Chapter 6
Introduction to Calculus

Archimedes of Syracuse (*c.* 287 BC – *c.* 212 BC)

Bhaskara II (1114 – 1185)

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6.0 Calculus

• Calculus is the mathematics of change.

• Two major branches: *Differential calculus* & *Integral calculus*, which are related by the *Fundamental Theorem of Calculus*.

• Differential calculus determines varying rates of change. It is applied to problems involving acceleration of moving objects (from a flywheel to the space shuttle), rates of growth and decay, optimal values, etc.

• Integration is the "inverse" (or opposite) of differentiation. It measures accumulations over periods of change. Integration can find volumes and lengths of curves, measure forces and work, etc. Older branch: Archimedes (c. 287−212 BC) worked on it.

• Applications in science, economics, finance, engineering, etc.

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6.0 Calculus: Early History

• The foundations of calculus are generally attributed to *Newton* and *Leibniz*, though Bhaskara II is believed to have also laid the basis of it. The Western roots go back to Wallis, Fermat, Descartes and Barrow.

• Q: How close can two numbers be without being the same number? Or, equivalent question, by considering the difference of two numbers: How small can a number be without being zero?

• Fermat's and Newton's answer: The *infinitessimal*, a positive quantity, smaller than any non-zero real number.

• With this concept differential calculus developed, by studying ratios in which both numerator and denominator go to zero simultaneously.

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6.1 Comparative Statics

- *Comparative statics*: It is the study of different equilibrium states associated with different sets of values of parameters and exogenous variables.

- Static equilibrium analysis: we start with $y^* = f(x)$
- Comparative static equilibrium analysis: $y_1^* - y_0^* = f(x_1) - f(x_0)$
  (subscripts 0 and 1: initial & subsequent points in time)

- Issues:
  - Quantitative & qualitative of change or
  - Magnitude & direction

  The rate of change –i.e., the derivative ($\partial Y / \partial G$)

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6.1 Comparative Statics: Application

• We uses differential calculus to study what happens to an equilibrium in an economic model when something changes.

Example: Macroeconomic Model

Given

$Y = C + I_0 + G_0$

$C = a + b\,(Y - T)$

$T = d + t\,Y$

Solving for Y $\Rightarrow Y^* = (a - bd + I_0 + G_0)/[1 - (b^*(1-t))]$

• <u>Question</u>: What happens to Y* when an exogenous variable, say G_0, changes in the model?

$$Y_1^* - Y_0^* = ?; \quad \frac{Y_1^* - Y_0^*}{G_1 - G_0} = ?$$

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6.2 Rate of Change & the Derivative

- *Difference quotient*:

  Let $y = f(x)$

  - Evaluate $y = f(x)$ at two points: x_0 and x_1: $y_0 = f(x_0)$ & $y_1 = f(x_1)$

  - Define: $\Delta x = x_1 - x_0 \quad \Rightarrow x_1 = x_0 + \Delta x$

    $\Delta y = y_1 - y_0 \quad \Rightarrow \Delta y = f(x_0 + \Delta x) - f(x_0)$

  - Then, we define the *difference quotient* as:

$$\frac{\Delta y}{\Delta x} = \frac{f(x_o + \Delta x) - f(x_0)}{\Delta x}$$

- Q: What happens to *f(.)* when *x* changes by a *very* small amount?

- That is, we want to describe the small-scale behavior of a function

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6.2 Rate of Change & the Derivative

• *Difference quotient*:

$$\frac{\Delta y}{\Delta x} = \frac{f(x_o + \Delta x) - f(x_0)}{\Delta x}$$

• To describe the small scale behavior of a function, we use the *derivative*.

• *Derivative* (based on Newton's and Leibniz's approach):
Let's take Δx as an infinitesimal (a positive number, but smaller than any positive real number) in the difference quotient:

$$f'(x) = \frac{\Delta y}{\Delta x}$$

Another interpretation: Infinitesimals are locations which are not zero, but which have zero distance from zero.

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6.2 Rate of Change & the Derivative

• The derivative captures the small-scale behavior of a function –i.e., what happens to *f(.)* when x changes by a very small amount.

• But, the infinitesimal approach was not elegant. A real number is either small but non-zero or is zero. Nothing is in between. Definition and manipulation of infinitesimals was not very precise.

• D'Alembert started to think about "vanishing quantities." He saw the tangent to a curve as a limit of secant lines. This was a revolutionary, though graphical, approach:

As the end point of the secant converges on the point of tangency, it becomes identical to the tangent "*in the limit*."

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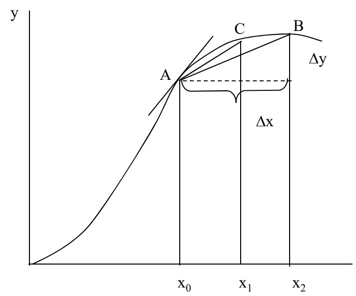
6.2 Rate of Change & the Derivative

• Graphical interpretation of the derivative:
Slope = tangent of the function at $x=x_0$:



Jean d'Alembert (1717 – 1783)

• As the lim of $\Delta x \to 0$, then $f'(x)$ measures the tangent (rise/run) of $f(x)$ at the initial point A. Again, the secant becomes the tangent.

• This is the traditional motivation in calculus textbooks.

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6.2 Rate of Change & the Derivative

• The graphical interpretation is subject to objections like those seen in Zeno's paradoxes (say, Achilles and the Tortoise).

• Based on the work of Cauchy and Weierstrass, we use limits to replace infinitesimals. Limits have a precise definition and nice properties. Then, we have an easier definition to work with:

• *Derivative* (based on Weierstrass's approach):
Let's take limits ($\Delta x \to 0$) in the difference quotient we get the derivative:

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 - \Delta x) - f(x_0)}{\Delta x} = \frac{dy}{dx}$$

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6.2 Rate of Change & Derivative: Economic Interpretation

• Product of Labor

Suppose $Q=f(L)$ is a production function, where L is the only input, labor. Let $L_1 = L_0 + \Delta L$, then, we define:

Average product of labor = Change in production/change in labor
$= [f(L_1) - f(L_0)] / (L_1 - L_0)$

Let $L_0 = 0$ and $f(L_0)=0$. Define $L_1 = \Delta L = 1$ hour

$\Rightarrow$ labor productivity $= f(L_1)$ /hour

• Marginal product of labor

The marginal product of labor at $L = L_0$ is

$\lim_{\Delta L \to 0} [f(L_0 + \Delta L) - f(L_0)]/\Delta L$ (if the limit exists)

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6.2 Rate of Change & Derivative: Power function

Example: $y = bx^2 + a$

$$f(x_0) = bx_0^2 + a$$

$$f(x_0 + \Delta x) = b(x_0 + \Delta x)^2 - 4$$

$$\frac{\Delta y}{\Delta x} = \frac{b(x_0 + \Delta x)^2 + a - (bx_0^2 + a)}{\Delta x}$$

$$= \frac{bx_0^2 + 2bx_0\Delta x + b\Delta x^2 + a - bx_0^2 - a}{\Delta x} = 2bx_0 + b\Delta x$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 2bx_0$$

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6.2 Rate of Change & Derivative: Power function

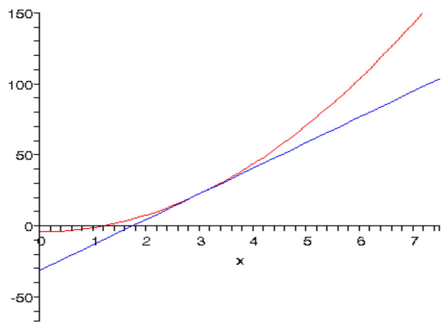
Now, evaluate derivative at x=3

$y = 3x^2 - 4$ (red)

$f'(x) = 6x$

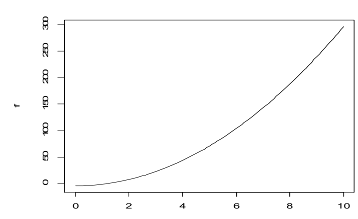
$f'(x_0=3) = 18$

$y = 18x - 31$, tangent at point (3, f(3)=23) (blue)



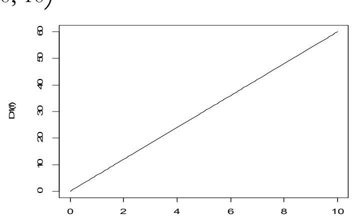
6.2 Rate of Change & Derivative in R

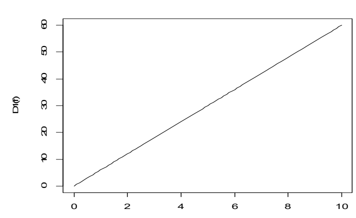
```
> f = function(x) {3*x^2 – 4}
> plot(f, 0, 7)
```



```
> D1 = function(f, delta=.000001){function(x){ (f(x+delta) - f(x))/(delta)} }
> plot(D1(f), 0, 10)

> D = function(f, delta=.000001){function(x){ (f(x+delta) - f(x-delta))/(2*delta)} }
> plot(D(f), 0, 10)
```





6.2 Differentials

• In the one dimensional case, we define the derivative as:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

Equivalently, we can write

$$f(x_0+\Delta x) = f(x_0) + \Delta x\, f'(x) + r_x(\Delta x),$$

where $r_x(\Delta x)$ represent the remainder, which is of *smaller order* than Δx, or $o(\Delta x)$. That is,

$$\lim_{\Delta x \to 0} \frac{r_x(\Delta x)}{\Delta x} = 0.$$

• The quantity $f(x_0+\Delta x) - f(x_0)$ is composed of two terms:
- $\Delta x\, f'(x)$, the part proportional to the change in x (Δx)
- $r_x(\Delta x)$, an "*error*," which gets smaller with Δx.

The expression $df(x) = \Delta x\, f'(x)$ is called the (first) *differential of f*.

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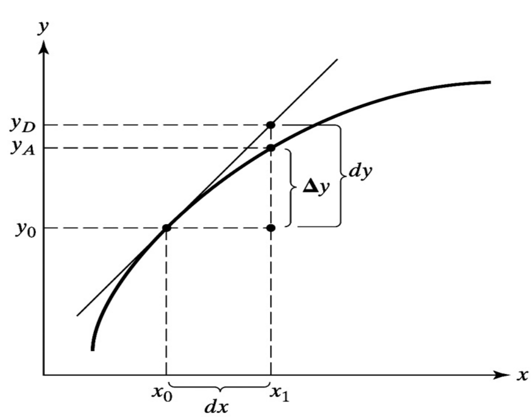
Figure 6.9 Differential Approximation and Actual Change of a Function

The differential $df(x) = \Delta x\, f'(x)$ is the linear part of the increment $f(x_0+\Delta x) - f(x_0)$. This is expressed by geometrically replacing the curve at point x_0 by its tangent.



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6.2 Differentials and Approximations

• The differential is used to linearly approximate changes in $f(x)$. The "*error*" –i.e., the quality of the approximation– depends on the curvature of is $f(x)$ and, of course, on the magnitude of Δx.

• For very *small* Δx, the approximation should be good, regardless of $f(x)$. But, when is Δx very small?

• An interesting case is when $\Delta x = 1$. In this case, $df(x) = f'(x)$. Then, the first derivative approximates the change in the function per additional unit of x.

In the production function example, $f'(L)$ measures the additional output that can be produced with an additional unit of labor.

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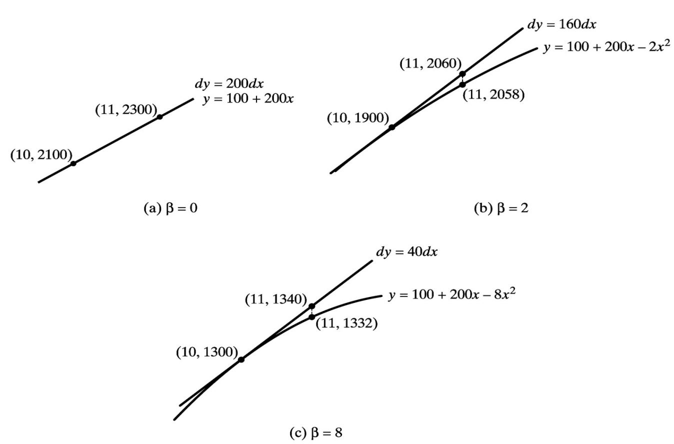
Figure 6.11 Differential Approximation for Beta with Different Functions



<u>Note</u>: Since $\Delta x = 1$ in all cases, $dy = f'(x)$. As the function has more curvature, the linear approximation becomes less precise.

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6.2 Differentials and Approximations: Example

• Recall the solution to Y (income) in the Macroeconomic model:

$$Y = f(I, G) = \frac{1}{1 - b(1-t)}(a - bd + I_0 + G_0)$$

We have a linear function in I (investment) and G (government spending) . Assume I is fixed, then we have y=*f*(G).

• <u>Comparative Static Question</u>:
What happens to Y* when G increases by ΔG? We approximate the answer by:

$$\Delta Y^* \approx \Delta G \; f'(G);$$

where $f'(G) = \frac{1}{1 - b(1-t)} > 0$ $\Rightarrow$ if ΔG =\$1, then dY = f'(G).

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6.2 Multivariate Calculus: Partial Differentiation

• It is straightforward to extend the concepts of derivative and differential to more than one variable. In this case, y depends on several variables: $x_1, x_2, \ldots, x_n$.

The derivative of y w.r.t. one of the variables –while the other variables are held constant– is called *partial derivative*.

$$y = f(x_1, x_2, \ldots, x_n)$$

$$\lim_{\Delta x_1 \to 0} \frac{\Delta y}{\Delta x_1} = \lim_{\Delta x_1 \to 0} \frac{f(x_1 + \Delta x_1, x_2, \ldots, x_n) - f(x_1, x_2, \ldots, x_n)}{\Delta x_1}$$

$$\equiv \frac{\partial y}{\partial x_1} \equiv f_1 \qquad \text{(partial derivative w.r.t. } x_1\text{)}$$

$$\text{In general,} \qquad \lim_{\Delta x_i \to 0} \frac{\Delta y}{\Delta x_i} \equiv \frac{\partial y}{\partial x_i} \equiv f_i, \qquad i = 1 \ldots n$$

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6.2 Partial derivatives: Example

Example: Cobb-Douglas production function

$$\text{Production function}: Q = AK^{\alpha} L^{\beta}$$

$$MPK = \frac{\partial Q}{\partial K} = \alpha A K^{-(1-\alpha)} L^{\beta}$$

$$MPL = \frac{\partial Q}{\partial L} = \beta A K^{\alpha} L^{-(1-\beta)}$$

• Interpretation: As usual:
MPL: Marginal product of labor
MPK: Marginal product of capital

• We use them to linearly approximate the change in production ΔQ in the face of unitary changes, one at a time, of inputs. When L and K change simultaneously, we need to use the *total derivative*:

$$\Delta Q \approx \text{MPL}\ \Delta L + \text{MPK}\ \Delta K$$

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6.3 Concept of Limit: Preliminaries

■ Let M be a metric space and let S be a subset of M. For example, $M=R^n$ and $S=Q$.

■ <u>Definition</u>: N-ball

Let c be a point in M and r be a positive number. The set of all points $x \in M$ whose distance is less than r is called an *n-ball* of radius r and center c. It is usually denoted by B(c) or B(c, r). Thus:

$$B(c, r) = \{ x : x \in M, d(x, c) < r \}$$

In Euclidian spaces, we have

$$B(c, r) = \{ x : x \in R^n, \| x - c \| < r \}$$

An n-ball is also called a *neighborhood of c*.

Example: B(4, 3) in R is the open interval (1, 7).

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6.3 Concept of Limit: Preliminaries

- Definition: Interior, accumulation & isolated points

Assume that $c \in$ S and $x \in M$. Then,

(a) if there is an n-ball B(c), all of whose points belong to S, c is called an *interior point* of S. The set of all interior points of S is called *the interior* of S

(b) if every n-ball B(x) contains at least one point of S different from x, then x is called an *accumulation point* of S. It is also called *limit point.*

(c) if B(c) ∩ S = {c}, then c is an *isolated point* of S.

- Definition: Boundary Point

(d) if every n-ball B(x) contains at least one point of S and at least one point of the complement of S, then x is called a *boundary point* of S. The set of all boundary points of S is called *the boundary* of S.

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6.3 Concept of Limit: Preliminaries

Note: Every non-isolated boundary point of a set S∈*R* is a limit point of S. A limit point is never an isolated point

Examples: Limit point (Let $M=R^n$ and S=*Q)*

- 2 is a *limit point* of S since {x_n = 2; 2; 2; 2; 2; ... } or {x_n = 1; 1+1/2; 1+2/3; 1+3/4; ...}. 2 is a limit point of S that belongs to S.
- π is a *limit point* of S since {x_n = 3, 3.14, 3.141, 3.1415, . . .}. π does not belong to S (though, it belongs to M).

Let's look at the interval (0, 4).

- The boundary of (0, 4) is the set consisting of the two elements {0, 4}. - The interior of the set (0, 4) is the set (0, 4) -i.e., itself.
- No points of either set are isolated, and each point of the set {0, 4} is an accumulation point. The same is true, incidentally for each of the sets (0, 4), [0, 4), (0, 4], and [0, 4].

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6.3 Concept of Limit: Open and Closed Sets

- Definition: Open and Closed sets

A set S ⊂ M is said to be:

(a) *open* (in M) if all its points are interior points

(b) *closed* (in M) if it contains all its accumulation points

(c) *bounded* (in M) if there is a real number $r>0$ and a point c in M such that S lies entirely within the n-ball B(c, r)

(d) *compact* (in M) if it is closed and bounded

Examples: Let A be an interval in R. For $a < b$ in R we have:

(a, b), (a, ∞), R are open intervals in R.

[a, b], [a, ∞), R are closed intervals in R.

[-3, 4] is bounded in R (it is contained in B(0, 5) or B(2, 6)). [-3, 4] is compact

- The set of all open sets on a space M is called the *topology* on M.

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6.3 Concept of Limit: Open and Closed Sets

Note: Some sets (like the M itself) are both closed and open, they are called *clopen* sets. But, [0; 1) ⊂ R is neither open nor closed. Thus, subsets of a metric space can be open, closed, both, or neither.

- Properties:

- The compliment of an open set is closed and the compliment of a closed set is open.
- Every union of open sets is again open.
- Every intersection of closed sets is again closed.
- Every finite intersection of open sets is again open
- Every finite union of closed sets is again closed.
- Every open set U ⊂ R can be expressed as a countable disjoint union of open intervals of the form (a; b), where a is allowed to take on the value -∞ and b is allowed to +∞.

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6.3 Concept of Limit: Cantor set C

Note: Open sets in *R* are generally easy. Closed sets can get complicated.

- Cantor Middle Third Set

Start with the unit interval $\boldsymbol{S}_0 = [0, 1]$.

Remove from $\boldsymbol{S}_0$ the middle third. Set $\boldsymbol{S}_1 = \boldsymbol{S}_0 \backslash (1/3, 2/3)$

Remove from $\boldsymbol{S}_1$ the 2 middle thirds. Set $\boldsymbol{S}_2 = \boldsymbol{S}_1 \backslash \{ (1/9, 2/9) \text{ U } (7/9, 8/9) \}$

Continue, where $\boldsymbol{S}_{n+1} = \boldsymbol{S}_n \backslash \{$ *middle thirds of subintervals of* $\boldsymbol{S}_n$ $\}$.

Then, the *Cantor set C* is defined as $\mathbf{C} = \ \mathbf{S}_n$

The Cantor set C is an indication of the complicated structure of closed sets in the real line.

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6.3 Concept of Limit: Cantor set C

The Cantor set C is an indication of the complicated structure of closed sets in the real line.

C has the following properties:

- C is compact (i.e., closed and bounded)
- C is perfect –i.e., it is closed and every point of C is a limit point of C.
- C is uncountable (since every non-empty perfect set is uncountable).
- C has length zero, but contains uncountably many points.
- C does not contain any open set.

This set is used to construct counter-intuitive objects in real analysis or to show lack of generalization of some results. For example, Riemann integration does not generalize to all intervals.

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6.3 Concept of Limit: Functions

■ <u>Definition</u>: Functions

Let S and T two sets (say, two metric spaces). If with each element x in S there is associated exactly one element y in T, denoted $f(x)$, then f is said to be a *function* from S to T. We write

$$f: S \rightarrow T,$$

and say that f is defined on S with values in T. The set S is called the *domain* of f; the set of all values of f is called the *range* of f, and it is a subset of T. T is called the *target* or *codomain*.

• The *image* of f is defined as

$$image(f) = \{t \in T : \text{there is an } s \in S \text{ with } f(s) = t\}.$$

If C is a subset of the range T, then the *preimage*, or *inverse image*, of C under the function f is the set defined as

$$f^{-1}(C) = \{x \in S : f(x) \in C\}$$

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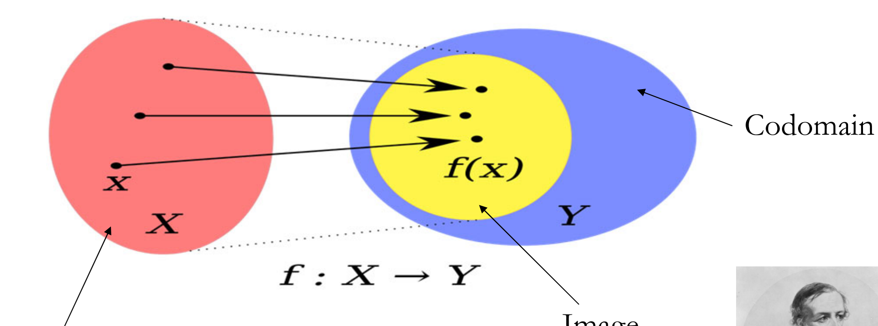
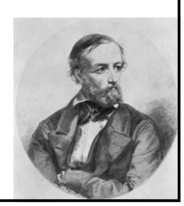
6.3 Concept of Limit: Functions

■ **Example**: Domain and image of $f: X \rightarrow Y$

f is a function from domain X to codomain Y. The smaller yellow oval inside Y is the image of f.





Peter G. Lejeune Dirichlet (1805 – 1859, Germany)

6.3 Concept of Limit: Functions

• A function f: S → R defined on set S with values in R is called *real-valued*. f: S → R^m (m>1) whose values are points in R is called a *vector function*.

A vector function is bounded if there is a real number B such that

$$\|f(x)\| \leq B \qquad \text{for all } x \text{ in S.}$$

■ A function f from S to T can be classified into three groups:

- *One-to-one* if whenever $f(s) = f(w)$, then s = w. Also called *injections*.
- *Onto* if for all $t \in$ T $\exists s \in$ S such that $f(s) = t$. Also called *surjections*.
- *Bijection* if it is one-to-one and onto -i.e., bijections are functions that are injective and *surjective*.

Examples: A linear function is a bijection. A periodic function is not one-to-one. Say, $g(x) = cos(x)$ is neither one-to-one nor onto in R.

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6.3 Concept of Limit: Inverse Functions

• When f: S → T is one-to-one on a set C in S, there is a function from $f(C)$ back to C, which assigns to each $t \in f(C)$ the unique point in C which mapped to it. This map is called the *inverse of f* on C and it written as:

$$f^{-1}: f(C) \rightarrow C.$$

Examples:

- Let f: R → T, say $f = 3x+2$ $\quad \Rightarrow f^{-1}$: $(y - 2)/3$
- The logarithm is the inverse of the exponential function.
- The demand function $q = D(p)$, under the usual assumptions, has as the inverse function $p = D^{-1}(q)$, which is called the *inverse demand function*.

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6.3 Concept of Limit: Composition Functions

• Let f: S → T and g: V → W be two functions. Suppose that T is a subset of V. Then, the *composition of f with g* is defined as the function:

$$(g \circ f)(x) = g(f(x)) \text{ for all } x \text{ in } S.$$

That is, *function composition* is the application of one function to the results of another. The functions f and g can be *composed* by computing the output of g when it has an argument of $f(x)$ instead of x. Intuitively, if z is a function $g(y)$ and y is a function $f(x)$, then z is a function $h(x)$.

Example: Define $f(x) = x^5$ and $g(x) = \exp(x)$. Then, $(g \circ f)(x) = \exp(x^5)$

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6.3 Concept of Limit: Sequences

■ <u>Definition</u>: Sequence

A *sequence* of real numbers is a function f: $N \rightarrow R$.

That is, a sequence can be written as $f(1), f(2), f(3), \ldots$.. Usually, we will denote such a sequence by the symbol $\{a_j\}$ where $a_j = f(j)$.

Example: The sequence ½, ¼, ⅛, ... is written as $\{1/2^j\}$.

■ <u>Definition</u>: Convergence

A sequence $\{a_j\}$ of real (or complex) numbers is said to *converge* to a real (or complex) number c if for every ε>0, there is an integer N>0 such that if $j > N$, then

$$|\, a_j - c \,| < \varepsilon.$$

The number c is called the *limit* of the sequence $\{a_j\}$ and we write $a_j \rightarrow c$.

If a sequence $\{a_j\}$ does not converge, then we say that it *diverges*.

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6.3 Concept of Limit: Sequences

Example: The sequence $\{\frac{1}{j}\}$ converges to zero.

We need to show that no matter which ε > 0 we choose, the sequence will eventually become smaller than this number. Take any ε > 0. Then, there exists a positive integer N such that $1/N < \varepsilon$.

Thus, for any $j > N$ we have:

$$\left|\frac{1}{j} - 0\right| = \left|\frac{1}{j}\right| < \frac{1}{N} < \varepsilon, \text{ whenever } j > N.$$

This is precisely the definition of the sequence $\{1/j\}$ converging to 0.

Note: Easy proof. A proper choice of N is the key.

■ If $\{a_j\}$ is a convergent sequence, $\{a_j\}$ is bounded & the limit is unique.

Example: The sequence of Fibonacci numbers is unbounded. Then, the sequence cannot converge (convergent sequence must be bounded). 35

6.3 Concept of Limit: Sequences

• Algebra of Convergent Sequences:

Let $\{a_j\}$ be a convergent sequence. Then, the sequence is bounded, and the limit is unique.

Suppose $\{a_j\}$ and $\{b_j\}$ are converging to a and b, respectively. Then,

- Their sum converges to $a + b$, and the sequences can be added term by term.

- Their product converges to $a * b$, and the sequences can be multiplied term by term.

- Their quotient converges to a/b, provide that $b \neq 0$, and the sequences can be divided term by term (if the denominators are not zero).

- If $a_n \leq b_n$ for all n, then $a \leq b$. (It does not work for strict inequalities).

• We know how to work with convergent sequences, we would like to have an easy criteria to determine whether a sequence converges. 36

6.3 Concept of Limit: Sequences

■ Definition: Monotonicity

A sequence $\{a_j\}$ is called *monotone increasing* if $a_{j+1} \geq a_j$ for all j.

A sequence $\{a_j\}$ is called *monotone decreasing* if $a_j \geq a_{j+1}$ for all j.

■ **Proposition:** Monotone Sequences

- If $\{a_j\}$ is a monotone increasing sequence that is bounded above, then the sequence must converge.
- If $\{a_j\}$ is a monotone decreasing sequence that is bounded below, then the sequence must converge.

Examples:

- $\{\frac{j}{j+1}\}$ is monotone increasing, bounded above by 1. It must converge.
- $\{\frac{1}{j}\}$ is monotone decreasing, bounded below by 0. It must converge.

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6.3 Concept of Limit: Cauchy Sequence

■ Often, it is hard to determine the actual limit of a sequence. We want to have a definition which only includes the known elements of a particular sequence and does not rely on the unknown limit.

■ Definition: Cauchy Sequence

Let $\{a_j\}$ be a sequence of real (or complex) numbers. We say that $\{a_j\}$ is *Cauchy* if for each ε>0 there is an integer N>0 such that if $j, k > N$ then

$$|a_j - a_k| < \varepsilon.$$

• Now, we know what it means for the elements of a sequence to get closer together, and to stay close together.

■ **Theorem**: Completeness Theorem in R.

Let $\{a_j\}$ be a Cauchy sequence in R. Then, $\{a_j\}$ is bounded.

Let $\{a_j\}$ be a sequence in R. $\{a_j\}$ is Cauchy iff it converges to some limit a.

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6.3 Concept of Limit: Subsequences

- By considering Cauchy sequences instead of convergent sequences we do not need to refer to the unknown limit of a sequence (in effect, both concepts are the same).

- Q: Not all sequences converge. How do we deal with these situation?

A: We change the sequence into a convergent one (extract subsequences) and we modify our concept of limit (*lim sup* and *lim inf*).

- Definition: Subsequence.

Let $\{a_j\}$ be a sequence. When we extract from this sequence only certain elements and drop the remaining ones we obtain a new sequences consisting of an infinite subset of the original sequence. That sequence is called a *subsequence* and denoted by $\{a_{j,k}\}$ $(k=1, 2, ..., \infty)$.

Note: We can think of a subsequence as a composition function.

6.3 Concept of Limit: Subsequences

Example: Take the sequence $\{(-1)^j\}$, which does not converge. The sequence is: {-1, 1, -1, 1,...}

Extract every even number in the sequence, we get: {-1, -1, -1, -1,...}

$\Rightarrow$ subsequence converges to -1.

Extract every odd number in the sequence, we get: {1, 1, 1, 1,...}

$\Rightarrow$ subsequence converges to 1.

Note: We can extract infinitely many subsequences from any given sequence

- **Proposition**: Subsequences from Convergent Sequence

Let $\{a_j\}$ be a convergent sequence, then every subsequence of $\{a_j\}$ converges to the same limit.

Let $\{a_j\}$ be a sequence such that every possible subsequence extracted from $\{a_j\}$ converge to the same limit, then $\{a_j\}$ also converges to that limit.

6.3 Concept of Limit: Subsequences

- **Theorem**: Bolzano-Weierstrass

Let $\{a_j\}$ be a sequence of real numbers that is bounded. Then, there exists a subsequence $\{a_{j,k}\}$ that converges.

This is one on the most important results of basic real analysis, and generalizes the above proposition. It explains why subsequences can be useful, even if the original sequence does not converge.

Example: The sequence $\{sin(j)\}$ does not converge, but since it is bounded, we can extract a convergent subsequence.

Note: The Bolzano-Weierstrass theorem does guarantee the existence of that subsequence, but it does not say how to obtain it. It can be difficult. We will extend the concept of limits to deal with divergent sequences.

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6.3 Concept of Limit: Lim Sup and Lim Inf

- Definition: Lim Sup and Lim Inf

Let $\{a_j\}$ be a sequence of real numbers. Define

$$A_j = inf\{a_j, a_{j+1}, a_{j+2}, \dots\}$$

and let $c = lim\ (A_j)$. Then, c is called the *limit inferior* of the sequence $\{a_j\}$.

Let $\{a_j\}$ be a sequence of real numbers. Define:

$$B_j = sup\{a_j, a_{j+1}, a_{j+2}, \dots\}$$

and let $d = lim\ (B_j)$. Then d is called the *limit superior* of the sequence.

Summary:

- $lim\ inf(a_j) = lim(A_j)$, where $A_j = inf\{a_j, a_{j+1}, a_{j+2}, \dots\}$.
- $lim\ sup(a_j) = lim(B_j)$, where $B_j = sup\{a_j, a_{j+1}, a_{j+2}, \dots\}$.

• These limits are often counter-intuitive, they have one very useful property: *lim sup* and *lim inf* always exist (possibly $\mp\infty$) for any sequence in R.

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6.3 Concept of Limit: Lim Sup and Lim Inf

Example 1: Consider $\{(-1)^j\}$. We find the numbers $A_j = inf\{a_j, a_{j+1}, a_{j+2}, ...\}$

$A_1 = inf\{-1, 1, -1, 1, ...\} = -1$

$A_2 = inf\{1, -1, 1, -1, ...\} = -1$

etc. It is clear that $lim\ inf\{(-1)^j\} = -1$. (also the infimum)

Similarly, $lim\ sup\{(-1)^j\} = 1$. (also the supremum)

Example 2: Consider $\{1/j\}$. The sequence is $\{1, 1/2, 1/3, 1/4, ...\}$. Then, the infimum is zero, while the supremum is 1. Let's get the A_j and B_j:

$A_1 = inf\{1, 1/2, 1/3, 1/4, ...\} = 0$ & $B_1 = sup\{1, 1/2, 1/3, 1/4, ...\} = 1$

$A_2 = inf\{1/2, 1/3, 1/4, 1/5, ...\} = 0$ & $B_2 = sup\{1/2, 1/3, 1/4, 1/5,.\} = 1/2$

$A_3 = inf(1/3, 1/4, 1/5, 1/6, ...\} = 0$ & $B_3 = sup(1/3, 1/4, 1/5, 1/6,.\} = 1/3$

etc. It is clear that $lim\ inf\{1/j\} = 0$. (also the infimum)

etc. It is clear that $lim\ sup\{1/j\} = 0$. (different from the supremum)

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6.3 Concept of Limit: Lim Sup and Lim Inf

- As can be seen from the previous example, as j increases, the *lim sup* decreases and the *lim inf* increases. We think of *lim sup* and *lim inf* as subsequential limits.

- **Theorem**: A finite lim sup $\{a_j\}$ exists iff

(i) $\{a_j\}$ is bounded above and

(ii) $\{a_j\} \nrightarrow -\infty$.

(Note: For the *lim inf* case we need (i) bounded below & (ii) $\nrightarrow \infty$.)

- **Theorem**: Let $\{a_j\}$ be a sequence. Suppose that $lim\ sup(a_j) = B$ exists. Then there is a subsequence $\{a_{jk}\}$ that converges to B. Moreover, B is the largest subsequential limit. (Similar theorem works for *lim inf.)*

Note: Both Theorems imply the Bolzano-Weierstrass Theorem.

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6.3 Concept of Limit: Lim Sup and Lim Inf

- *lim sup* $\{a_j\}$ is the largest limit of convergent subsequences of $\{a_j\}$. (Reverse results holds for *lim inf*.)

Define a sequence $\{b_j\}$ by $\{b_j\} = -\{a_j\}$. Then, there is a subsequence $\{b_{jk}\}$ of $\{b_j\}$ converging to *lim sup* $\{b_j\}$, and $\{a_{jk}\}$ is a subsequence of $\{a_j\}$, satisfying:

$$\lim_{k\to\infty} \{a_{jk}\} = -\lim_{k\to\infty} \{b_{jk}\} = -\textit{lim sup}\ \{b_j\} = \textit{lim inf}\ \{a_j\},$$

- **Corollary.** Let $\{a_j\}$ be a bounded sequence in R. Then, $\{a_j\}$ converges –i.e., $\lim_{j\to\infty} \{a_j\}$ exists– if and only if *lim sup* $\{a_j\}$ = *lim inf* $\{a_j\}$.

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6.3 Concept of Limit: Definition

- Definition: Limit

Let f: S → R^m. Let c be an accumulation point of S. Suppose there exists a point b in R^m with the property that for every ε>0 there is a δ>0 such that

$$\|f(x) - b\| < \varepsilon \qquad \forall x \text{ in S}, x \neq c, \text{ for which } \|x - c\| < \delta.$$

Then, we say the *limit* of $f(x)$ is b, as x tends to c, and we write

$$\lim_{x\to c} f(x) = b$$

Note: This is the *(ε, δ)-definition* of limit, introduced by Bolzano/Cauchy and perfected by Weierstrass.

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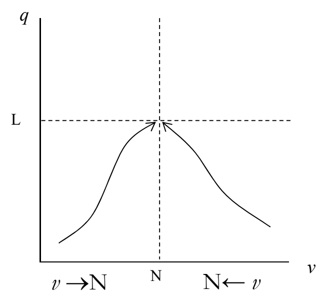
6.3 Concept of Limit: Right- & Left-hand Limit

- The limit $(f(x), x \to c$, direction) function attempts to compute the limiting value of $f(x)$ as x approaches c from left, c^- (the *left-hand limit*) or right, c^+ (the *right-hand limit*).
- When the left-hand and the right-hand limits are equal, say to L, we say the limit exists and equals L.
- If $q = f(v)$, what value does q approach as $v \to$ N? Answer: L
- As $v \to$ N from either side, $q \to$ L.

Then, both the left-side limit and the right side-limit are equal.



- Therefore, lim q = L.

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6.4 Evaluation of a Limit

- To take a limit, substitute successively smaller values that tend to N from both the left and right sides since N may not be in the domain of the function.
- If v is in both the numerator and denominator remove it from either depending on the function
- Taking limits sometimes is not straightforward.

Example: Given $q = (2v + 5)/(v + 1)$, find the limit of q as $v \to +\infty$. Dividing the numerator by denominator:

$$q = \frac{2v + 5}{v + 1} = 2 + \frac{3}{v + 1}$$

$$\lim_{v \to +\infty} q = 2$$

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6.4 Limit Theorems

- If $q = a\,v + b$, $\Rightarrow$ $\lim_{v \to N} q = a\,N + b$
- If $q = g(v) = b$, $\Rightarrow$ $\lim_{v \to N} q = b$
- If $q = v$, $\Rightarrow$ $\lim_{v \to N} q = N$
- If $q = v^k$, $\Rightarrow$ $\lim_{v \to N} q = N^k$
- $\lim_{v \to N} (q_1 \pm q_2) = \lim_{v \to N} (q_1) \pm \lim_{v \to N} (q_2)$
- $\lim_{v \to N} (q_1 * q_2) = \lim_{v \to N} (q_1) * \lim_{v \to N} (q_2)$
- $\lim_{v \to N} (q_1 / q_2) = \lim_{v \to N} (q_1) / \lim_{v \to N} (q_2)$

Example: Find lim $(1+v)/(2+v)$ as $v \to 0$

$$\frac{L_1}{L_2} = \frac{\lim_{v \to 0} (1 + v)}{\lim_{v \to 0} (2 + v)} = \frac{1}{2}$$

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6.4 L'Hôpital's Rules

- If f and g are differentiable in a neighborhood of $x=c$, and $f(c)=g(c)=0$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

provided the limits exist.

Note: The same result holds for one-sided limits.

- If f and g are differentiable and $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$, then

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)}$$

provided the last limit exists.

• In other situations L'Hôpital's rules may also apply, but often a problem can be rewritten so that one of these two cases will apply.

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6.4 Limit Jokes

After explaining to a student through various lessons and examples that:

$$\lim_{x \to 8} \frac{1}{x-8} = \infty$$

I tried to check if she really understood that, so I gave her a different example. This was the result:

$$\lim_{x \to 5} \frac{1}{x-5} = 5$$

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6.5 Continuous Function: Definition

- Definition: Continuous function

  Let M and T be two metric spaces (two sets). We say $f: M \to T$ is *continuous* if for each convergent sequence $x_n \to x$ in M, we have $f(x_n) \to f(x)$ in T.

- Requirements for continuity
  - $f(x)$ is defined and belongs to T.
  - f has a limit as $x_n \to x$ –i.e., the limit exists.
  - limit equals $f(x)$ in value

Note: Continuity preserves limits.

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6.5 Continuous Function: (ε, δ) Definition

- (ε, δ) Definition: Continuous function

  Let M and T be two metric spaces (two sets). A function f: $M \rightarrow$ T is *continuous* at $x \in M$ if $\forall \varepsilon > 0$, $\exists \delta > 0$, such that $y \in M$ and $d(x, y) < \delta \Rightarrow d(f(x), f(y)) < \varepsilon$.

We say that f is continuous on M if it is continuous at every $x \in$ S. If δ does not depend on x *and* y *(or* , then we say f: $M \rightarrow$ T is *uniformly continuous*.

That is, f is uniformly continuous if it is possible to guarantee that $f(x)$ and $f(y)$ be as close to each other as we want by requiring only that x and y are sufficiently close to each other. With ordinary continuity, the maximum distance between $f(x)$ and $f(y)$ may depend on x and y themselves.

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6.5 Continuous Function: (ε, δ) Definition

- Continuous functions can fail to be uniformly continuous if they are unbounded on a finite domain.

Example: $f(x) = 1/x$ on (0; 1) is continuous, but is not uniformly continuous, since it does not matter how small we choose δ, there are always points (x, y) in the interval (0; δ) such that $|f(x) - f(y)| > \varepsilon$).

$f(x) = x^2$ on R is not uniform continuous. This function becomes arbitrarily steep as x approaches infinity.

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6.5 Continuous Function: (ε, δ) Definition

- Continuity preserves limits, but, it does not preserve Cauchy sequences. For example, $f(x) = 1/x$ on (0; 1] → R maps 1, 1/2, 1/3, 1/4,... (Cauchy sequence) to 1, 2, 3, 4,... (non-Cauchy sequence) in R. We need uniform continuity to preserve Cauchy sequences.

- **Theorem**:

Let M and T be two metric spaces, $f: M \to T$ a uniform continuous function and $\{x_n\}$ a Cauchy sequence in M. Then, $f(x_n)$ is a Cauchy sequence in T.

Proof: Let ε>0 be given. Since f is uniformly continuous, ∃δ>0 s.t. $d_M(x, y) < \delta \Rightarrow d_N(f(x), f(y)) < \varepsilon$. Since $\{x_n\}$ is Cauchy, $\exists L$ s.t. $m; n > L \Rightarrow d_M(x_m, x_n) < \delta \Rightarrow d_N(f(x_m), f(x_n)) < \varepsilon \Rightarrow f(x_n)$ is Cauchy.

Example: Consistent Estimates (Slutzky's Theorem).

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6.5 Continuous Function: (ε, δ) Definition in *R*

- (ε, δ) definition: (ε, δ) Continuous function in R

  Let $f: S \to R$ be a real valued function on a set S in R^n. Let c be a point in S. We say that f is *continuous* at c if for every ε>0, there exists a δ>0 such that

$$\| f(c+u) - f(c) \| < \varepsilon$$

  for all points c+u for which $\| u \| < \delta$. If f is continuous at every point of S, we say f is continuous on S.

  Note: f has to be defined at the point c to be continuous at c.

• Continuous functions can be added, multiplied, divided, and composed with one another and yield again continuous functions.

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6.5 Continuity and Differentiability of a Function

■ If c is an accumulation (limit) point of S, the definition of continuity implies that

$$\lim_{u\to 0} f(c+u) = f(c).$$

■ Intuition is tricky: Geometry seems to show that if f is continuous at c, it must be continuous near c. This is wrong!

Example: The Dirichlet function

Let f: $R \to R$ defined by

$$f(x) = x \quad \text{if } x \text{ is rational}$$
$$\quad\;\; = 0 \quad \text{if } x \text{ is irrational}$$

is continuous at x=0, but at no other point.

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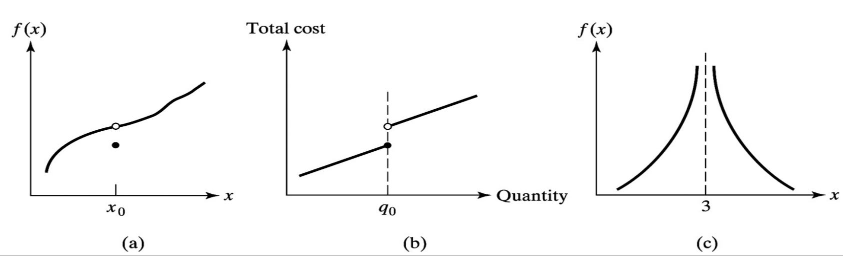
6.5 Continuity and Differentiability of a Function

■ Almost all the basic functions in mathematical econ models are assumed to be continuous.

■ For example, a production function is continuous if a small change in inputs yields a small change in output. (A reasonable assumption.)

■ If a function fails to be continuous at a point c, then the function is called *discontinuous* at c, (c is called a *point of discontinuity*).

Examples: Non-continuous functions:



6.5 Continuity and Differentiability of a Function

- We say $f: R^n \rightarrow R$ is *differentiable* at x^* if $f'(x^*)$ *exists.*
- Not every continuous function has a derivative at every point. For example: $f(x) = |x|$.
  - $|x|$ is not differentiable at $x=0$, the left and the right-handed limits are different (-1 and +1). Then, there is no unambiguous tangent line defined at $x=0$.
  - We need the function $f(.)$ to be *smooth* –i.e., no kinks.
  - If f is differentiable at x^*, then f is continuous at x^*. (Converse is, of course, not true.)

- As with continuous functions, differentiable functions can be added, multiplied, divided, and composed with each other to yield again differentiable functions.

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6.5 Continuity and Differentiability of a Function

- A function $f: R^n \rightarrow R$ is *continuously differentiable* on an open set U of R^n if and only if for each x, df/dx_i exists for all x in U and is continuous in x.

- This rational function is not defined at $v = \pm 2$, even though the limit exists as $v \rightarrow \pm 2$. It is discontinuous and thus does not have continuous derivatives --i.e., it is not continuous differentiable.

$$q = \frac{v^3 + v^2 - 4v - 4}{v^2 - 4}$$

- This continuous function is not differentiable at $x=3$ and, thus, does not have continuous derivatives (it is not continuously differentiable):

$$y = \begin{cases} 5 - x, \text{ where} & (x \leq 3) \\ x - 1, \text{ where} & (x > 3) \end{cases}$$

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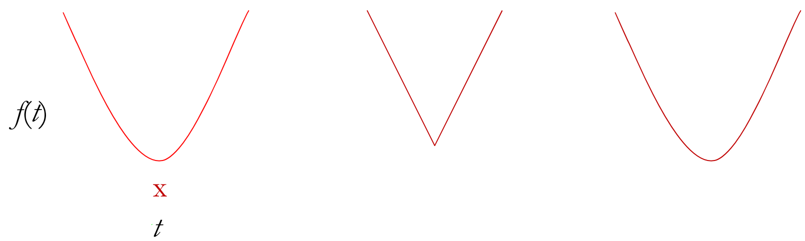
6.5 Continuity and Differentiability of a Function



not continuous:

$f(t) = -1$ if $t = 0$
$f(t) = t^2$ otherwise

continuous but not differentiable:

$f(t) = |t|$

continuous & differentiable:

$f(t) = t^2$

6.5 Continuity and Differentiability of a Function

- For a function to be continuous differentiable
  - All points in the domain of f defined
  - The limit is taken on the difference quotient at $x=x_0$ as $\Delta x \to 0$ from both directions. The continuity condition is necessary, not sufficient.
  - The differentiability condition (smoothness) is both necessary and sufficient for whether f is differentiable.

- **Theorem**: Rolle's Theorem

If f is continuous on $[a, b]$ and differentiable on (a, b), and $f(a)=f(b)=0$, then there exists a number x in (a, b) such that $f'(x) = 0$.

Note: An extension of Rolle's theorem that removes the conditions on $f(a)$ and $f(b)$ is the Mean-Value Theorem. These theorems form the basis for the familiar test for local extrema of a function.

6.6 Continuity and Differentiability: Examples

- Here we want to list some functions that illustrate more or less subtle points for continuous and differentiable functions.
- *Dirichlet function*: A function that is not continuous at any point in R
- *Countable discontinuities*: A function that is continuous at the irrational numbers and discontinuous at the rational numbers.

$$y = \begin{cases} 1/q, & \text{if } x = p/q \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

- *C^1 function*: A function that is differentiable, but the derivative is not continuous.

$$y = \begin{cases} x^2 \sin(1/x), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

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6.6 Continuity and Differentiability: Examples

- *C^n function*: A function that is n-times differentiable, but not (n+1)-times differentiable

$$y = \sqrt[3]{x^{3n+1}}$$

- *C^{inf} function*: A function that is not zero, infinitely often differentiable, but the n-th derivative at zero is always zero.

$$y = \begin{cases} \exp(-1/x^2), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

- *Weierstrass function*: A function that is continuous everywhere and nowhere differentiable in R.

$$y = \sum_{n=0}^{\infty} (3/4)^n \, |\sin(4^n x)|$$

- *Cantor function*: A continuous, non-constant, differentiable function whose derivative is zero everywhere except on a set of length zero 64

6.7 The Limit and the Quotient Ratio

- Let $q \equiv \Delta y/\Delta x$ and $v \equiv \Delta x$ such that $q = f(v)$ and

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{v \to 0} q$$

- Q: What value does variable q approach as v approaches 0?

A: If the function is differentiable, we move from the quotient ratio to the derivative.

Note: This definition may not work well when $\boldsymbol{x}$ is a vector, say $\boldsymbol{x}=(y,z)$. Measuring $\Delta \boldsymbol{y}$ is not a problem (the difference between two functions), but measuring $\Delta \boldsymbol{x} = (\Delta y, \Delta z)$ is not clear.

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6.8 Resolution of a Controversy: Butter Biscuits or Fruit Chewy Cookies?