{1+r-r}{r}\right) = \frac{d}{r}$$

where $x = \frac{1}{(1+r)}$.

Example: d= USD 1; r = 8% $\Rightarrow$ P = USD 1/.08 = USD 12.50

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7.7 Application: Geometric series & PV Models

- Now, we assume dividends (d) grow at a constant (g) and the discount rate is r. Then:

$$P = \sum_{t=1}^{\infty} d\frac{(1+g)^t}{(1+r)^t} = d(1\frac{1}{1-x} - 1), \qquad \text{where } x = \frac{(1+g)}{(1+r)}$$

$$P = d\left(\frac{1}{1 - \frac{(1+g)}{(1+r)}} - 1\right) = d\left(\frac{1}{\frac{(1+r)-(1+g)}{(1+r)}} - 1\right)$$

$$= d\left(\frac{1+r}{(r-g)} - 1\right) = d\left(\frac{(1+g)}{(r-g)}\right)$$

Example: d = USD 1; r = 8% & g = 2%
$\Rightarrow$ P = USD 1 * (1.02)/(.08 – .02) = USD 17.

Note: NPV of dividend growth = USD 17 – USD 12.5 = USD 4.5

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Q: What is the first derivative of a cow?

A: Prime Rib!