Chapter 14
Differential Equations

Johann Bernoulli (1667–1748)

Leonhard Paul Euler (1707–1783)

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14.1 Differential Equations: Definitions

• Difference equations work well when we model variables that change incrementally in value. When variables change continuously we use differential equations.

• We start with a continuous time series $\{x(t)\}$.

• Ordinary Differential Equation (ODE): It relates the values of variables at a given point in time and the changes in values over time.

Example: $G(t, x(t), x'(t), x''(t), ...) = 0\ \forall t$. t: scalar, usually time

• An ODE depends on a single independent variable. A *partial differential equation* (PDE) depends on many independent variables.

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14.1 Differential Equations: Definitions

• ODEs are classified according to the highest degree of derivative.
- First-Order ODE: $x'(t) = F(t, x(t)) \quad \forall t.$
- Nth-Order ODE: $G(t, x(t), x'(t), x''(t), \ldots) = 0 \ \forall t.$

Examples: First-order ODE $x'(t) = a\,x(t) + \varphi(t)$
Second-order ODE $x''(t) = a\,x'(t) + b\,x(t) + \varphi(t)$

• If $G(.)$ is linear, we have a *linear* ODE. If $G(.)$ is anything but linear, then we have a *non-linear* ODE.

• In this lecture, we emphasize linear ODE.

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14.1 Differential Equations: Definitions

• A differential equation not depending directly on t is called *autonomous*.
Example: $x'(t) = a\,x(t) + b$ is autonomous.

• A differential equation is *homogeneous* if $\varphi(t) = 0$
Example: $x'(t) = a\,x(t)$ is homogeneous.

• If starting values, say $x(0)$, are given. We have an *initial value problem*.
Example: $x'(t) + 2\,x(t) = 3 \quad x(0) = 2.$

• If values of the function and/or derivatives at *different* points are given, we have a *boundary value problem*.
Example: $x'(t) + 4\,x(t) = 0 \quad x(0) = -2, \ x(\pi/4) = 10.$

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14.1 Differential Equations: Definitions

• A *solution* of an ODE is a *function* $x(t)$ that satisfies the equation for all values of *t*. Many ODE have no solutions.

• *Analytic solutions* -i.e., a closed expression of x in terms of t- can be found by different methods. Example: conjectures, integration.

• Most ODE's do not have *analytic solutions*. *Numerical solutions* will be needed.

• If for some initial conditions a differential equation has a solution that is a constant function (independent of t), then the value of the constant, x_∞, is called an *equilibrium state* or *stationary state*.

• If, for all initial conditions, the solution of the differential equation converges to x_∞ as $t \rightarrow\infty$, then the equilibrium is *globally stable*.

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14.1 ODE: Classic Problem

• *Problem*: "The rate of growth of the population is proportional to the size of the population."

Quantities: t = time, $P(t)$ = population, k = proportionality constant (growth-rate coefficient)

• The differential equation representing this problem:

$$\frac{dP(t)}{dt} = k\ P(t)$$

Note that $P_0 = 0$ is a solution because $\frac{dP(t)}{dt} = 0$ forever (trivial!).

• If $P_0 \neq 0$, how does the behavior of the model depend on P_0 and k? In particular, how does it depend on the *signs* of P_0 and k?

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14.1 ODE: Classic Problem

• The differential equation: $\frac{dP(t)}{dt} = k\,P(t)$

• Guess a solution: The first derivative should be "similar" to the function. Let's try an exponential: $P(t) = c\,e^{kt}$

$$\frac{dP(t)}{dt} = c\,ke^{kt} = k\,P(t) \qquad \text{–it works!}$$

(and, in fact, $c = P_0$.)

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14.2 1st-order ODE: Notation and Steady State

• A first-order ODE:

$$x'(t) = f(t, x(t)) \qquad \forall t.$$

Notation: $\dot{x} = x'(t) = \frac{dx}{dt}$

• The *steady state* represents an equilibrium where the system does not change anymore. When $x(t)$ does not change anymore, we call its value x_∞. That is,

$$x'(t) = 0$$

Example: $x'(t) = a\,x(t) + b,$ with $a \neq 0$.

When $x'(t) = 0$, $x_\infty = -\,b/a$.

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14.2 Separable first-order ODE

• A 1st-order ODE is *separable* if it can be written as: $x'(t) = f(t)\, g(x)$ $\forall t$. Easier to solve (discussed first by Leibniz and Bernoulli in 1694).

Example: $x'(t) = \dfrac{e^{x(t)+t}}{x(t)}\sqrt{(1+t^2)}$ is *separable*.

It can be written as: $x'(t) = [e^{x(t)}/x(t)]\cdot[e^{t}\sqrt{(1+t^2)}]$.

• $x'(t) = f(t) + g(x(t))$ is not separable unless either $f(.)$ or $g(.)$ is identically 0: it cannot be written in the form $x'(t) = f(t)g(x)$.

• If $g(.)$ is a constant, then the general solution of the equation is simply the indefinite integral of $f(.)$.

• If $g(.)$ is not constant, the equation may be easily solved. Assume $g(x) \neq 0$ for all values that x assumes in a solution, we may write:

$$dx/g(x) = f(t)\, dt.$$

• Then, we may integrate both sides: $\int_0^t 1/g(x)\, dx = \int_0^t f(t)\, dt.$ 9

14.2 Separable first-order ODE

Example: $x'(t) = x(t)\, t$.

• Steps to solve equation:

1) Write the equation as: $dx/x = t\, dt$.

2) Integrate both sides: $\ln x = t^2/2 + \text{C}$. (C always consolidates the constants of integration).

3) Isolate x: $x(t) = \text{C}\, e^{t^2/2}$ $\forall\, t$. ($\text{C} = e^{C}$).

Note: If $x(t) \neq 0$ $\forall t$, in all the solutions we need $\text{C} \neq 0$.

• With an initial condition $x(t_0) = x_0$, the value of C is determined:

$$x_0 = Ce^{t_0^2/2} \quad \Rightarrow \quad C = x_0 e^{-t_0^2/2}$$

• *Definite solution* (no unknowns): $x(t) = [x_0 e^{-t_0^2/2}]e^{t^2/2}$

• Suppose at $x(t_0{=}2) = x_0 = 1.78$. Then, at $t = 3.103$,

$x(t = 3.103) = [1.78 * \exp(-2^2/2)] * \exp(3.103^2/2) = 29.693194$ 10

14.2 Linear first-order ODE: Case I - $a(t) = a$

- A linear first-order differential equation takes the form
  $x'(t) = a(t)\,x(t) + b(t) \quad \forall t,$ for some functions a and a.

- **Case I**. $a(t) = a \neq 0$ for all t.
  $\Rightarrow \qquad x'(t) + a\,x(t) = b(t) \qquad \forall t.$
  - The LHS looks like the derivative of a product. But, not *exactly* the derivative of $f(t)\,x(t) = f'(t)\,x(t) + f(t)\,x'(t)$

We would need $f(t) = 1$ and $f'(t) = a \;\; \forall t$, which is not possible.

- *Trick*: Multiply both sides by $g(t)$ for each t:
  $g(t)\,x'(t) + a\,g(t)\,x(t) = g(t)\,b(t) \qquad \forall t.$
- Now, we need $f(t) = g(t)$ and $f'(t) = ag(t)$.
  If $f(t) = e^{at} \;\Rightarrow\; f'(t) = a\,e^{at} = a\,f(t)$

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14.2 Linear first-order ODE: Case I - $a(t) = a$

- Set $g(t) = e^{at}$ $\qquad \Rightarrow e^{at}\,x'(t) + a\,e^{at}\,x(t) = e^{at}\,b(t)$
- The integral of the LHS is $e^{at}\,x(t)$
- *Solution:*

$$e^{at}\,x(t) = C + \int_0^t e^{as}\,b(s)\,ds, \qquad \text{or}$$

$$x(t) = e^{-at}[C + \int_0^t e^{as}\,b(s)\,ds]$$ -- $\int_0^t f(s)\,ds$ is the indefinite integral of $f(s)$ evaluated on $(0, t)$

- *Proposition*
  The general solution of the differential equation
  $x'(t) + a\,x(t) = b(t) \qquad \forall t,$
  where a is a constant and b is a continuous function, is given by
  $x(t) = e^{-at}\,[C + \int_0^t e^{as}\,b(s)\,ds] \qquad \forall t.$

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14.2 Linear first-order ODE: Case I - $a(t) = a$

- **Special Case**: $b(s) = b$

The differential equation is $x'(t) + a\,x(t) = b$

Solution:

$$x(t) = e^{-at}\,[C + \int^t e^{as}\,b\,ds] = e^{-at}\,[C + b\int^t e^{as}\,ds]$$

$$x(t) = e^{-at}[C + \frac{b}{a}e^{as}\,|_0^t] = e^{-at}[C + \frac{b}{a}(e^{at} - 1)]$$

$$= e^{-at}(C - \frac{b}{a}) + \frac{b}{a};$$

Note: If $x(t = 0) = x_0$, $\Rightarrow x_0 = C$

Stability: If $a > 0$ $\Rightarrow x(t)$ is stable (and $x_\infty = b/a$)

If $a < 0$ $\Rightarrow x(t)$ is unstable

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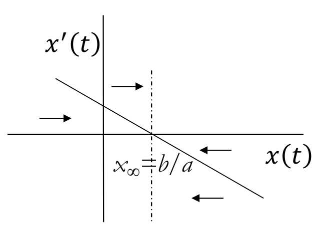
14.2 Linear first-order ODE: Phase Diagram

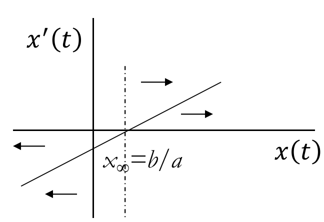
- A phase diagram graphs the first-order ODE. That is, plots $x'(t)$ and $x(t)$.

Example: $x'(t) = -a\,x(t) + b$

$a > 0$



$a < 0$



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14.2 Linear first-order ODE: Examples

- Solution: $x(t) = C^*\, e^{-at} + \frac{b}{a}$

Example: $u'(t) + 0.5\, u(t) = 2$.

Solution:

$u(t) = C^*\, e^{-0.5} + 4$ (Solution is stable ⇒ a=0.5>0)

Steady state: $u_\infty = b/a$ = 2/0.5 = 4

If $u(0) = 20$ ⇒ $C^* = 16$, ⇒ Definite solution: $u(t) = 16\, e^{-.5t} + 4$.

Example: $v'(t) + 0.5\, v(t) = 2$.

Solution:

$v(t) = C^*\, e^{2t} + 2$ (Solution is unstable ⇒ a =-2<0)

Steady state: $v_\infty = b/a$ = -4/-2 = 2

If $v(0) = 3$ ⇒ $C^* = 1$, ⇒ Definite solution: $v(t) = 1\, e^{2t} + 2$.

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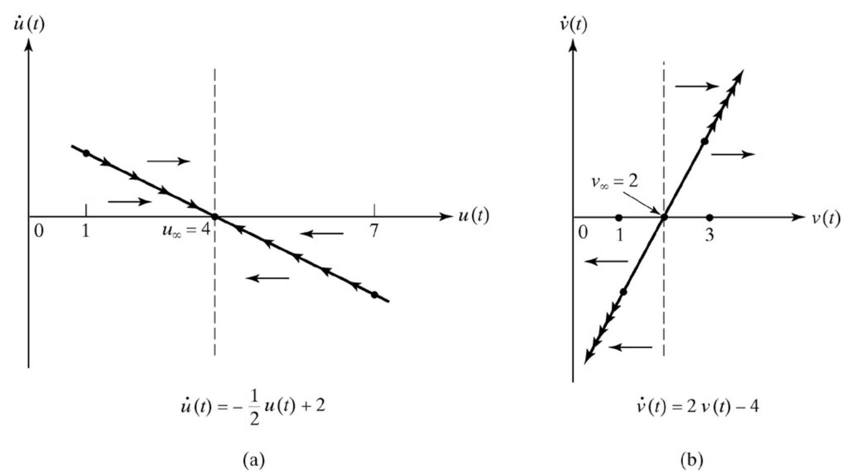
Figure 14.1 Phase Diagrams for Equations (14.6) and (14.7)



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14.2 Linear first-order ODE: Price Dynamics

- Let p be the price of a good.
- Total demand: $D(p) = c - d\,p$
- Total supply: $S(p) = \alpha + \beta\,p$,
- c, d, α, and β are positive constants.
- Price dynamics: $p'(t) = \theta\,[D(p) - S(p)]$, with $\theta > 0$.
- Replacing supply and demand:

$$p'(t) + \theta\,(d + \beta)\,p(t) = \theta\,(c - \alpha) \qquad \text{(a 1st-order linear ODE)}$$

- *Solution:* $x(t) = C^*\,e^{-at} + \frac{b}{a}$

$$p(t) = C^*\,e^{-\theta(d+\beta)t} + (c - \alpha)/(d + \beta).$$

$$p_\infty = \frac{b}{a} = (c - \alpha)/(d + \beta),$$

Given $a = \theta\,(d + \beta) > 0$, this equilibrium is globally stable.

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14.2 Linear first-order ODE: Case II - $a(t) \neq a$

- **Case II**. $a(t) \neq a$ (a is a function!)
  - Then, $x'(t) + a(t)\,x(t) = b(t) \quad \forall t.$
  - Recall we need to recreate $f(t)x(t)$ to apply product rule:
  - We need $f(t) = g(t)$ and $f'(t) = a(t)\,g(t) \quad \forall t$:
  - Try: $g(t) = e^{\int_0^t a(s)ds}$ (the derivative of $\int_0^t a(s)ds = a(t)$).
  - Multiplying the ODE equation by $g(t)$:

$$e^{\int_0^t a(s)ds}\,x'(t) + a(t)\,e^{\int_0^t a(s)ds}\,x(t) = e^{\int_0^t a(s)ds}\,b(t)$$

$$\Rightarrow \quad \frac{d}{dt}\left[e^{\int_0^t a(s)ds}\,x(t)\right] = e^{\int_0^t a(s)ds}\,b(t).$$

$$\Rightarrow \quad e^{\int_0^t a(s)ds}\,x(t) = C + \int_0^t e^{\int_0^t a(s)ds}\,b(u)du$$

$$\Rightarrow \quad x(t) = e^{-\int_0^t a(s)ds}\,[C + \int_0^t e^{\int_0^t a(s)ds}\,b(u)du]$$

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14.2 Linear first-order ODE: Case II - Example

- Solution: $x(t) = e^{-\int_0^t a(s)ds}[C + \int_0^t e^{\int_0^t a(s)ds} b(u)du]$

Example: $x'(t) + \frac{1}{t}x(t) = e^t$

Note: $a(t) = \frac{1}{t}$ $\Rightarrow \int_0^t \frac{1}{s} ds = \ln(t)$ $\Rightarrow e^{\int_0^t \frac{1}{s} ds} = t.$

Solution:

$$x(t) = (1/t)(C + \int_0^t u\, e^u\, du)$$

$$= (1/t)(C + t\, e^t - \int_0^t e^u\, du) \qquad \text{(use integration by parts.)}$$

$$= (1/t)(C + t\, e^t - e^t) = C/t + e^t - e^t/t.$$

We can check that this solution is correct by differentiating:

$$x'(t) + x(t)/t = [-C/t^2 + e^t - e^t/t + e^t/t^2] + C/t^2 + e^t/t - e^t/t^2 = e^t$$

As usual, an initial condition determines the value of C.

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14.2 Linear ODE: Analytic Solution Revisited

- Suppose, we have the following form:

  $x''(t) + ax'(t) + bx(t) = f(t)$ (a and b are constants)
- Let x_1 be a solution of the equation. For any other solution of this equation x, define $z = x - x_1$.
- Then z is a solution of the homogeneous equation:

  $x''(t) + ax'(t) + bx(t) = 0.$

  $\Rightarrow z''(t) + az'(t) + bz(t) = [x''(t) + ax'(t) + bx(t)] - [x_1''(t) + ax_1'(t) + bx_1(t)] = f(t) - f(t) = 0.$
- Further, for every solution z of the homogeneous equation, $x_1 + z$ is clearly a solution of original equation.
- That is, the set of **all solutions** of the original equation may be found by finding *one solution of this equation* and adding to it the *general solution of the homogeneous equation*.

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14.2 Linear ODE: Analytic Solution Revisited

- Thus, we can follow the same strategy used for difference equations to generate an *analytic general solution*

- Steps:
  1) Solve homogeneous equation (constant term is equal to zero.)
  2) Find a particular solution, for example x_∞.
  3) Add homogenous solution to particular solution

Example: $x'(t) + 2\,x(t) = 8.$

Step 1: Guess a solution to homogeneous equation: $x(t) = C\,e^{-2t}$

Step 2: Find a particular solution, say $x_\infty = \frac{8}{2} = 4$

Step 3: Add both solutions: $x(t) = C\,e^{-2t} + 8$

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14.3 Non-linear ODE: Back to Population Model

- The population model presented before was very simple. Let's complicate the model:

  1. If the population is small, growth is proportional to size.
  2. If the population is too large for its environment to support, it will decrease.

  Now, we have quantities: t = time, P = population, k = growth-rate coefficient for small populations, N = "carrying capacity."

- Let's restate 1. and 2. in terms of derivatives:
  1. dP/dt is approximately kP when P is "small."
  2. dP/dt is negative when $P > N$.

- Logistic Model (Pierre-François Verhulst):

$$\frac{dP}{dt} = k\left(1 - \frac{P}{N}\right)P$$

Pierre François Verhulst (1804 – 1849, Belgium)

14.3 Non-linear ODE: Back to Population Model

- Let's divide both sides of the equation by N:

$$\frac{d}{dt}\frac{P}{N} = k\left(1 - \frac{P}{N}\right)\frac{P}{N}$$

- Let $x(t) = P/N$ $\quad\Rightarrow x'(t) = k[1 - x(t)]\, x(t) = k\, x(t) - k\, x(t)^2$
- The logistic equation can be integrated and has a solution (the *logistic function*).

Solution: $\quad P(t) = \dfrac{N}{1 + CNe^{-kt}}; \qquad \lim_{t\to\infty} P(t) = N.$

where $C = 1/P(0) - 1/N,$ $\qquad$ with $P(0)$ = initial condition.

Note: Analytic solutions to non-linear ODEs are rare.

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14.4 Second-order Differential Equations

• A second-order ordinary differential equation is a differential equation of the form:

$$G(t, x(t),\ x'(t), x''(t)) = 0 \qquad \forall\, t,$$

involving only t, $x(t)$, and the first and second derivatives of x.

• We can write such an equation in the form:

$$x''(t) = F\big(t, x(t),\ x'(t)\big)$$

• Note that equations of the form $x''(t) = F\big(t, x(t),\ x'(t)\big)$ can be reduced to a first-order equation by making the substitution:

$$z(t) = x'(t).$$

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14.4 2nd Order ODE: Risk Aversion Application

- The function $\varrho(w) = -wu''(w)/u'(w)$ is the *Arrow-Pratt measure of relative risk aversion*, where $u(w)$ is the utility function for wealth w.
- Question: What $u(w)$ has a degree of risk-aversion that is independent of the level of wealth? Or, for what u do we have

$$a = -wu''(w)/u'(w) \text{ for all } w?$$

This is a 2nd-order ODE in which the term $u(w)$ does not appear. (The variable is w, rather than t.) It can be solved by 1st-order methods.

- Let $z(w) = u'(w) \quad \Rightarrow a = -wz'(w)/z(w)$ (a 1st-order ODE)

$\Rightarrow az(w) = -wz'(w)$ (a separable equation)

$\Rightarrow a \cdot z = -w\, dz/dw.$

$\Rightarrow a \cdot dw/w = -dz/z.$

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14.4 Second Order Differential Equations: Risk Aversion Application

$\Rightarrow a \cdot dw/w = -dz/dz.$

- *Solution*: $a \cdot \ln w = -\ln z(w) + C$, or

$z(w) = C^* w^{-a}$ (C*=exp(C))

- Now, $z(w) = u'(w)$, so to get u we need to integrate:

$$\Rightarrow u(w) = C^* \ln w + B \qquad \text{if } a = 1$$
$$= C^* w^{1-a}/(1-a) + B \qquad \text{if } a \neq 1$$

- That is, a utility function with a *constant degree of relative risk-aversion* (CRRA) equal to a takes this form.

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14.4 Linear 2nd-order ODE with constant coefficients: Finding a Solution

• Based on the solutions for first-order ODE, we guess that the homogeneous equation has a solution of the form $x(t) = Ae^{rt}$.

• Check: $x(t) = Ae^{rt}$

$$x'(t) = rAe^{rt}$$

$$x''(t) = r^2Ae^{rt},$$

$$\Rightarrow x''(t) + ax'(t) + bx(t) = r^2Ae^{rt} + arAe^{rt} + bAe^{rt} = 0$$

$$\Rightarrow \quad Ae^{rt}(r^2 + ar + b) = 0.$$

• For $x(t)$ to be a solution of the equation we need $r^2 + ar + b = 0$.

• This equation is the *characteristic equation* of the ODE.

• Similar to second-order difference equations, we have 3 cases:

- If $a^2 > 4b$ $\Rightarrow$ 2 distinct real roots
- If $a^2 = 4b$ $\Rightarrow$ 1 real root
- If $a^2 < 4b$ $\Rightarrow$ 2 distinct complex roots.

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14.4 Linear 2nd-order ODE with constant coefficients: Finding a Solution

• Case 1: If $a^2 > 4b$ $\Rightarrow$ Two distinct real roots: r and s.

$\Rightarrow x_1(t) = Ae^{rt}$ & $x_2(t) = Be^{st}$, for any values of A and B, are solutions.

$\Rightarrow$ also $x(t) = Ae^{rt} + Be^{st}$ is a solution. (It can be shown that *every* solution of the equation takes this form.)

• Case 2: If $a^2 = 4b$ $\Rightarrow$ One single real root: r

$\Rightarrow (A + Bt)e^{rt}$ is a solution ($r = -(1/2)a$ is the root).

• Case 3: If $a^2 < 4b$ $\Rightarrow$ Two complex roots: $r_j = \alpha \pm i\,\beta$ j=1,2.

$\Rightarrow x_1(t) = e^{(\alpha+i\beta)t}$ and $x_2(t) = e^{(\alpha-i\beta)t}$ ($\alpha=-a/2$, $\beta=\sqrt{(b-a^2)/4}$)

Use *Euler's formula* to eliminate complex numbers: $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.
Adding both solutions and after some algebra:

$$\Rightarrow x(t) = A\, e^{(\alpha+i\beta)t} + B\, e^{(\alpha-i\beta)t} = A\, e^{\alpha t}\cos(\beta t) + B\, e^{\alpha t}\sin(\beta t).$$

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14.4 Linear second-order equations with constant coefficients: Finding a Solution

Example 1: $x''(t) + x'(t) - 2x(t) = 0.$ ($a^2 > 4b = 1 > 4*(-2)=8$)
Characteristic equation: $r^2 + r - 2 = 0 \Rightarrow$ roots are 1 and -2.

Solution: $x(t) = Ae^t + Be^{-2t}$.

Example 2: $x''(t) + 6x'(t) + 9x(t) = 0.$ ($a^2 = 4b = 6^2 = 4*9$)
Characteristic equation: $r^2 + 6r + 9 = 0$ $\Rightarrow$ unique root is -3.

Solution: $x(t) = (A + Bt)e^{-3t}$.

Example 3: $x''(t) + 2x'(t) + 17x(t) = 0.$ ($a^2 < 4b = 4<4*(17)=68$)
Characteristic equation: $r^2 + 2r + 17 = 0$ $\Rightarrow$ roots are complex with $\alpha = -a/2 = -1$ and $\beta = \sqrt{(b - a^2/4)} = 4$.

Solution: $[A\cos(4t) + B\sin(4t)]e^{-t}$.

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14.4 Linear second-order equations with constant coefficients: Stability

- Consider the homogeneous equation $x''(t) + ax'(t) + bx(t) = 0$.
  If $b \neq 0$, there is a single equilibrium, namely 0 –i.e., the only constant function that is a solution is equal to 0 for all t.

- 3 cases:

- Characteristic equation with two real roots: r and s.
  Solution: $x(t) = Ae^{rt} + Be^{st}$ $\Rightarrow$ equilibrium is stable iff $r < 0$ and $s < 0$.
- Characteristic equation with one single real root: r
  Solution: $(A + Bt)e^{rt}$ $\Rightarrow$ equilibrium is stable iff $r < 0$.
- Characteristic equation with complex roots
  Solution: $(A\cos(\beta t) + B\sin(\beta t))e^{\alpha t}$, where $\alpha = -a/2$, the *real* part of each root. $\Rightarrow$ equilibrium is stable iff $\alpha<0$ (or $a>0$).

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14.4 Linear second-order equations with constant coefficients: Stability

• The real part of a real root is simply the root. We can combine the three cases:

The equilibrium is stable if and only if the real parts of both roots of the characteristic equation are negative. A bit of algebra shows that this condition is equivalent to $a > 0$ and $b > 0$.

• *Proposition*

An equilibrium of the homogeneous linear second-order differential equation $x''(t) + ax'(t) + bx(t) = 0$ is stable if and only if the real parts of both roots of the characteristic equation $r^2 + ar + b = 0$ are negative, or, equivalently, if and only if $a > 0$ and $b > 0$.

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14.4 Linear second-order equations with constant coefficients: Example

• *Stability of a macroeconomic model.*

- Let Q be aggregate supply, p be the price level, and π be the expected rate of inflation.
- $Q(t) = a - bp + c\pi$, where $a > 0$, $b > 0$, and $c > 0$.
  - Let be Q^* the long-run sustainable level of output.
  - Assume that prices adjust according to the equation:
    $p'(t) = h(Q(t) - Q^*) + \pi(t)$, where $h > 0$.
  - Finally, suppose that expectations are adaptive:
    $\pi'(t) = k(p'(t) - \pi(t))$ for some $k > 0$.

<u>Question</u>: Is this system stable?

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14.4 Linear second-order equations with constant coefficients: Example

Question: Is this system stable?

– Reduce the system to a second-order ODE:

1) Differentiate equation for $p'(t)$ $\Rightarrow$ get $p''(t)$

2) Substitute in for $\pi'(t)$ and $\pi(t)$.

– We obtain: $p''(t) - h(kc - b)\, p'(t) + khb\, p(t) = kh(a - Q^*)$

$\Rightarrow$ System is stable iff $kc < b$. ($khb > 0$ as required.)

Note:

If $c = 0$ –i.e., expectations are ignored– $\Rightarrow$ system is stable.

If $c \neq 0$ and k is large –inflation expectations respond rapidly to changes in the rate of inflation– $\Rightarrow$ system may be unstable.

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14.5 System of Equations: 1st-Order Linear ODE - Substitution

• Consider the 2x2 system of linear homogeneous differential equations (with constant coefficients)

$$x'(t) = ax(t) + by(t)$$
$$y'(t) = cx(t) + dy(t)$$

• We can solve this system using what we know:

1. Isolate $y(t)$ in the first equation $\Rightarrow y(t) = x'(t)/b - ax(t)/b$.
2. Differentiate this $y(t)$ equation $\Rightarrow y'(t) = x''(t)/b - ax'(t)/b$.
3. Substitute for $y(t)$ and $y'(t)$ in the second equations in our system:

$$x''(t)/b - ax'(t)/b = cx(t) + d[x'(t)/b - ax(t)/b],$$
$$\Rightarrow \quad x''(t) - (a + d)x'(t) + (ad - bc)x(t) = 0.$$

This is a linear second-order ODE in $x(t)$. We know how to solve it.

4. Go back to step 1. Solve for $y(t)$ in terms of $x'(t)$ and $x(t)$.

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14.5 System of Equations: 1st-Order Linear ODE - Substitution

- **Example**:

  $x'(t) = 2x(t) + y(t)$
  $y'(t) = -4x(t) - 3y(t).$

1. Isolate $y(t)$ in the first equation: $\Rightarrow y(t) = x'(t) - 2x(t).$
2. Differentiate in 1. $\Rightarrow y'(t) = x''(t) - 2x'(t).$
3. Substitute these expressions into the second equation:

   $x''(t) - 2x'(t) = -4x(t) - 3x'(t) + 6x(t)$, or
   $x''(t) + x'(t) - 2x(t) = 0.$

Solution:

$$x(t) = Ae^{t} + Be^{-2t}.$$

4. Using the expression $y(t) = x'(t) - 2x(t)$ we get

   $y(t) = Ae^{t} - 2Be^{-2t} - 2Ae^{t} - 2Be^{-2t} = -Ae^{t} - 4Be^{-2t}.$

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14.5 System of Equations: 1st-Order Linear ODE - Diagonalization

- Consider the 2x2 system of linear differential equations (with constant coefficients)

  $x'(t) = ax(t) + by(t) + m$
  $y'(t) = cx(t) + dy(t) + n$

- Let's rewrite the system using linear algebra:

$$z'(t) = \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} m \\ n \end{bmatrix} = Az(t) + \kappa$$

- Diagonalize the system (A must have independent eigenvectors):

  $H^{-1} z'(t) = H^{-1} A (H H^{-1}) z(t) + H^{-1} \kappa$
  $H^{-1} A H = \Lambda$
  $H^{-1} z(t) = u(t)$ and $H^{-1} \kappa = s$
  $u'(t) = \Lambda u(t) + s$ $\Rightarrow u'_1(t) = \lambda_1 u_1(t) + s_1$
  $u'_2(t) = \lambda_2 u_2(t) + s_2$

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14.5 System of Equations: First-Order Linear Differential Equations - Diagonalization

- Now, we have $u'(t) = \Lambda\, u(t) + s$

$$\Rightarrow u'_1(t) = \lambda_1\, u_1(t) + s_1$$
$$u'_2(t) = \lambda_2\, u_2(t) + s_2$$

- *Solution:*

$$u_1(t) = e^{-\lambda 1t}\, [u_1(0) - s_1/\lambda_1] + s_1/\lambda_1$$
$$u_2(t) = e^{-\lambda 2t}\, [u_2(0) - s_2\, /\lambda_2] + s_2/\lambda_2$$

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14.5 System of Equations: First-Order Linear Differential Equations – General Approach

- We start with an nxn system $\mathbf{z'}(t) = \mathbf{A}\,\mathbf{z}(t) + \mathbf{b}(t)$.
- First, we solve the homogenous system:

Theorem: Let $\mathbf{z'} = \mathbf{Az}$ be a homogeneous linear first-order system. If $\mathbf{z} = \mathbf{v}e^{\lambda t}$ is a solution to this system (where $\mathbf{v} = [v_1, v_2, ..., v_n]$), then λ is an eigenvalue of $\mathbf{A}$ and $\mathbf{v}$ is the corresponding eigenvector.

Proof: Start with $\mathbf{z} = \mathbf{v}e^{\lambda t}$ $\Rightarrow$ $\mathbf{z'} = \lambda\, \mathbf{v}e^{\lambda t}$
Substitute for $\mathbf{z}$ and $\mathbf{z'}$ in $\mathbf{z'} = \mathbf{Az}$, $\Rightarrow \lambda\, \mathbf{v}e^{\lambda t} = \mathbf{Av}e^{\lambda t}$
Divide $e^{\lambda t}$ both sides $\Rightarrow \lambda\, \mathbf{v} = \mathbf{Av}$ or $(\mathbf{A} - \lambda\, \mathbf{I})\mathbf{v} = 0$.

Thus, for a non-trivial solution, it must be that $|\mathbf{A} - \lambda\mathbf{I}| = 0$, which is the characteristic equation of matrix $\mathbf{A}$. Thus, λ is an eigenvalue of $\mathbf{A}$ and $\mathbf{v}$ is its associated eigenvector. ■

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14.5 System of Equations: First-Order Linear Differential Equations – General Approach

- **A** has n eigenvalues, $\lambda_1, .., \lambda_n$ and n eigenvectors, $\mathbf{v}_1, \mathbf{v}_2, .., \mathbf{v}_n$
  $\Rightarrow$ each term $\mathbf{v}_i e^{\lambda_i t}$ is a solution to **z'** = **Az**.

- Any linear combination of these terms are also solutions to **z'** = **Az**. Thus, the *general solution* to the homogeneous system **z'** = **Az** is:

$$\mathbf{z}(t) = \sum_{i=1}^{i=n} c_i \, \mathbf{v}_i e^{\lambda_i t}$$

where $c_1, .., c_n$ are arbitrary, possibly complex, constants.

- If the eigenvalues are not distinct, things get a bit complicated but nonetheless, as repeated roots are not robust, or "*structurally unstable*" −i.e., do not survive small changes in the coefficients of **A**−, then these can be generally ignored for practical purposes.

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14.5 System of Equations: First-Order Linear Differential Equations – General Approach

Example: $x'(t) = x(t) + 2\,y(t)$
$y'(t) = 3\,x(t) + 2\,y(t)$ $\quad x(0) = 0, \; y(0) = -4$

- Rewrite system:

$$z'(t) = \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = Az(t)$$

- Eigenvalue equation: $\lambda^2 - 3\lambda - 4 = 0$ $\quad \Rightarrow \lambda_1, \lambda_2 = (-1, 4)$
- Find Eigenvectors: $\lambda_1 = -1$ $\;\Rightarrow \mathbf{v}_1 = (v_{1,1}\, v_{1,2})$ $\; v_{1,1} = -v_{1,2}$
  Let $v_{1,2} = 1$ $\;\Rightarrow v_1 = (-1, 1)$
  $\lambda_2 = 4$ $\;\Rightarrow \mathbf{v}_2 = (v_{2,1}\, v_{2,2})$ $\; v_{2,1} = (2/3)\, v_{2,2}$
  Let $v_{2,2} = 3$ $\;\Rightarrow \mathbf{v}_2 = (2, 3)$
- *Solution*:

$$\mathbf{z}(t) = \sum_{i=1}^{i=n} c_i \, \mathbf{v}_i e^{\lambda_i t} = c_1 e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

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14.5 System of Equations: First-Order Linear Differential Equations – General Approach

• Find constants:

$$\mathbf{z}(0)=\begin{bmatrix}0\\-4\end{bmatrix}=c_1\begin{bmatrix}-1\\1\end{bmatrix}+c_2\begin{bmatrix}2\\3\end{bmatrix}$$

$\Rightarrow$ 2x2 system: $c_1 = -(8/5)$; $c_2 = -(4/5)$

• Definite solution:

$$\mathbf{z}(t)=-(8/5)e^{-t}\begin{bmatrix}-1\\1\end{bmatrix}-(4/5)e^{4t}\begin{bmatrix}2\\3\end{bmatrix}$$

$$\begin{bmatrix}x(t)\\y(t)\end{bmatrix}=\begin{bmatrix}(8/5)e^{-t}-(8/5)e^{4t}\\-(8/5)e^{-t}-(12/5)e^{4t}\end{bmatrix}$$

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14.5 System of Equations: First-Order Linear Differential Equations – Phase Plane

• In the single ODE we sketch the solution, *x(t)*, in the *x-t* plane. This will be difficult in this case since our solutions are actually vectors.

• Think of the solutions as points in the *x-y* plane. Plot the points. The *steady state* corresponds to (x_∞, y_∞). The *x-y* plane is called the *phase plane*.

• Phase diagrams are particularly useful for non-linear systems, where analytic solution may not possible. Phase diagrams provides qualitative information about the solution paths of nonlinear systems.

• For the linear case, plot points in the *x-y* plane when $z'(t) = 0$. Trajectories of $z(t)$ are easy to deduce from the parameters *a, b, c, and d*.

• For the non-linear case, we need to be more creative.

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14.5 System of Equations: First-Order Linear Differential Equations – Phase Plane

• First, we start with the non-linear system:

$x'(t) = f(x(t), y(t))$
$y'(t) = g(x(t), y(t))$

• Second, we establish the slopes of the singular curves by totally differentiating the singular curves:

$$f_x(x,y)dx + f_y(x,y)dy = 0$$
$$g_x(x,y)dx + g_y(x,y)dy = 0$$
$$\left.\frac{\partial y}{\partial x}\right|_{\dot{x}=0} = -\frac{f_x}{f_y} > 0 \text{ say} \quad \left.\frac{\partial y}{\partial x}\right|_{\dot{y}=0} = -\frac{g_x}{g_y} < 0 \text{ say}$$

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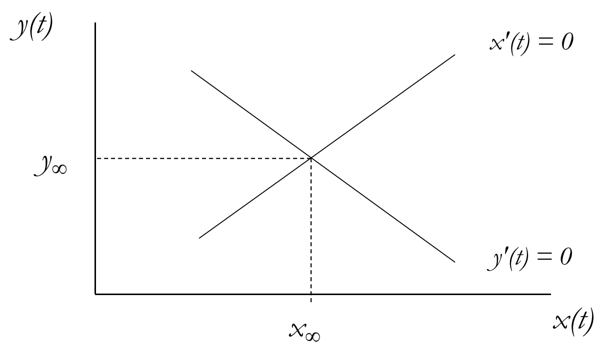
14.5 System of Equations: First-Order Linear Differential Equations – Phase Plane



• Now, establish the directions of motion. Suppose that

$x'(t) = f_x < 0$
$y'(t) = g_y < 0$

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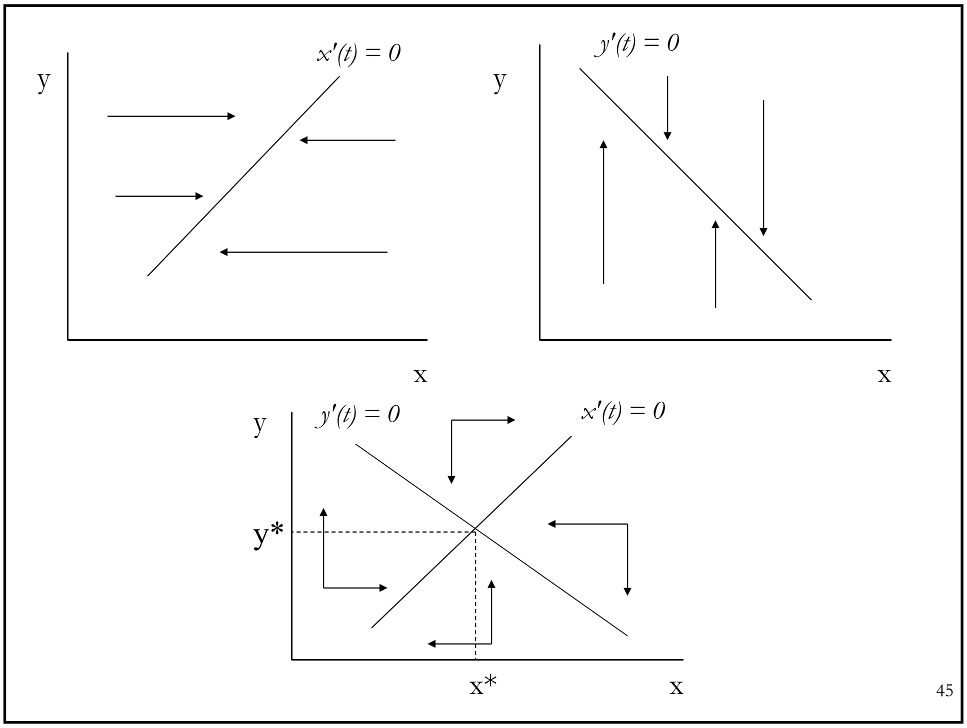
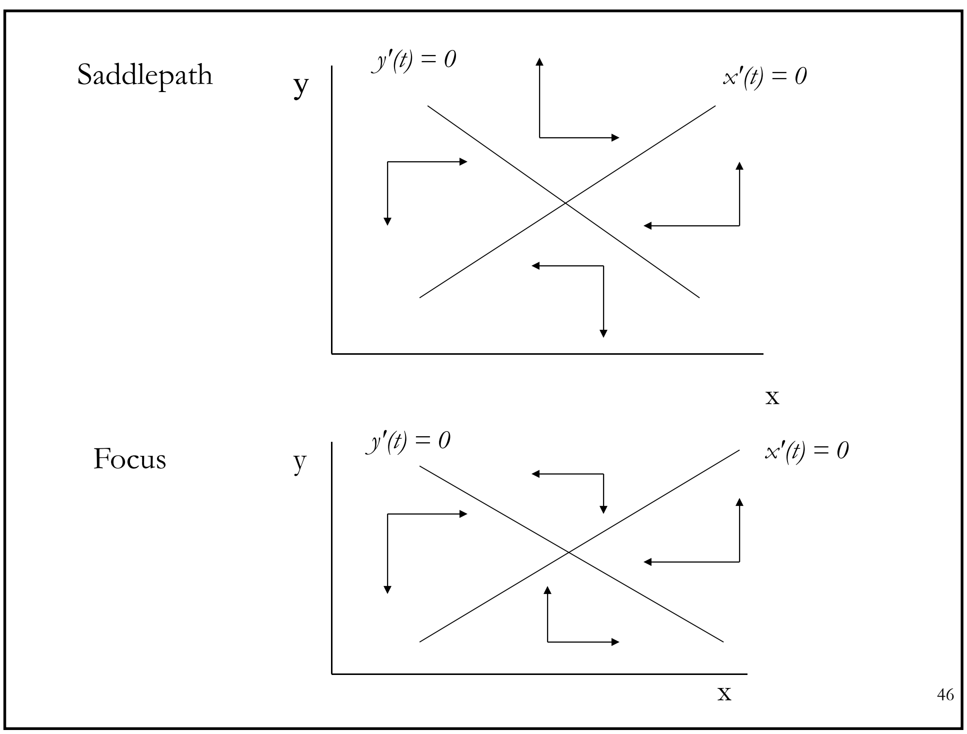


y
x'(t) = 0
x
y
y'(t) = 0
x
y
y'(t) = 0
x'(t) = 0
y*
x*
x
45



Saddlepath
y
y'(t) = 0
x'(t) = 0
x
Focus
y
y'(t) = 0
x'(t) = 0
x
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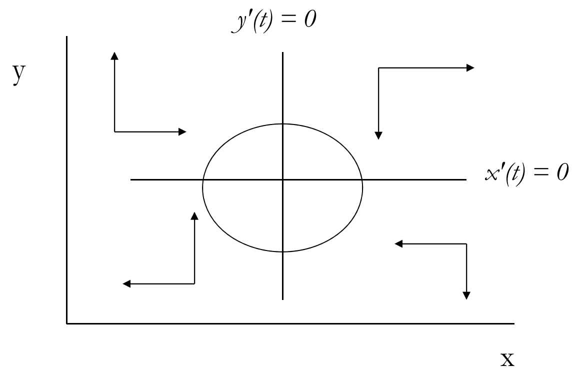
Limit Cycle



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14.5 System of Equations: First-Order Linear Differential Equations – Phase Plane

- **Example**:

$x'(t) = x(t) + 2\,y(t)$

$y'(t) = 3\,x(t) + 2\,y(t)$ $\qquad x(0) = 0,\ y(0) = -4 \quad (\Rightarrow \lambda_1, \lambda_2 = (-1, 4))$

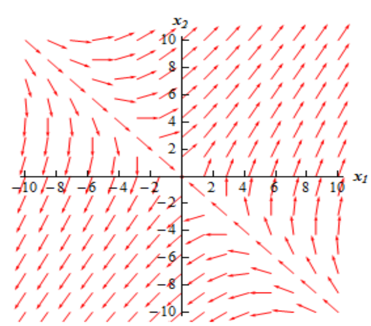
Plot some points in the *x-y* plane: (-2, 4); (1, 0); (2, -2); (-3, -1)

$$z'(t) = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

$$z'(t) = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$z'(t) = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

$$z'(t) = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \end{bmatrix} = \begin{bmatrix} -5 \\ -11 \end{bmatrix}$$



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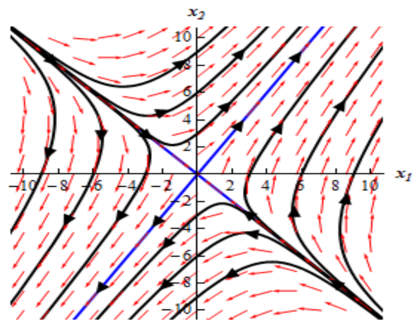
14.5 System of Equations: First-Order Linear Differential Equations – Phase Plane

• Plot the trajectories of the solutions in black and blue. In blue, the lines that follow the direction of the eigenvectors:



• With the exception of two trajectories, the trajectories in red move away from the equilibrium solution (0, 0).
• These equilibrium points are called *saddle point*, which is *unstable*.

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14.5 System of Equations: First-Order Linear Differential Equations – Stability

• The general solution of the homogeneous equation:

$$\mathbf{z}(t) = \sum_{i=1}^{i=n} c_i \, \mathbf{v}_i e^{\lambda_i t}$$

• The stability depends on the eigenvalues. Recall eigenvalue equation:

$$\lambda^2 - \text{tr}(\mathbf{A})\,\lambda + |\mathbf{A}| = 0$$

• Three cases:
• 1. $[\text{tr}(\mathbf{A})]^2 > 4|\mathbf{A}|$ $\Rightarrow$ 2 real distinct roots
- signs of λ_1, λ_2
1) $\lambda_1 < 0$, $\lambda_2 < 0$ if $\text{tr}(\mathbf{A})<0, |\mathbf{A}|>0$
2) $\lambda_1 > 0$, $\lambda_2 > 0$ if $\text{tr}(\mathbf{A})>0, |\mathbf{A}|>0$
3) $\lambda_i > 0$, $\lambda_j < 0$ if $|\mathbf{A}|<0$

• Under **Situation 1** ($\lambda_1 < 0$, $\lambda_2 < 0$), the system is globally stable. There is convergence towards (x_∞, y_∞), which is called a *tangent node*.

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14.5 System of Equations: First-Order Linear Differential Equations – Stability

• **Example**: $x'(t) = -5(t) + 1\, y(t)$
$y'(t) = 4\, x(t) - 2\, y(t) \qquad x(0)=1, y(0)=2$

Eigenvalue equation: $\lambda^2 - 7\,\lambda + 6 = 0 \qquad \Rightarrow \lambda_1, \lambda_2 = (-1, -6)$

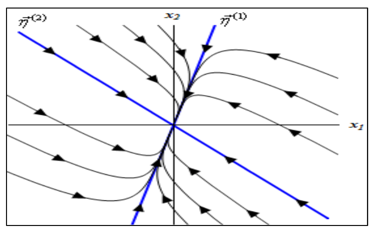
Eigenvectors:

$\lambda_1 = -6 \qquad \Rightarrow \mathbf{v}_1 = (v_{1,1}\, v_{1,2}) \quad v_{1,1} = -v_{1,2}$

Let $v_{1,2} = 1 \qquad \Rightarrow \mathbf{v}_1 = (1, -1)$

$\lambda_2 = -1 \qquad \Rightarrow \mathbf{v}_2 = (v_{2,1}\, v_{2,2}) \quad v_{2,1} = (1/4)v_{2,2}$

Let $v_{2,2} = 4 \qquad \Rightarrow \mathbf{v}_2 = (1, 4)$



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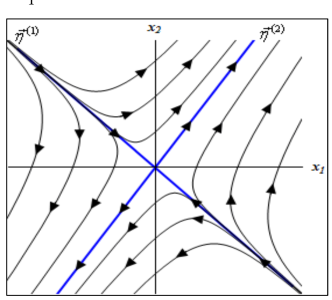
14.5 System of Equations: First-Order Linear Differential Equations – Stability

• Under **Situation 2** ($\lambda_1 > 0$, $\lambda_2 > 0$), the system is globally unstable. There is no convergence towards (x_∞, y_∞). A shock will move the system away from the tangent node, unless we are lucky and the system jumps to the new tangent node.

• Under **Situation 3** ($\lambda_i > 0$, $\lambda_j < 0$), the system is saddle path unstable. We need $C_i = 0$ when $\lambda_i > 0$.



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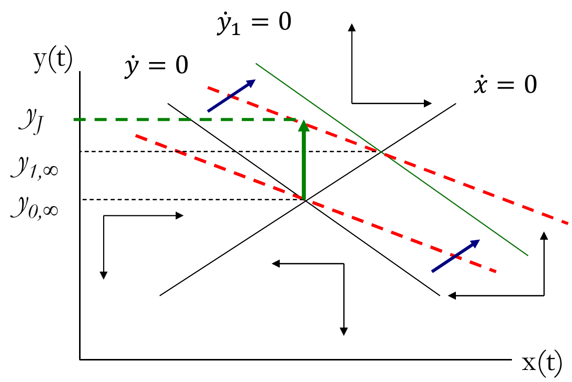
14.5 System of Equations: First-Order Linear Differential Equations – Stability - Application

• In economics, it is common to assume that the economy is stable. If a model determines an equilibrium with a saddle path, the saddle path trajectory is assumed. If the equilibrium is perturbed, the economy jumps to the new saddle path.

• This model displays "*overshooting*" in *y*(t). The economy jumps from $y_{0,\infty}$ to y_J immediately, then it converges to $y_{1,\infty}$.



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14.5 System of Equations: First-Order Linear Differential Equations – Stability

• 2. $[\text{tr}(\mathbf{A})]^2 = 4|\mathbf{A}|$ $\Rightarrow$ 1 real root, equal to $\lambda = \text{tr}(\mathbf{A})/2 = (a+d)/2$
System cannot be diagonalized (eigenvectors are the same!).

$$x(t) = C_1\, e^{\lambda t} + C_2\, t\, e^{\lambda t} + x_\infty$$
$$y(t) = [(\lambda - a)/b\, (C_1 + C_2\, t\,) + C_2/b]e^{\lambda t} + y_\infty$$

The stability of the system depends on λ. If $\lambda<0$, the system is globally stable.

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14.5 System of Equations: First-Order Linear Differential Equations – Stability

• 3. $[\text{tr}(\mathbf{A})]^2 > 4|\mathbf{A}|$ $\Rightarrow$ 2 complex roots $r_i = \lambda \pm i\mu$

Two solutions:

Similar to what we did for second-order DE, we can use Euler's formula to transform the $e^{i\lambda t}$ part and eliminate the complex part:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta).$$

Example: $x'(t) = 3x(t) - 9\,y(t)$

$y'(t) = 4\,x(t) - 3\,y(t)$ $\quad x(0) = 2, y(0) = -4$

Eigenvalue equation: $\lambda^2 + 27 = 0$ $\Rightarrow$ $\boldsymbol{\lambda_1, \lambda_2 = (3\sqrt{3}i, -3\sqrt{3}i)}$

Eigenvectors: $\boldsymbol{\lambda_1 = 3\sqrt{3}i}$ $\Rightarrow v_{1,2} = 1/3(1 - \sqrt{3}i)v_{1,1}$

Let $v_{1,1} = 3$ $\Rightarrow \mathbf{v}_1 = (1, (1 - \sqrt{3}i))$

$\boldsymbol{\lambda_2 = -1}$ $\Rightarrow \mathbf{v}_2 = (v_{2,1}, v_{2,2})$ $\quad v_{2,1} = (1/4)v_{2,2}$

Let $v_{2,2} = 4$ $\Rightarrow \mathbf{v}_2 = (1, 4)$

The solution from the first eigenvalue $\lambda_1 = 3\sqrt{3}i$: $\quad \mathbf{z}_1(t) = \mathbf{v}_1\, e^{3\sqrt{3}it}$

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14.5 System of Equations: First-Order Linear Differential Equations – Stability

• Using Euler's formula:

$$\mathbf{z}_1(t) = e^{3\sqrt{3}\,it}\begin{bmatrix} 3 \\ 1-\sqrt{3}\,i \end{bmatrix} = \cos(3\sqrt{3}\,t) + i\sin(3\sqrt{3}\,t)\begin{bmatrix} 3 \\ 1-\sqrt{3}\,i \end{bmatrix}$$

$$\mathbf{z}_1(t) = \begin{bmatrix} 3\cos(3\sqrt{3}\,t) \\ \cos(3\sqrt{3}\,t) + \sqrt{3}\sin(3\sqrt{3}\,t) \end{bmatrix} + i\begin{bmatrix} 3\sin(3\sqrt{3}\,t) \\ \sin(3\sqrt{3}\,t) - \sqrt{3}\cos(3\sqrt{3}\,t) \end{bmatrix} = \mathbf{u}(t) + i\mathbf{v}(t)$$

• It can be shown that both $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are independent solutions. We can use them to get a general solution to the homogeneous system:

$$\mathbf{z}(t) = c_1\mathbf{u}(t) + c_2\mathbf{v}(t)$$

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14.5 System of Equations: First-Order Difference Equations - Example

- Now, we have a system

$$x'(t) = 4\,x(t) + 5\,y(t) + 2$$
$$y'(t) = 5\,x(t) + 4\,y(t) + 4$$

- Let's rewrite the system using linear algebra.

$$z'(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

- Eigenvalue equation: $\lambda^2 - 8\lambda - 9 = 0 \quad \Rightarrow \lambda_1, \lambda_2 = (9,-1)$

$u'_1(t) = 9\,u_1(t) + s_1$ (unstable equation)

$u'_2(t) = -1\,u_2(t) + s_2$ (stable equation)

- *Solution:*

$$u_1(t) = e^{9t}\,[u_1(0) - s_1/9] + s_1/9$$
$$u_2(t) = e^{-t}\,[u_2(0) - s_2/(-1)] + s_2/(-1)$$

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14.5 System of Equations: First-Order Difference Equations - Example

- Use the eigenvector matrix, H, to transform the system:

$$H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}; \quad H^{-1} = \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}(-1/2)$$

$$s = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = H^{-1}\kappa = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$z(t) = Hu(t) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} u_1(t) + u_2(t) \\ u_1(t) - u_2(t) \end{bmatrix}$$

$$\begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} [e^{9t}(u_1(0) - \frac{3}{9}) + \frac{3}{9}] + [e^{-t}(u_2(0) - 1) - 1] \\ [e^{9t}(u_1(0) - \frac{3}{9}) + \frac{3}{9}] - [e^{-t}(u_2(0) - 1) - 1] \end{bmatrix}$$

- We need $[x(0), y(0)] = (x_0, y_0)$ to obtain $u_1(0)$ and $u_2(0)$.

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14.6 Analytical Solutions

• A function y is called a *solution in the extended sense* of the differential equation $y'(t) = f(t,y)$ with $y(t_0) = y_0$ if y is absolutely continuous, y satisfies the differential equation a. e. and y satisfies the initial condition.

• **Theorem**: Carathéodory's existence theorem
Consider the differential equation $y'(t) = f(t, y), y(t_0) = y_0$,
with $f(t, y)$ defined on the rectangular domain

$$R = \{(t, y) \mid |t - t_0| \leq a, |f(t, y)| \leq m(t)\}$$

If the function $f(t,y)$ satisfies the following three conditions:
- $f(t, y)$ is continuous in y for each fixed t,
- $f(t, y)$ is measurable in t for each fixed y,
- there is an L-integrable function $m(t)$, $|t - t_0| \leq a$, such that
$$|f(t,y)| \leq m(t) \text{ for all } (t, y) \in R,$$

then, the differential equation has a solution in the extended sense in a neighborhood of the initial condition.

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14.6 Analytical Solutions

• The Carathéodory's existence theorem states than an ODE has a solution, under some mild conditions.

• It is a generalization of the *Peano's existence theorem*, which requires the right hand side of the first-order ODE to be continuous. Peano's theorem also applies to higher dimensions, when the domain of $f(.)$ is an open subset of RxR^n.

• These theorems are general, imposing mild restrictions on $f(.)$. The *Picard–Lindelöf theorem* (or *Cauchy–Lipschitz theorem*) establishes conditions for the existence of a uniqueness of solutions to first-order equations with given initial conditions. Under this theorem, $f(.)$ is Lipschitz continuous (with bounded derivatives) in y and continuous in t.

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14.6 Numerical Solutions

• As the previous theorems show, under mild conditions, an ODE has a solution, though it may not be easy to find it. For these cases, we have to satisfy ourselves with an approximation to the solution.

• *Numerical ordinary differential equations* is the part of numerical analysis which studies the numerical solution of ODE. This field is also known under the name *numerical integration,* but some people reserve this term for the computation of integrals.

• There are several algorithms to compute an approximate solution to an ODE.

• A simple method is to use techniques from calculus to obtain a series expansion of the solution. An example is the *Taylor Series Method.*

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14.6 Numerical Solutions

• We focus on solving a first degree ODE, with a boundary condition. That is, we will be given an ODE with the derivative a function of the dependent and independent variable and an initial condition (point):

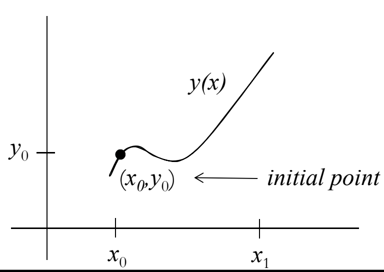
$$\frac{dy}{dx} = f(x, y) \quad and \quad y(x_0) = y_0$$

• The solution *y(x)* can be pictured graphically. The point (x_0, y_0) must be on the graph. The function *y(x)* would also satisfy the differential equation if you plugged *y(x)* in for *y*:

$$y'(x) = f(x, y(x))$$

• Now, given x_1, we want to find y_1.

Problem: y_1 can only be estimated.



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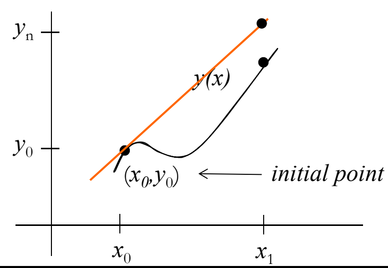
14.6 Numerical Solutions

• Simple (Euler's) idea: follow the tangent! That is, use the usual discrete estimation of the slope to approximate y_1 (a 1st-order Taylor expansion):

$$\Delta y = f'(x, y)\Delta x$$

$$y_1 \approx y_0 + f'(x_0, y_0)(x_1 - x_0)$$

• Depending on the curvature of $f(.)$ and how far x_1 is from x_0, this approximation may not work well. We can do better.



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14.6 Numerical Solutions: Taylor Series Method

• The Taylor series method is a simple adaptation of classic calculus to develop the solution as an infinite series. The method is not strictly a numerical method but it is used in conjunction with numerical schemes.

• *Problem*: Computers usually cannot be programmed to construct the terms and the order of the expansion is *a priori* unknown.

• From the Taylor series expansion:

$$y(x) = y(x_0) + \Delta h\, y'(x_0) + \frac{\Delta h^2}{2!} y''(x_0) + \frac{\Delta h^3}{3!} y'''(x_0) + \frac{\Delta h^4}{4!} y^{IV}(x_0) + \cdots$$

The step size is defined as: $\Delta h = x - x_0$

• Using the ODE to get all the derivatives and the initial conditions, a solution to the ODE can be approximated.

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14.6 Numerical Solutions: Taylor Series Method

• There are two errors in numerical methods: truncation error (practitioner related, from the discretization process) and rounding error (computer related).

• The truncation error is estimated using the remainder in Taylor's theorem. For example, if we decide to truncate at n, then:

$$\text{error} = \frac{(\Delta h)^n}{n+1!}\, y^{(\mathrm{n})}(\xi), \quad 0 < \xi < \Delta h$$

• This error is a *local error*, it occurs at each point. The accumulation of local errors is the *global error,* more difficult to compute.

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14.6 Numerical Solutions: Taylor Series Method

Example: ODE $y'(x) = x + y$ $x = 0,\ y_0 = 1,$

Analytical solution: $y(x) = 2\,e^x - x - 1$

• We are interested in $y(1)$ (exact solution: 2*exp(1) – 1 – 1 = **3.43656**)

Let's try to approximate $y(x)$ using a Taylor series expansion.

- First, we need the j$^{\text{th}}$-order derivatives for j=1, 2, 3, …

$$\begin{aligned} y'(x) &= x + y(x) & \Rightarrow\ y'(0) &= x + y(0) = 0 + 1 = 1 \\ y''(x) &= 1 + y'(x) & y''(0) &= 1 + y'(0) = 1 + 1 = 2 \\ y'''(x) &= y''(x) & y'''(0) &= y''(0) = 2 \\ y^{(4)}(x) &= y'''(x) & y^{(4)}(0) &= y'''(0) = 2 \end{aligned}$$

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14.6 Numerical Solutions: Taylor Series Method

- Second, replace in the Taylor series expansion

$$y(x) = y(x_0) + \Delta h\, y'(x_0) + \frac{\Delta h^2}{2!} y''(x_0) + \frac{\Delta h^3}{3!} y'''(x_0) + \frac{\Delta h^4}{4!} y^{(4)}(x_0) + \ldots$$

Note: The Taylor series is a function of x_0 and Δh. Plug in the initial conditions (n=4):

$$y(\Delta h) = 1 + \Delta h\,(1) + \frac{\Delta h^2}{2!}(2) + \frac{\Delta h^3}{3!}(2) + \frac{\Delta h^4}{4!}(2) + \text{ Error}$$

Resulting in the equation:

$$y(\Delta h) = 1 + \Delta h + \Delta h^2 + \frac{\Delta h^3}{3} + \frac{\Delta h^4}{12} + \text{ Error}$$

• Then,

$y(1) = 1 + 1 + 1^2 + 1^3/3 + 1^4/12 = 3.41667$ (< **3.43656**)

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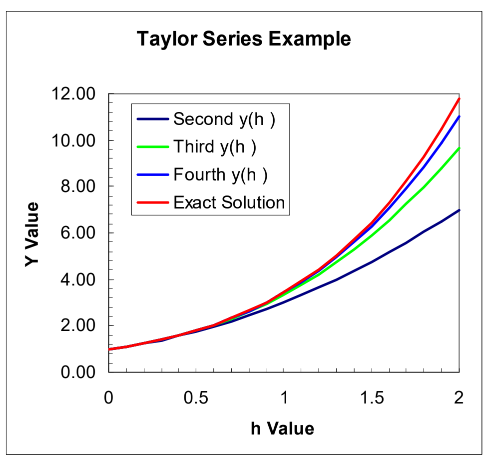
14.6 Numerical Solutions: Taylor Series Method

• The results (x=0)

Δh	Second y(Δh)	Third y(Δh)	Fourth y(Δh)	Exact Solution
0	1.00000	1.00000	1.00000	1.00000
0.1	1.11000	1.11033	1.11034	1.11034
0.2	1.24000	1.24267	1.24280	1.24281
0.3	1.39000	1.39900	1.39968	1.39972
0.4	1.56000	1.58133	1.58347	1.58365
0.5	1.75000	1.79167	1.79688	1.79744
0.6	1.96000	2.03200	2.04280	2.04424
0.7	2.19000	2.30433	2.32434	2.32751
0.8	2.44000	2.61067	2.64480	2.65108
0.9	2.71000	2.95300	3.00768	3.01921
1	3.00000	3.33333	3.41667	**3.43656**
1.1	3.31000	3.75367	3.87568	3.90833
1.2	3.64000	4.21600	4.38880	4.44023
1.3	3.99000	4.72233	4.96034	5.03859
1.4	4.36000	5.27467	5.59480	5.71040
1.5	4.75000	5.87500	6.29688	6.46338
1.6	5.16000	6.52533	7.07147	7.30606
1.7	5.59000	7.22767	7.92368	8.24789
1.8	6.04000	7.98400	8.85880	9.29929
1.9	6.51000	8.79633	9.88234	10.47179
2	7.00000	9.66667	11.00000	11.77811



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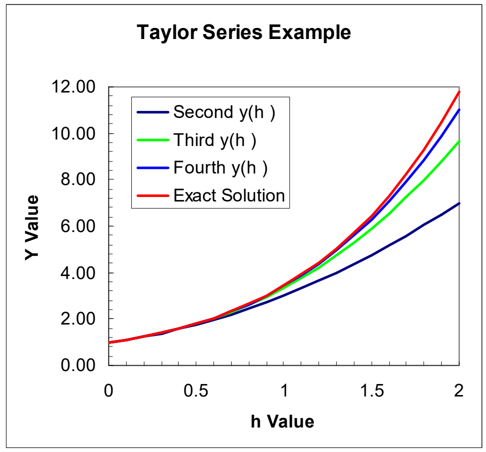
14.6 Numerical Solutions: Taylor Series Method

Note that the last set of terms, we start to lose accuracy for the 4th order with big Δh:

$$\text{Error} = \frac{\Delta h^5}{5!} y^{(5)}(\xi), \quad 0 < \xi < \Delta h$$

Difficult to estimate. All we know is that it is in the range of $0 < \xi < \Delta h$.



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14.6 Numerical Solutions: Taylor Series Method

• Numerical analysis is an art. The number of terms, we chose is a mater of judgment and experience.

• We usually truncate the Taylor series, when the contribution of the last term is negligible to the number of decimal places to which we are working.

• Things can get complicated for higher-order ODE.

• Example: $y''(x) = 3 + x - y^2, \quad y(0) = 1, \; y'(0) = -2$

$$y''' = 1 - 2yy'$$

$$y^{\text{IV}} = -2yy'' - 2y'^2$$

$$y^{\text{V}} = -2y'y'' - 2yy''' - 4y'y'' = -6y'y'' - 2yy'''$$

• The higher order terms can be calculated from previous values and they are difficult to calculate. *Euler method* can be used in these cases.

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14.6 Numerical Solutions: Euler Method

• One feature of the Taylor series method is that the error is small when Δh is small and only a few terms are need for good accuracy.

• The Euler method may be thought of an extreme of the idea for a Taylor series having a small error when Δh is extremely small. The Euler method is a 1st-order Taylor series with each step having an upgrade of the derivative and y term changed:

$$y(x + \Delta h) = y(x_0) + \Delta h\ y'(x_0) + \text{Error}$$

$$\text{Error} = \frac{\Delta h^2}{2!} y''(\xi), \qquad x_0 < \xi < x_0 + \Delta h$$

• The Euler method's algorithm upgrades the coefficients in each time step:

$$y_{n+1} = y_n + \Delta h\ y_n' + O(\Delta h^2) \text{ error}$$

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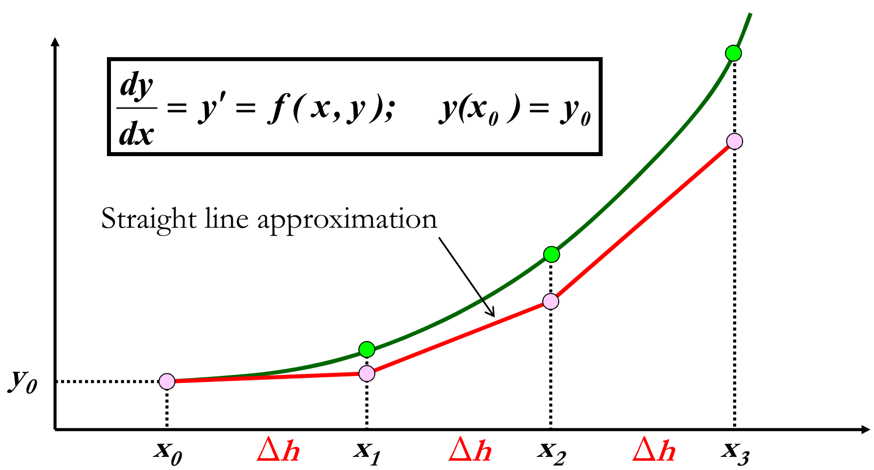
14.6 Numerical Solutions: Euler Method

• The first derivative and the initial y values are updated for each iteration.



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14.6 Numerical Solutions: Euler Method

• Consider: $y'(x) = x + y$

The initial condition is: $y(0) = 1$

The step size is: $\Delta h = .02$

The analytical solution is: $y(x) = 2\,e^{x} - x - 1$

• The algorithm has a loop using the initial conditions (x= 0; $y(0)$=1) and definition of the derivative: $y_i'(x) = [\,y_{i+1}(x) - y_i\,]/\ \Delta h$

Loop:

The derivative is calculated as: $y_i'(x) = y_i + x_i$

The next y value is calculated: $y_{i+1}(x) = y_i + \Delta h\, y_i'(x)$

Take the next step: $x_{i+1} = x_i + \Delta h$

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14.6 Numerical Solutions: Euler Method

• First iterations: $y_i'(x) = y_i + x_i$ & $y_{i+1}(x) = y_i + \Delta h\, y_i'(x)$

- #1: $x(1) = 0$; $y(0) = 1$; $y'(0) = 1 + 0 = 1$

$\Rightarrow y(1) = 1 + 0.02*1 =$ **1.02** & *error* = **1.02** – 1.020403 = –.020403

exact solution: $y(x{=}.02) = 2e^{.02} - .02 - 1 = 1.020403$

- #2: $x(2) = 0.02$; $y(1) =$ **1.02**; $y'(1) =$ **1.02** + .02 = **1.04**

$\Rightarrow y(2) =$ ***1.02*** + *0.02**___***1.04***___ = ***1.0408*** & *error* = – .00082

- #3: $x(3) = 0.04$; $y(2) =$ **1.0408**; $y'(2) =$ **1.0408** + .04 = **1.0808**

$\Rightarrow y(3) =$ ***1.0408*** + *0.02*1.0808 = 1.062416* & *error* = – .00126

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14.6 Numerical Solutions: Euler Method

• Code in R and results

```
dif_sol <- function(N,x0,y0,dh){
  Z <- matrix(0, N, 4)
  Z[1,1] <- x0                              #initialize x
  Z[1,2] <- y0                              #initialize y
  Z[1,3] <- y0 + x0                         #initialize derivative
  Z[1,4] <- 2*exp(Z[1,1]) + Z[1, 1] -1      #exact solution
 for (i in 2:N) {
     Z[i, 1] <- Z[i-1, 1] + dh
     Z[i, 2] <- Z[i-1, 2] + dh * Z[i-1, 3]
     Z[i, 3] <- Z[i, 1] + Z[i, 2]
     Z[i, 4] <- 2*exp(Z[i,1]) - Z[i, 1] - 1
}
return(Z)
}
```

```
xy <- dif_sol(51,0,1,0.02)
xy
      [,1]     [,2]     [,3]     [,4]
 [1,] 0.00 1.000000 1.000000 1.000000
 [2,] 0.02 1.020000 1.040000 1.020403
 [3,] 0.04 1.040800 1.080800 1.041622
 [4,] 0.06 1.062416 1.122416 1.063673
 [5,] 0.08 1.084864 1.164864 1.086574
 [6,] 0.10 1.108162 1.208162 1.110342
 [7,] 0.12 1.132325 1.252325 1.134994
 [8,] 0.14 1.157371 1.297371 1.160548
 [9,] 0.16 1.183319 1.343319 1.187022
[10,] 0.18 1.210185 1.390185 1.214435

[50,] 0.98  3.297624  4.27762  3.348912
[51,] 1.00  3.383176  4.38317  3.436564
```

$y(1)$ = 3.297624 + .02* (4.27762) = 3.383176

error(1)= 3.436564 - 3.383176 = 0.053388

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14.6 Numerical Solutions: Euler Method

• Recall exact solution: $y(1) = 2*\exp(1) – 1 – 1= 3.43656$

• With $\Delta h=.02$ $(N=50)$

$\Rightarrow$ $y(1) = 3.297624 + .02* (4.27762) = 3.383176$

• With $\Delta h=.01$ $(N=100)$

$\Rightarrow$ $y(1) = 3.366067 + .02* (4.356067) = 3.409628$

• With $\Delta h=.005$ $(N=200)$

$\Rightarrow$ $y(1) = 3.401054 + .02* (4.396054) = 3.423034$

Remark: As Δh gets smaller, we get a lower error.

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14.6 Numerical Solutions: Euler Method

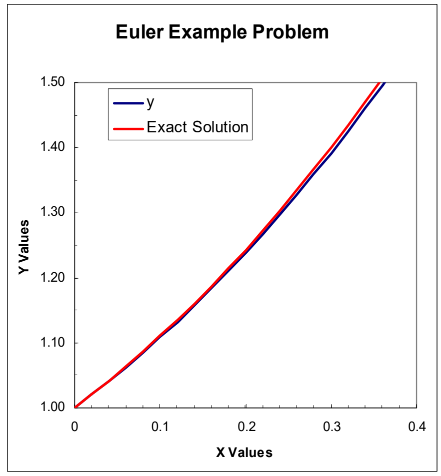
• Compare the error at $y(0.1)$ with a $\Delta h=0.02$

Error $= \mathbf{1.1103} - \mathbf{1.1081}$
$= 0.0022$

If we want the error to be smaller than 0.0001

$$\text{Reduction} \quad = \frac{0.0022}{0.0001} = 22$$

We need to reduce the step size by 22 to get the desired error.



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14.6 Numerical Solutions: Euler Method - Notes

• The trouble with this method is

– Small step size to get good accuracy.

– Numerical unstable for *stiff equations* –i.e., diff. equations where numerical solutions only work well for very small step sizes.

Example: $y'(x) = -2\,y$, $y(0) = 1$, & $\Delta h = 1$.

• Euler method only uses the previously computed value y_n to determine y_{n+1}. This can be generalized to include more past values. These methods are called *multi-steps.*

Note: For the simple Euler method, we use the slope at the beginning of the interval y_n', to determine the increment to the function, but this is always wrong. One way to reduce this error is to evaluate the derivative at the midpoint of the interval.

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14.6 Numerical Solutions: Midpoint Method

• We want to calculate the slope, y'_{i+1}, not at beginning of the interval (x_i, y_i), but at midpoint $(x_{i+\Delta h/2}, y_{i+\Delta h/2})$. But, we do not know $y'_{i+\Delta h/2}$ at that point, since we need $(x_{i+\Delta h/2}, y_{i+\Delta h/2})$ to calculate it.

• But, we can approximate the value of at midpoint, $y_{i+\Delta h/2}$, as usual:

$$y_{i+\Delta h/2} = y_i + y_i'(x)\, \Delta h/2.$$

• Then, we use this approximation to compute the slope at midpoint. Using the previous example, $y_i'(x) = y_i + x_i$, we find:

$$y'_{i+\Delta h/2} = y_{i+\Delta h/2} + x_{i+\Delta h/2} = [y_i + y_i'(x)\, \Delta h/2] + [x_i + \Delta h/2].$$

• Finally, we use this approximation to calculate y_{i+1}:

$$y_{i+1} = y_i + y'_{i+\Delta h/2}\, \Delta h.$$

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14.6 Numerical Solutions: Midpoint Method

• Code in R and results

```
dif_sol <- function(N,x0,y0,dh){
  Z <- matrix(0, N, 4)
  Z[1,1] <- x0                          #initialize x
  Z[1,2] <- y0                          #initialize y
  Z[1,3] <- y0 + x0                     #initialize derivative
  Z[1,4] <- 2*exp(Z[1,1]) + Z[1, 1] -1           #exact solution
 for (i in 2:N) {
     Z[i, 1] <- Z[i-1, 1] + dh
     Z[i, 3] <- (Z[i-1, 3]*dh/2 + Z[i-1, 2]) + (Z[i-1, 1] + dh/2)
     Z[i, 2] <- Z[i-1, 2] + dh*Z[i, 3]
     Z[i, 4] <- 2*exp(Z[i,1]) - Z[i, 1] -1
}
return(Z)
}
```

```
xy <- dif_sol(51,0,1,0.02)
xy
      [,1]     [,2]     [,3]     [,4]
 [1,] 0.00 1.000000 1.000000 1.000000
 [2,] 0.02 1.020400 1.020000 1.020403
 [3,] 0.04 1.041612 1.060600 1.041622
 [4,] 0.06 1.063656 1.102218 1.063673
 [5,] 0.08 1.086550 1.144679 1.086574
 [6,] 0.10 1.110310 1.187997 1.110342
 [7,] 0.12 1.134954 1.232190 1.134994
 [8,] 0.14 1.160499 1.277276 1.160548
 [9,] 0.16 1.186965 1.323272 1.187022
[10,] 0.18 1.214369 1.370197 1.214435

[50,] 0.98 3.348063 4.274276 3.348912
[51,] 1.00 3.435680 4.380806 3.436564
```

$y(1)$ = **3.348063** + .02* (**4.274276**) = **3.435680**

error(1)= **3.436564** - **3.435680** = 0.000884

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14.6 Numerical Solutions: Midpoint Methods

• We have presented two simple methods within a simple example. But, there are more advanced methods, which are more complex to derive, but are based on the ideas we have introduced.

• The standard workhorses for solving ODEs is the called the *Runge-Kutta method*. This method is simply a higher order approximation to the midpoint method.

• Instead of relying to the midpoint to estimating the derivative, we can do better, by using more points in the interval to calculate an average.

• This is what the (2nd-Order) Runge-Kutta method does: It takes four steps (one quarter of the interval, the midpoint, etc.) to estimate the derivative.

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Extra
Introduction to Stochastic Processes and Calculus

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Preliminaries: Sigma-algebra

Definition: A *sigma-algebra* F is a set of subsets ω of Ω s.t.:

- $\Phi \in F$.
- If $\omega \in F$, then $\omega^c \in F$.
- If $\omega_1, \omega_2, \ldots, \omega_n, \ldots \in F$, then $\text{U}_{(I >= 1)}\ \omega_i \in F$.

(A σ-algebra is a mathematical model of a state of partial knowledge about an outcome of a "probability experiment").

– The set (Ω, F) is called a *measurable space*.

- There may be certain elements in Ω that are not in F.

– A *filtration* is an increasing sequence of *σ*-algebras on a measurable space. Usually, filtratrions are used to form conditional expectations.

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Preliminaries: Probability Measure

Definition: Probability measure

A *probability measure* is the triplet (Ω, F, P) where P: $F \rightarrow [0,1]$ is a function from F to [0,1].

- P(Ø) = 0 and P(Ω) = 1 always.
- The elements in Ω that are not in F have no probability.
  –We can extend the probability definition by assigning a probability of zero to such elements.

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Preliminaries: Stochastic Process

Definition: Random variable *x* (or X) w.r.t. (Ω, *F*, P)

– $x : F \rightarrow R^n$ is a measurable function (*i.e.* $x^{-1}(z) \in F$ for all z in R^n).

– Hence, P: $F \rightarrow [0,1]$ is translated to an equivalent function

$\mu_x : R^n \rightarrow [0,1]$, which is the distribution of *x*.

Definition: Stochastic Process X(t, ω)

A *stochastic process* is a parameterized collection of random variables *x(t)*, or X(t, ω) = $\{x(t)\}_{t.}$

– Normally, *t* is taken as time.

– Think of ω as one outcome from a set of possible outcomes of an experiment. Then, X(t, ω) is the state of an outcome ω of the experiment at time t.

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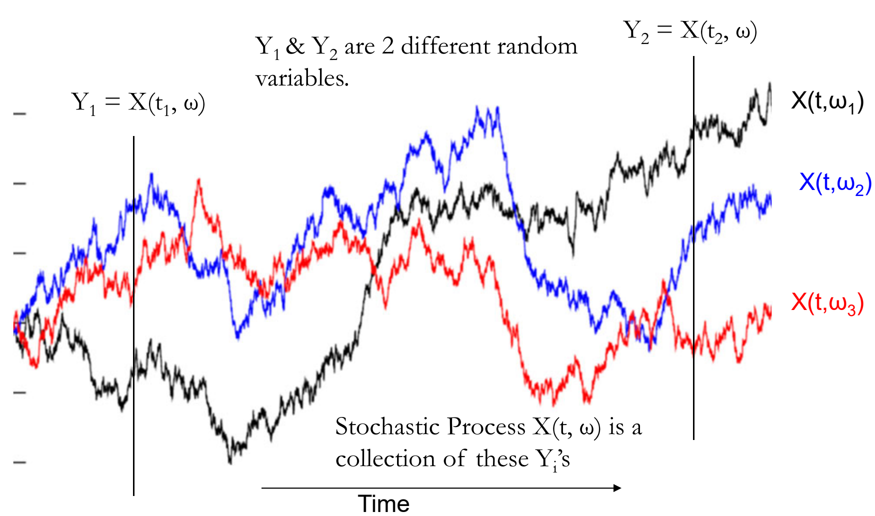
Stochastic Process - Illustration



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Stochastic Process: Brownian motion (or Wiener process)

• Long history: In 1827 the botanist Robert Brown observed that grains of pollen suspended in water have a continuous jittery, erratic movement, now known as *Brownian motion (BM), B_t*.

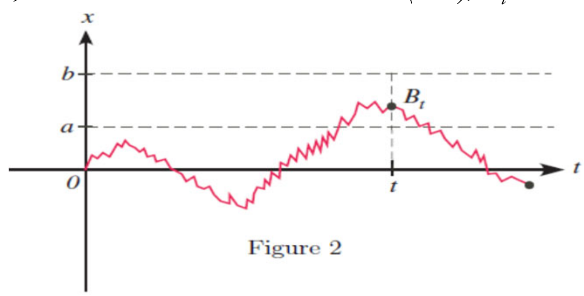

Figure 2

• We think of Brownian motion (also called *Wiener process*) as a model of *random continuous motion*.

Robert Brown (1773–1858, Scotland)

Stochastic Process: Brownian motion – Normal

• Einstein (1905) show that that the probability of the pollen to be in an interval [*a; b*] at time t is given by:

$$P(a \leq B_t \leq b) = \frac{1}{\sqrt{2\pi t}} \int_a^b e^{-\frac{x^2}{2t}} dx$$

• Note: Einstein derived the pdf: $f(x,t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$

as a solution to the diffusion equation: $\frac{\partial f}{\partial t} = \frac{1}{2} \frac{\partial^2 f}{\partial x^2}$

• That is, a BM has increments driven by the standard normal distribution.

Typical notation: W_t, B_t; W(t), B(t); z(t).

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Stochastic Process: Brownian motion - Definition

• **Definition. Brownian motion (or Wiener process)**

A BM$\{W(t)\}$ is a family of RVs $W(t)$: $\Omega \rightarrow R$, where Ω is a probability space, satisfying the following properties:

1. $W(0) = 0$. (A convenient assumption; can be relaxed.)
2. *Continuous path*. The function $t \rightarrow W(t)$ is a *continuous function* of t.
3. *Stationary increments*. $W(t) - W(s) \sim N(0, t-s)$, where $t>s$.
4. *Independent increments*. If $s < t$, the random variable $W(t) - W(s)$ is independent of the values $W(r)$ for $r \leq s$.

• Note: The sample paths are continuous, but they are nowhere differentiable since increments are random ("*normally distributed*"). Brownian motions are a special case of *Lévy processes,* which can be discountinuous.

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Stochastic Process: A few considerations

• A stochastic process is a function of a continuous variable (most often: time).

• The question now becomes how to determine the continuity and differentiability of a stochastic process?

– It is not simple as a stochastic process is not deterministic.

• We use the same definitions of continuity, but now look at the expectations and probabilities.

– A deterministic function f(t) is continuous if:

–$\| f(t_1) - f(t_2) \| \leq \delta \, \| t_1 - t_2 \|$.

– To determine if a stochastic process X(t,ω) is continuous, we need to determine:

–$P(\| X(t_1, \omega) - X(t_2, \omega) \|) \leq \delta \, \| t_1 - t_2 \|$ or

$E(\| X(t_1, \omega) - X(t_2, \omega) \|) \leq \delta \, \| t_1 - t_2 \|$

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Stochastic Process: Kolomogorov Continuity Theorem

• If for all T > 0, there exist a, b, $\delta > 0$ such that:

$E(|X(t_1, \omega) - X(t_2, \omega)|^a) \leq \delta \, |t_1 - t_2|^{(1 + b)}$

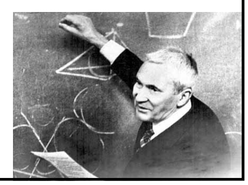
Then $X(t, \omega)$ can be considered as a continuous stochastic process.

• Summary:

– BM is a continuous stochastic process.

– BM (Wiener process): $W(t, \omega)$ is almost surely continuous, has independent normal distributed ($N(0, t - s)$) increments and $W(t=0, \omega) = 0$.

– The limit of random walks. Informally, we say "continuous random walk (motion)."



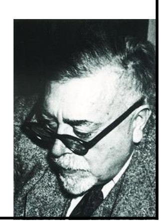
Andrey Kolmogorov (1903-1987, Russia/USSR)

Stochastic Process: *W(t)* – Drift and Variance

• A stochastic process $W(t)$ is called a (one-dimensional) Brownian motion (*generalized Wiener process*) with *drift m* and *variance* (parameter) σ^2 starting at the origin if it satifies the following:

- $W(t=0)=0$.
- For $s<t$, the distribution of $\Delta W = W(t) - W(s) \sim N(m(t-s), \sigma^2\,(t-s))$.
- The values of ΔW for any 2 different (non-overlapping) periods of time are independent.
- With probability 1, the function $t \to W(t)$ is a continuous function of t.

If $m = 0$; $\sigma^2 = 1$, then $W(t)$ is called a *standard BM* or, just, a *Wiener process*.



Norbert Wiener (1894 – 1964, USA)

Stochastic Process: $W(t)$ – Drift and Variance

• A property of the normal distribution is invariance under addition:

If Z ∼ N ⇒ Y = σ Z + μ ∼ N.

In particular, if Z ∼ N(0,1) ⇒ Y ∼ N(μ ; σ^2).

• Then, if $W(t)$ is a standard BM and Y(t) = σ $W(t)$ + μ , then, Y(t) is a BM with drift m and variance σ^2.

• In finance and economics, a Brownian motion is used to describe the continuous process behind the change in value of financial assets. For example, stocks or bonds.

For example, we say that IBM returns, Y(t), follow a BM with drift 10% and variance $(15\%)^2$.

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Stochastic Process: $W(t)$ - Properties

• Properties of a BM:

– E[ΔW] = 0 & Var[ΔW] = Δt (standard deviation is √Δt).

– Let N = T/Δt, and ε_i ∼ N(0,1) then

$$W(T) - W(0) = \sum_{i=1}^{N} \varepsilon_i \sqrt{\Delta t}$$

• Thus, $W(t)$ has independent increments, ΔW, with ΔW ∼ N(0, Δt).

Note: We denote the continuous change with the operator d.

Example: x follows a BM with a drift rate μ and a variance rate σ^2 if

$$dx = \mu\, dt + \sigma\, dW$$

Interpretation: - Mean change in x in time T is μT

- Variance of change in x in time T is $\sigma^2 T$

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Stochastic Process: $W(t)$ - Itô process

• In an Itô process the drift and the variance rates are functions of time

$$dx = a(x,t)\, dt + b(x,t)\, dz$$

$$\Delta x = a(x,t)\Delta t + b(x,t)\varepsilon\sqrt{\Delta t}$$

(the discrete time equivalent is only true in the limit as Δt tends to 0.)

Example: Itô process for stock prices (S)

$$dS = \mu\, S\, dt + \sigma\, S\, dz$$

where μ is the expected return and σ is the volatility.

• The discrete time equivalent is $\Delta S = \mu S \Delta t + \sigma S \varepsilon \sqrt{\Delta t}$
where $\Delta S/S \sim N(\mu\Delta t, \sigma^2\Delta t)$.

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Stochastic Process: $W(t)$ - Properties

Theorem (Levy): Quadratic variation.
As the partition of [0,T] becomes finer (a smaller norm), say $\|P\| \to 0$,

$$\lim_{\|P\|\to 0} \sum_{t=1}^{N} (W_t - W_{t-1})^2 = T$$

That is, in the limit (Riemann integral), the sum of square increments is equal to T.

Intuition: As time passes, we observe the random changes $W(t) - W(t\text{-}1)$. The accumulation of squared random changes is equal to T. This is the internal *clock* of a random process. It is a special feature of a BM that the internal clock works keeps up with normal time.

In fact, a BM is almost entirely defined by this property: If a continuous martingale has quadratic variation, then it is a BM.

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Stochastic Processes: Applications (1)

• We saw several systems expressed as differential equations.

Example: Population growth, say $dN/dt = a(t)N(t)$.
There is no stochastic component to N(t), given initial conditions, we can derive without error the evolution of N(t) over time.

• However, in real world applications, several factors introduce a random factor in such models:

$$a(t) = b(t) + \sigma(t) \text{ x "Noise"} = b(t) + \sigma(t)\, W(t),$$

where W(t) is a stochastic process that represents the source of randomness (for example, "white noise").

• A simple differential equation becomes a stochastic differential equation.

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Stochastic Processes: Applications (2)

- Other applications where stochastic processes are used :
  - Filtering problems (*Kalman filter*)
    - Minimize the expected estimation error for a system state.
  - *Optimal Stopping Theorem*
  - *Financial Mathematics*
    - Theory of *option pricing* uses the *differential heat equation* applied to a *geometric Brownian motion* or *GBM* ($e^{\mu t+\sigma W(t)}$).

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Stochastic Process and Calculus: Motivation

- Consider a process which is the square of a BM:

$$Y(t) = W(t)^2$$

This process is always non-negative, $Y(0) = 0$, Y(t) has infinitely many zeroes on $t > 0$ and $E[Y(t)] = E[W(t)^2] = t$.

Question: What is the stochastic differential of Y(*t*)?

- Using standard calculus: $dY(t) = 2W(t)\, dW(t)$

$$\Rightarrow Y(t) = \int^t dY = \int^t 2W(t)\, dW(t)$$

- Consider $\int^t 2W(t)\, dW(t)$:

$$\int_0^t 2W(t)dW(t) \approx \sum_{i=1}^{n} 2W((i-1)t/n)[W(it/n) - W((i-)t/n)]$$

- By definition, the increments of W(t) are independent, with constant mean.

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Stochastic Process and Calculus: Motivation

- Therefore, the expected value, or mean, of the summation will be zero:

$$\begin{aligned}
\mathbb{E}[Y(t)] &= \mathbb{E}\left[\int_0^t 2W(t)dW(t)\right] \\
&= \mathbb{E}\left[\lim_{n\to\infty}\sum_{i=1}^{n} 2W((i-1)t/n)(W(it/n) - W((i-1)t/n))\right] \\
&= \lim_{n\to\infty}\sum_{i=1}^{n} 2\mathbb{E}[W((i-1)t/n)[W(it/n) - W((i-)t/n)]] \\
&= 0.
\end{aligned}$$

- But the mean of $Y(t) = W(t)^2$ is t which is definitely not zero! The two stochastic processes do not agree even in the mean, so something is not right! If we want to keep the integral definition and limit processes, then the rules of calculus will have to change.

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Stochastic calculus: Introduction (1)

• Let us consider:

$$dx/dt = b(t,x) + \sigma(t,x)\, W(t)$$

- White noise assumptions on $W(t)$ would make $W(t)$ discontinuous.

• This is bad news.

• Hence, we consider the discrete version of the equation:

$$\Delta x_{k+1} = x_{k+1} - x_k = b(t_k,x_k)\Delta t_k + \sigma(t_k,x_k) W(t_k)\Delta t_k \qquad (x_k = x(t_k,\omega))$$

- We can make white noise assumptions on B_k, where

$$\Delta B_k = W(t_k)\Delta t_k.$$

- It turns out that B_k can only be a BM

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Stochastic calculus: Introduction (2)

• Now we have another problem:
  – $x(t) = \sum b(t_k,x_k)\Delta t_k + \sum\sigma(t_k,x_k)\, \Delta B_k$
  – As $\Delta t_k \to 0$, $\sum b(t_k,x_k)\Delta t_k \to$ time integral of $b(t,x)$

• What about $\lim \sum\sigma(t_k,x_k)\, \Delta B_k$?
  – Hence, we need to find expressions for "integral" and "differentiation" of a function of stochastic process.

• Again, we have a problem.
  • BM is continuous, but not differentiable (Riemnann integrals will not work!)
  • *Stochastic Calculus* provides us a mean to calculate "integral" of a stochastic process but not "differentiation."

  – This is OK, as most stochastic processes are not differentiable.

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Stochastic calculus: Introduction (3)

• We use the definition of "integral" of deterministic functions as a base:

$\int \sigma(t,\omega)\, dB = \lim \sum \sigma(t_{\underline{k}},\omega)\, \Delta B_k$, where $t_{\underline{k}} \in [t_k, t_{k+1})$ as $t_{k+1} - t_k \to 0$.

• But, we cannot chose any $t_{\underline{k}}$; $t_{\underline{k}} \in [t_k, t_{k+1}]$.

Example: if $t_{\underline{k}} = t_k$, then $E(\sum B_{\underline{k}}\, \Delta B_k) = 0$.

Example: if $t_{\underline{k}} = t_{k+1}$, then $E(\sum B_{\underline{k}}\, \Delta B_k) = t$.

• We need to be careful (and consistent) in choosing $t_{\underline{k}}$.

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Stochastic calculus: Itô and Stratonovich

• Two choices for $t_{\underline{k}}$ are popular:-

– If $t_{\underline{k}} = t_k$, then it is called Itô's integral.

– If $t_{\underline{k}} = (t_k + t_{k+1})/2$, then it is called Stratonovich integral.

• We will concentrate on Itô's integral as it provides computational and conceptual simplicity.

- Itô's and Stratonovich integrals differ by a simple time integral only.

- In economics, Stratonovich integrals are not popular, since it requires at time t_k knowledge of t_{k+1}. In general, we like to integrate over values we know.

Kiyoshi Itô (1915–2008, Japan)

Stochastic calculus: Itô's Theorem (1)

- For a given $f(t,\omega)$ if:

1. $f(t,\omega)$ is F_t *adapted* ("a process that cannot look into the future")
- $f(t,\omega)$ can be determined by t and values of $B_t(\omega)$ up to t.
- $B_{t/2}(\omega)$ is F_t *adapted* but $B_{2t}(\omega)$ is not F_t *adapted.*

2. $E[\int f^2(t,\omega)\, dt] < \infty$ $\quad (\Rightarrow E[\int (f(t,\omega) - \Phi_n(t,\omega))^2 dt\,] \to 0$ as $n \to \infty)$
- This implication from (2) is a result from measure theory, needed for the convergence in L^2 of the sequence of Itô integrals.

Then,

$$\int f(t,\omega)\, dB_t(\omega) = \sum \Phi(t_k,\omega)\,(B_{k+1} - B_k) \text{ and}$$

$$E[(\int f(t,\omega)\, dB_t(\omega))^2] = E[\int f^2(t,\omega)\, dt] \quad \text{(Itô isometry)}$$

$\Rightarrow$ the integral $f(t,\omega)$ dB can be defined. $f(t,\omega)$ is said to be *B-integrable.*

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Stochastic calculus: Itô's Theorem (1)

- Under (1) and (2):

$$\int f(t,\omega)\, dB_t(\omega) = \sum \Phi(t_k,\omega)\,(B_{k+1} - B_k) \quad \text{and}$$

$$E(|\int f(t,\omega)\, dB_t(\omega)|^2) = E(\int f^2(t,\omega)\, dt) \text{ (Itô isometry)}$$

$\Rightarrow$ the integral $f(t,\omega)$ dB can be defined. $f(t,\omega)$ is said to be *B-integrable*
(*integrable* = bounded integral)

- Remarks:
  - $\Phi(t,\omega)$ are called elementary (simple) functions. Their values are constant in the interval $[t_k, t_{k+1}]$.
  - $E[\int |f(t,\omega) - \Phi_n(t,\omega)|^2 dt\,] \to 0$ as $n \to \infty$. This result is an implication from (2). It is used to get the convergence in L^2 of the sequence of Itô integrals $I_n(\omega) = \int \Phi_n(t,\omega)\, dB_t(\omega)$ to the RV $I(\omega) = \int f(t,\omega)\, dB_t(\omega)$.

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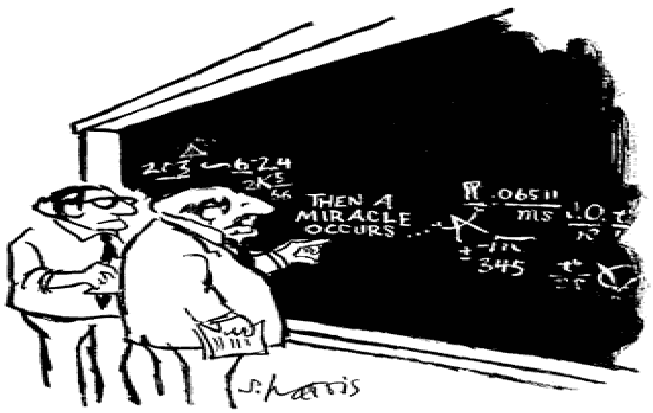

"I think you should be more explicit here in step two."

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Stochastic calculus: Itô's Theorem (2)

• If $f(t,\omega) = B(t,\omega)$ $\Rightarrow$ select $\Phi(t,\omega) = B(t_k,\omega)$ when $t \in [t_k, t_{k+1})$

Then, we have: $\int B(t,\omega)\, dB(t,\omega) = \lim \sum B(t_k,\omega)\, (B(t_{k+1},\omega) - B(t_k,\omega))$

Some algebra (recalling $2b(a-b) = a^2 - b^2 - (a-b)^2$) and results:

(1) $B(t_k,\omega)\,(B(t_{k+1},\omega) - B(t_k,\omega)) = ½\, \{B^2(t_{k+1},\omega) - B^2(t_k,\omega) - [B(t_{k+1},\omega) - B(t_k,\omega)]^2\}$

(2) $B^2(t_{k+1},\omega) - B^2(t_k,\omega) = [B(t_{k+1},\omega) - B(t_k,\omega)]^2 + 2\, B(t_k,\omega)\,(B(t_{k+1},\omega) - B(t_k,\omega))$

(3) $B^2(t) - B^2(0) = \sum B^2(t_{k+1},\omega) - B^2(t_k,\omega)$ (accumulation of Brownian motion)

(4) $\lim_{\Delta t \to 0} \sum [\, (B(t_{k+1},\omega) - B(t_k,\omega))^2] = T$ (*quadratic variation* property of B(t))

Then,

$$\int B(t,\omega)\, dB(t,\omega) = ½ \lim_{\Delta t \to 0} \sum [B^2(t_{k+1},\omega) - B^2(t_k,\omega) - (B(t_{k+1},\omega) - B(t_k,\omega))^2]$$
$$= B^2(t,\omega)/2 - t/2.$$

• Note: Itô's integral gives us more than the expected $B^2(t,\omega)/2$. This is due to the time-variance of the Brownian motion.

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Stochastic calculus: Itô's Theorem (3)

• Simple properties of Itô's integrals:

- $\int [a\, X(t,\omega) + b\, Y(t,\omega)]\, dB(t) = \int a\, X(t,\omega)\, dB(t) + \int b\, Y(t,\omega) dB(t)$
- $E[\int a\, X(t,\omega)\, dB(t)] = 0$
- $\int a\, X(t,\omega)\, dB(t)$ is F_t measurable

• It will be easier to calculate stochastic integrals using *Itô's lemma*, the fundamental theorem of stochastic calculus.

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Stochastic calculus: Review of FTC

• Simple derivation of the FTC through a 1st-order Taylor expansion of a function *f(t),* which is C^1 (with continuous 1st derivatives), on [0,1]:

$f(t+s) = f(t) + f'(t)\, s + o(s)$ $\qquad$ $(o(s)/s \rightarrow 0$, as $s \rightarrow 0)$

We write $f(1)$ as an accumulation of n increments starting at $f(0)$:

$f(1) = f(0) + \sum_{j=1 \text{ to } n} \{ f(j/n) - f((j-1)/n) \}$

Then, using a Taylor expansion for each of the $f(j/n)$

$\{ f(j/n) - f((j-1)/n) \} = f'((j-1)/n)\, (1/n) + o(n)$

Then,

$f(1) = f(0) + \lim_{n\rightarrow\infty} \sum_{j=1 \text{ to } n} f'((j-1)/n)\, (1/n) + \lim_{n\rightarrow\infty} \sum_{j=1 \text{ to } n} o(n).$

Using the definition of Rienmann intergral, we are done:

$f(1) = f(0) + \int_0^t f'(t)\, dt.$

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Stochastic calculus: Itô's Process (1)

- For a general process $x(t,\omega)$, how do we define integral $\int f(t,x)\, dx$?
  - If x can be expressed by a stochastic differential equation, we can calculate $\delta f(t,x)$.

- **Definition**:
  An *Itô's process* is a stochastic process on (Ω, F, P), which can be represented in the form:
  $$x(t,\omega) = x(0) + \int \mu(s)\, ds + \int \sigma(s) dB(s)$$
  where μ and σ may be functions of x and other variables. Both are processes with finite (square) Riemann integrals.

  Alternatively, we have already said $x(t,\omega)$ is called an Itô's process if
  $$dx(t) = \mu(t)\, dt + \sigma(t)\, dB(t).$$

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Stochastic calculus: Itô's Process and Lemma

• Itô's Formula 1

Let $B(t)$ be a standard BM.

Let $f(t,x)$ be a C^2 function –i.e., twice continuously differentiable.

Then, $$f(B_t) = f(B_0) + \int_0^t f'(B_s)\, dB_s + \int_0^t f''(B_s)\, ds$$

The theorem is usually written as $\partial f(B_t) = f'(B_s)\, dB_t + ½ f''(B_s)\, dt.$

That is, the process $x(t) = f(B_t)$ at time t evolves like a BM with drift $f''(B_s)/2$ and variance $[f'(B_s)]^2$.

Derivation: Similar to the previous derivation of the FTC. Now, use a 2nd-order Taylor expansion of $f(B_t)$.

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Stochastic calculus: Itô's Process and Lemma

• **Itô's Formula 2 (Itô's Lemma)**

Let $x(t,\omega)$ be an Itô process: $dx(t) = \mu(t)\, dt + \sigma(t)\, dB(t)$.

Let $f(t,x)$ be a C^2 function.

Then, $f(t,x)$ is also an Itô process and

$$\partial f(t,x) = (df/dt)\, dt + (df/dx)\, dx(t) + ½\, d^2f/dx^2\, (dx(t))^2$$

$$= [(df/dt) + (df/dx)\, \mu(t) + ½\, d^2f/dx^2\, \sigma^2(t)]dt + (df/dx)\, \sigma(t)\, dB(t)$$

This result is called *Itô's Lemma.*

Note: Itô processes is closed under twice continuously differentiable transformations.

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Stochastic calculus: Itô's Process and Lemma

• To do quick calculations in calculus, we write down differentials and discard all terms that are of smaller order than *dt*. In stochastic calculus, we can do the same using the following rules:

$$dB(t)*dB(t) = (dB(t))^2 = dt$$

$$dt*dt = (dt)^2 = 0$$

$$dt*dB(t) = dB(t)*dt = 0$$

• Then, applying these rules, we have that Itô's lemma implies:

Itô's lemma: $\partial f(t,x) = [(df/dt) + (f'(x)\, \mu(t) + ½\, f''(x)\, \sigma^2(t)]\, dt + (df/dx)\, \sigma(t)\, dB(t)$

$$dX(t) = \mu(t)\, dt + \sigma(t)\, dB$$

$$\Rightarrow \quad (dX(t))^2 = \mu(t)^2\, dt^2 + 2\, \mu(t)\, dt\, \sigma(t)\, dB + \sigma(t)^2\, dB^2$$

$$= 0 + 0 + \sigma(t)^2\, dt$$

Note: Non-stochastic! A square of an Itô process leaves the variance. 114

Itô's Lemma – Check

Itô's lemma: $\partial f(t,x) = (df/dt)\, dt + (df/dx)\, dx(t) + ½\, d^2f/dx^2\, (dx(t))^2$

• Check:

Let $B(t,\omega) = X(t)$ (think of $\mu = 0$, $\sigma = 1$).

Define: $f(t,\omega) = B^2(t,\omega)/2$.

Now,

$$\partial(B^2(t,\omega)/2) = 0\, dt + B(t,\omega)\, dB(t) + ½\, d^2f/dx^2\, (dB(t))^2$$
$$= B(t,\omega)\, dB(t) + ½\, dt$$

$$\Rightarrow B^2(t,\omega)/2 = \int B(t,\omega)\, dB_t + \int ½\, dt$$
or $$\int B(t,\omega)\, dB_t = B^2(t,\omega)/2 - t/2$$

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Itô's Lemma – Derivation

• Let Δx be a small change in x and ΔG be the resulting small change in $G = f(t,x)$.

• Let's do a Taylor expansion of G:

$$\Delta G = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial t}\Delta t + ½\frac{\partial^2 G}{\partial x^2}\Delta x^2 + \frac{\partial^2 G}{\partial x\,\partial t}\Delta x\,\Delta t + ½\frac{\partial^2 G}{\partial t^2}\Delta t^2 + \dots$$

• Note:

– In ordinary calculus we have: $\Delta G = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial t}\Delta t$

– In stochastic calculus this becomes: $\Delta G = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial t}\Delta t + ½\frac{\partial^2 G}{\partial x^2}\Delta x^2$

because Δx has a component which is of order $\sqrt{\Delta t}$

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Itô's Lemma – Derivation

• Let x be an Itô process: $dx = a(x,t)\, dt + b(x,t)\, dz$
then,

$$\Delta x = a(x,t)\Delta t + b(x,t)\varepsilon\sqrt{\Delta t}$$

• Ignoring term of higher order than Δt in ΔG:

$$\Delta G = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial t}\Delta t + ½\,\frac{\partial^2 G}{\partial x^2} b^2 \varepsilon^2 \Delta t$$

• Let's focus on the $\varepsilon^2\Delta t$ term:
- Since $\varepsilon \sim$ *i.i.d.* N(0,1) $\Rightarrow E[\varepsilon^2] = 1$. Then, $E[\varepsilon^2\Delta t] = \Delta t$
- The variance is proportional to Δt^2. As $\Delta t \to 0$, it collapses to a point.

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Itô's Lemma – Derivation

• Now, we take limits as $\Delta t \to 0$:

$$dG = \frac{\partial G}{\partial x} dx + \frac{\partial G}{\partial t} dt + ½\,\frac{\partial^2 G}{\partial x^2} b^2 dt$$

• Replacing $dx = a\, dt + b\, dz$ in dG, we get:

$$dG = \left(\frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + ½\frac{\partial^2 G}{\partial x^2} b^2\right) dt + \frac{\partial G}{\partial x} b\, dz$$

• This is Itô's Lemma.

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Itô's Lemma: Examples – Discounting

• Recall Itô's lemma:

$$\partial f(t,x) = (df/dt)\,dt + (df/dx)\,dx(t) + ½\,d^2f/dx^2\,(dx(t))^2$$

Example: Stochastic Discounting I

$f(t,\omega) = e^{tB(t)}$.

Now, $\partial(f(t,\omega)) = e^{tB(t)}\,B(t)\,dt + t\,e^{tB(t)}\,dB(t) + ½\,t^2\,e^{tB(t)}\,dB(t))^2$

$= e^{tB(t)}\,(B(t) + ½t^2)\,dt + t\,e^{tB(t)}\,dB(t)$

Example: Stochastic Discounting II

$Z(t) = f(t,\omega) = e^{rt+\sigma B(t)}$.

Now, $\partial(f(t,\omega)) = Z(t)\,r\,dt + \sigma\,Z(t)\,dB(t) + ½\,\sigma^2\,Z(t)\,dt$

$= (r + ½\,\sigma^2)\,Z(t)\,dt + \sigma\,Z(t)\,dB(t)$

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Itô's Lemma: Examples – Forward Contracts

• Recall Itô's lemma:

$$\partial f(t,x) = (df/dt)\,dt + (df/dx)\,dx(t) + ½\,d^2f/dx^2\,(dx(t))^2$$

Let $dS = \mu\,S\,dt + \sigma\,S\,dz$

Then,

$df(t,x) = [(df/dt) + \mu S(t)\,(df/dS) + ½\,d^2f/dS^2\,\sigma^2 S(t)^2]dt + (df/dS)\,\sigma S(t)dB(t)$

Example: Forward Contracts

$F(t) = f(t,\omega) = S(t)\,e^{r(T-t)}$,

$\Rightarrow\ d(F(t)) = [-r\,S(t)\,e^{r(T-t)} + \mu S(t)\,e^{r(T-t)} + ½\,0\,\sigma^2 S(t)^2]dt + e^{r(T-t)}\,\sigma S(t)dz$

$= (\mu - r)\,F(t)\,dt + \sigma\,F(t)\,dz$

$\Rightarrow d(F(t))/F(t) = (\mu - r)\,dt + \sigma\,dz$

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Itô's Lemma: Examples – Lognormal Property

Let $dS = \mu\, S\, dt + \sigma\, S\, dz$ and using

$df(t,x) = [(df/dt) + \mu S(t)\,(df/dS) + \frac{1}{2}\, d^2f/dS^2\, \sigma^2 S(t)^2]dt + (df/dS)\, \sigma S(t) dB(t)$

Example: Lognormal Property

$$G(t) = f(t,\omega) = \ln S(t),$$

$$\Rightarrow \quad d(G(t)) = [0 + \mu\, S\,(1/S) + \tfrac{1}{2}\,(-1/S^2)\,\sigma^2 S^2]\, dt + (1/S)\,\sigma\, S\, dz$$
$$= (\mu - \sigma^2/2)\, dt + \sigma\, dz$$

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Stochastic calculus: Application – Black-Scholes

- Let S(t), a (non-dividend) stock price, follow a geometric BM:
  $dS(t) = \mu\, S(t)\, dt + \sigma\, S(t)\, dB(t).$

- The payoff of an option $f(S,T)$ is known at T.

- Applying Ito's formula:
  $$d(f(S,t)) = (df/dt)\, dt + (df/dS)\, dS(t) + \tfrac{1}{2} d^2f/dS^2\, (dS(t))^2$$
  $$= [(df/dt) + \mu S(t)\,(df/dS) + \tfrac{1}{2} d^2f/dS^2\, \sigma^2 S(t)^2]dt + (df/dS)\, \sigma S(t) dB(t)$$

- Form a (*delta-hedge*) portfolio: hold one option and continuously trade in the stock in order to hold ($-df/dS$) shares. At *t*, the value of the portfolio:
  $$\pi(t) = f(S, t) - S(t)\, df/dS$$

- We want to accumulate profits from this portfolio.

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Stochastic calculus: Application – Black-Scholes

- Let R be the accumulated profits from the portfolio. Then, over the time period $[t, t + dt]$, the instantaneous profit or loss is:

  $dR = df(S, t) – df/dS\ dS(t)$

- Substituting using Itô's lemma for $df(S,t)$ and for $dS(t)$, we get:

  $dR = [(df/dt)\ dt + (df/dS)\ dS(t) + ½\ d^2f/dS^2\ (dS(t))^2] – df/dS\ dS(t)$

  $= [(df/dt) + ½\ d^2f/dS^2\ \sigma^2\ S(t)^2]\ dt$

  Note: This is not a SDE ($dB(t)$ has disappeared: riskless portfolio!)

- Since there is no risk, the rate of return of the portfolio should be r, the rate on a riskless asset.

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Stochastic calculus: Application – Black-Scholes

- That is,

  $dR = r\,\pi(t)\ dt = r\ [f(S,t) – S(t)\ df/dS]\ dt$

  $\Rightarrow\ r\ [f(S,t) – S(t)\ df/dS]\ dt = [(df/dt) + ½\ d^2f/dS^2\ \sigma^2\ S(t)^2]\ dt$

  $\Rightarrow\ (df/dt) + ½\ d^2f/dS^2\ \sigma^2\ S(t)^2 + r\,S(t)\ df/dS – r\,f(S,t)=0$

  This is the *Black-Scholes* PDE. Given the boundary conditions for a call option, $C(S,t)$, it can be solved using the standard methods.

- Boundary conditions:

  $C(0, t) = t$ for all t

  $C(S, t) \rightarrow S$, as $S \rightarrow \infty$.

  $C(S, T) = \max(S – K, 0)$; K = strike price

- Solution (already seen in Chapter 7):

  $$C_t = S_t\,N(d1) - K\,e^{-i(T-t)}N(d2)$$

Stochastic calculus: Solving a stochastic DE

- Make a guess (Hope you are lucky!)

Example: We want to solve the stochastic DE:

$$dZ(t) = \sigma\, Z(t)\, dB(t).$$

Guess: $Y(t) = e^{rt+\sigma B(t)}$ (Stochastic Discounting II example)

with SDE: $dZ(t) = (r + \frac{1}{2}\sigma^2)\; Z(t)\, dt + \sigma\, Z(t)\, dB(t).$

Replace in given SDE:

$$\Rightarrow (r + \tfrac{1}{2}\sigma^2)\; Z(t)\, dt + \sigma\, Z(t)\, dB(t) = \sigma\, Z(t)\, dB(t).$$

$$\Rightarrow r = -\tfrac{1}{2}\sigma^2$$

Solution: $Y(t) = \exp(-\frac{1}{2}\sigma^2\, t + \sigma\, dB(t))$ (This solution is called the *Dolèan's exponential of BM*.)

Note: SDE with solutions are rare.

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For Man U fans: The Black Scholes