



Chapter 10

Optimization: More than One Choice Variable



William Stanley Jevons (1835-1882)



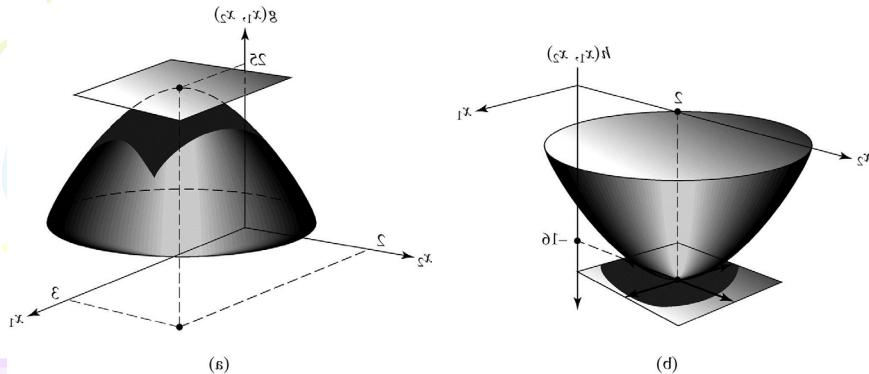
Carl Menger (1840–1921)

10. Optimization Problems

- Chapter 9: $\max_x u(x)$ one choice variable: x (consumption)
- Chapter 10: $\max_{x,y} u(x, y)$ two choice variables: x, y (leisure)
s.t $x p_x + y p_y = I$, where p_x, p_y are exogenous prices and I is exogenous income.
- With one choice variable, the f.o.c. is defined by setting $u'(x)$ equal to zero. Note that
 - $u = u(x) \Rightarrow du = u'(x) dx$.
 - when $u'(x) = 0 \Rightarrow du=0$.
- With two or more choice variables, the f.o.c. is defined by a differential, when du equals zero.

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Figure 10.1 Stationary Points and the Tangent Planes of Bivariate Functions



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10.2 First-order condition

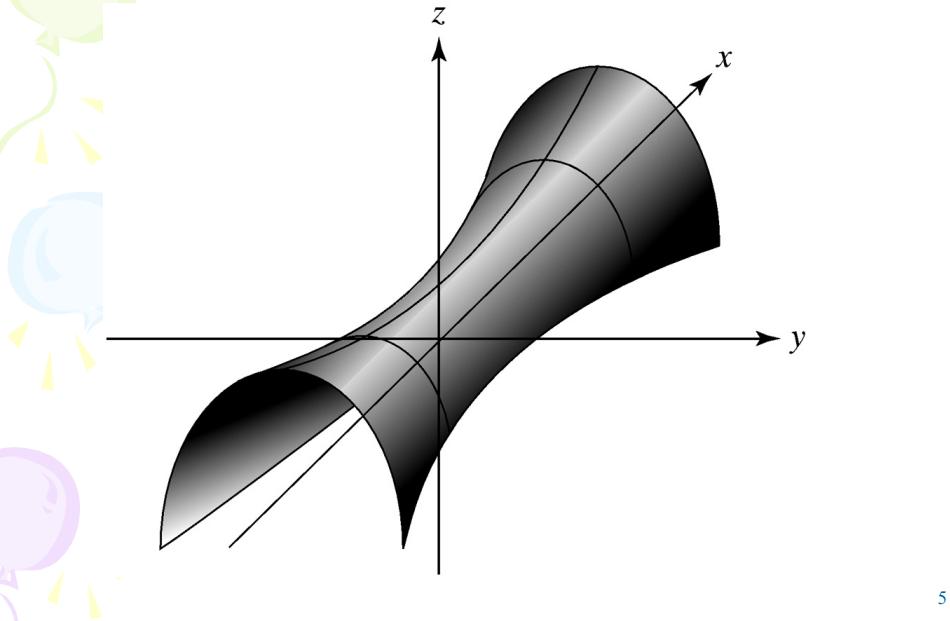
- Given the function $z = f(x, y)$
- Find the total differential

$$dz = f_x dx + f_y dy$$
- Set $dz = 0$. Then $f_x = f_y = 0$ (f.o.c.)
- $dz = 0$ is a necessary condition for an extreme point. But, like in the one variable case, it is not sufficient. See Figure 10.4.
- Find the second - order differential :

$$d^2z = f_{xx}dx^2 + 2f_{xy}dxdy + f_{yy}dy^2$$
- Find partial derivatives : $f_x, f_y, f_{xx}, f_{xy}, f_{yy}$

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Figure 10.4 A Saddle Point



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10.2 Second-order & cross partial derivatives

1) $z = f(x, y)$

2) $f_x = \frac{\partial z}{\partial x} \quad f_y = \frac{\partial z}{\partial y}$

3) $f_{xx} = \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}$

4) $f_{xy} = \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y}$

5) $f_{yy} = \frac{\partial}{\partial y}(f_y) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2}$

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10.2 Second-order total differential

$$1) \quad z = f(x, y)$$

$$2) \quad dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = f_x dx + f_y dy$$

$$\begin{aligned} 3) \quad d(dz) &= d^2 z = \frac{\partial(dz)}{\partial x} dx + \frac{\partial(dz)}{\partial y} dy \\ &= \frac{\partial(f_x dx + f_y dy)}{\partial x} dx + \frac{\partial(f_x dx + f_y dy)}{\partial y} dy \\ &= f_{xx} dx^2 + f_{xy} dx dy + f_{xy} dx dy + f_{yy} dy^2 \\ &= f_{xx} dx^2 + 2f_{xy} dx dy + f_{yy} dy^2 \end{aligned}$$

- Note: $d^2 z$ is a quadratic form in dy and dx .

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10.2 Second-order total differential

- Example: $z = x^3 + 2xy + y^2$

$$f_x = 3x^2 + 2y; \quad f_y = 2x + 2y$$

$$f_{xx} = 6x; \quad f_{xy} = 2; \quad f_{yy} = 2$$

$$d^2 z = (6x)dx^2 + 4dxdy + 2dy^2$$

First order condition (f.o.c.)

$$f_x = 3x^* + 2y^* = 0; \quad f_y = 2x^* + 2y^* = 0$$

$$x^* = 2/3; \quad y^* = 2/3$$

Question : what kind of critical point is (x^*, y^*) ?

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10.2 Second-order total differential

- If $d^2z < 0$, evaluated at (x^*, y^*) , in every x and y direction, then, (x^*, y^*) is a maximum.
- On the other hand, if $d^2z > 0$, evaluated at (x^*, y^*) , in every x and y direction, then, (x^*, y^*) is a minimum.
- We can use d^2z to state the second order condition:
 - $d^2z(x^*, y^*) < 0$ for x^* and y^* to be a maximum
 - $d^2z(x^*, y^*) > 0$ for x^* and y^* to be a minimum
- When can we know the sign of d^2z ?
 - When $f_{xy} = 0$, sign(d^2z) is determined by the sign of f_{yy} and f_{xx} :
 - If $f_{xx} < 0$ and $f_{yy} < 0$, then $\text{sign}(d^2z) < 0$
 - If $f_{xx} > 0$ and $f_{yy} > 0$, then $\text{sign}(d^2z) > 0$
 - But, in general, $f_{xy} \neq 0$, then getting the sign(d^2z) is not that simple. 9

10.2 Quadratic forms (Again)

- A form is a polynomial expression in which each component term has a uniform degree. A quadratic form has a uniform 2nd degree.

Examples:

$$\begin{array}{ll} 9x + 3y + 2z & \text{-first degree form.} \\ 6x^2 + 2xy + 2y^2 & \text{-second degree (quadratic) form.} \\ d^2z & \text{-second degree form in } dx \text{ and } dy. \end{array}$$

- Let q be a quadratic form. We say q is:

Positive definite if q is invariably positive ($q > 0$)
Positive semi-definite if q is invariably non-negative ($q \geq 0$)
Negative semi-definite if q is invariably non-positive ($q \leq 0$)
Negative definite if q is invariably negative ($q < 0$)

A quadratic form is said to be indefinite if q changes signs.

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10.2 Quadratic forms

- d^2z is a quadratic form in dx and dy :

$$d^2z = f_{xx}dx^2 + 2f_{xy}dxdy + f_{yy}dy^2$$

Let $u = dx$; $v = dy$

Let $a = f_{xx}$; $b = f_{yy}$; $f_{yx} = h$

Rewriting d^2z :

$$q = au^2 + 2huv + bv^2$$

Trick to determine sign : complete the square :

$$q = au^2 + 2huv + bv^2 + \frac{h^2}{a}v^2 - \frac{h^2}{a}v^2$$

$$q = a(u^2 + 2\frac{h}{a}uv + \frac{h^2}{a^2}v^2) + bv^2 - \frac{h^2}{a}v^2 = a(u + \frac{h}{a}v)^2 + (b - \frac{h^2}{a})v^2$$

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10.2 Quadratic forms

- Now, we are ready to determine $\text{sign}(d^2z)$:

$$d^2z = q = a(u + \frac{h}{a}v)^2 + (b - \frac{h^2}{a})v^2$$

$$\text{sign}(q) \text{ depends on } a = f_{xx}, \text{ sign}\left(\frac{f_{yy}f_{xx} - f_{xy}^2}{f_{xx}}\right)$$

- We can write d^2z in a more compact form, using linear algebra.

$$d^2z = f_{xx}dx^2 + 2f_{xy}dxdy + f_{yy}dy^2$$

$$d^2z = [dx \quad dy] \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} = u^T Hu$$

where u is the vector of 1st derivatives and H is the matrix of second derivatives (the *Hessian*).

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10.2 Sign(d^2z) as a Determinant

- Rewrite d^2z , using linear algebra:

$$d^2z = [dx \quad dy] \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} [dx \quad dy]$$

$f_{yy}f_{xx} - f_{xy}^2$ is the determinant of the matrix above.

Let $|H| = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$

If $\begin{cases} f_{xx} > 0 \\ f_{xx} < 0 \end{cases}$ and $|H| > 0$, then d^2z is $\begin{cases} \text{positive definite} \\ \text{negative definite} \end{cases}$

Notation: $|H_i|$: $i \times i$ subdeterminant of H ($k \times k$) for $i = 1, 2, \dots, k$.

In this case, $i = 2$. Then, $|H_1| = f_{xx}$ & $|H_2| = |H|$.

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10.2 Sign(d^2z) as a Determinant: Example

$$Q = 4u^2 + 4uv + 3v^2$$

$$f_u = 8u + 4v; \quad f_{uu} = 8; \quad f_{uv} = 4$$

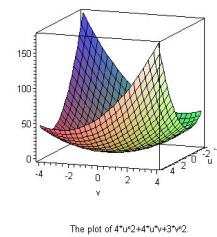
$$f_v = 4u + 6v; \quad f_{vu} = 4; \quad f_{vv} = 6$$

$$d^2Q = [du \quad dv] \begin{bmatrix} 8 & 4 \\ 4 & 6 \end{bmatrix} [du \quad dv]$$

If $\begin{cases} f_{xx} > 0 \\ f_{xx} < 0 \end{cases}$ and $|H| > 0$, then d^2Q is $\begin{cases} \text{positive definite} \\ \text{negative definite} \end{cases}$

$$f_{uu} = 8 > 0 \quad |H| = 48 - 16 = 32 > 0$$

Q is positive definite \rightarrow minimum



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10.2 Optimization conditions: Summary

- 1) $z = f(x, y)$ where $f_i = f_i(x, y)$, $i = 1, 2$
- endog. (z) exog. (x, y) parameters ($f_1, f_2, f_{11}, f_{12}, f_{21}, f_{22}$)
- 2) $dz = f_1 dx + f_2 dy = 0$ (FOC) solve for (x^*, y^*)
- 3) $d^2 z = f_{11} dx^2 + f_{12} dxdy + f_{21} dydx + f_{22} dy^2$
- 4) $|H| = \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix}$
 max if $f_{11} & f_{22} < 0, |H| > 0$
 min if $f_{11} & f_{22} > 0, |H| > 0$
 saddle point if $f_{11} \neq f_{22}, |H| < 0$

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10.2 Optimization conditions: Summary

Condition	Maximum	Minimum	Saddlepoint
1 st order Necessary	$f_x(x^*, y^*) = 0$ $f_y(x^*, y^*) = 0$	$f_x(x^*, y^*) = 0$ $f_y(x^*, y^*) = 0$	$f_x(x^*, y^*) = 0$ $f_y(x^*, y^*) = 0$
2 nd Order Necessary	$f_{xx}(x^*, y^*) < 0$ $f_{yy}(x^*, y^*) < 0$	$f_{xx}(x^*, y^*) > 0$ $f_{yy}(x^*, y^*) > 0$	$f_{xx}(x^*, y^*) \pm$ $f_{yy}(x^*, y^*) \mp$
Sufficient Condition	$f_{xx}f_{yy} > f_{xy}^2$ $ H_2 > 0$	$f_{xx}f_{yy} > f_{xy}^2$ $ H_2 > 0$	$f_{xx}f_{yy} < f_{xy}^2$ $ H_2 < 0$

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10.2 Optimization: Example

Find the extreme value(s) of $z = f(x, y)$. Determine whether they are maxima or minima

$$z = x^2 + xy + 2y^2 + 3 = x^2 + 1/2xy + 1/2xy + 2y^2 + 3$$

First - order total differential

$$dz = 2xdx + ydx + xdy + 4ydy$$

$$dz = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}$$

F.o.c.

$$\begin{cases} f_x = 2x + y = 0 \\ f_y = x + 4y = 0 \end{cases} \Rightarrow \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

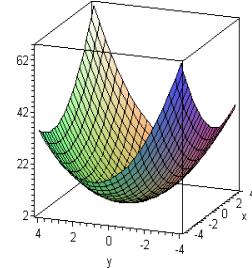
$$y^* = 0, x^* = 0, z^* = 3$$

Second - order total differential

$$d^2z = 2d^2x + 2dydx + 4d^2y = \begin{bmatrix} dx & dy \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}$$

$$|H| = \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix}, \quad |H_1| = f_{11} = 2 > 0, \quad |H_2| = 7 > 0$$

$$\Rightarrow d^2z \text{ is pd, } z^* \text{ is minimum}$$



The plot of $x^2 + xy + 2y^2 + 3$.

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10.2 Optimization: ML Example

We assume 2 explanatory variables. The log likelihood function we want to maximize w.r.t. β is :

$$\log L = f(y_1, y_2, \dots, y_T | \beta, \sigma^2) = -\frac{T}{2} \ln 2\pi - \frac{T}{2} \ln \sigma^2 - \frac{1}{2\sigma^2} \sum_{t=1}^T (y_t - x_{1,t}\beta_1 - x_{2,t}\beta_2)^2$$

1st derivatives :

$$\frac{\partial \ln L}{\partial \beta_1} = -\frac{1}{2\sigma^2} \sum_{t=1}^T 2(y_t - x_{1,t}\beta_1 - x_{2,t}\beta_2)(-x_{1,t}) = \frac{1}{\sigma^2} \sum_{t=1}^T (y_t x_{1,t} - x_{1,t}^2 \beta_1 - x_{2,t} x_{1,t} \beta_2)$$

$$\frac{\partial \ln L}{\partial \beta_2} = \frac{1}{\sigma^2} \sum_{t=1}^T (y_t x_{2,t} - x_{1,t} x_{2,t} \beta_1 - x_{2,t}^2 \beta_2)$$

F.o.c.

$$\begin{cases} f_x = \frac{1}{\sigma^2} \sum_{t=1}^T (y_t x_{1,t} - x_{1,t}^2 \hat{\beta}_{1,MLE} - x_{2,t} x_{1,t} \hat{\beta}_{2,MLE}) = 0 \\ f_y = \frac{1}{\sigma^2} \sum_{t=1}^T (y_t x_{2,t} - x_{1,t} x_{2,t} \hat{\beta}_{1,MLE} - x_{2,t}^2 \hat{\beta}_{2,MLE}) = 0 \end{cases}$$

$$\Rightarrow \begin{bmatrix} \sum_{t=1}^T (x_{1,t}^2) & \sum_{t=1}^T (x_{2,t} x_{1,t}) \\ \sum_{t=1}^T (x_{1,t} x_{2,t}) & \sum_{t=1}^T (x_{2,t}^2) \end{bmatrix} \begin{bmatrix} \hat{\beta}_{1,MLE} \\ \hat{\beta}_{2,MLE} \end{bmatrix} = \begin{bmatrix} \sum_{t=1}^T y_t x_{1,t} \\ \sum_{t=1}^T y_t x_{2,t} \end{bmatrix} \Rightarrow (X' X) \hat{\beta}_{MLE} = X' y$$

Solution :

$$\Rightarrow \hat{\beta}_{MLE} = (X' X)^{-1} X' y$$

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10.2 Optimization: ML Example

S.o.c.

$$f_{xx} = -\frac{1}{\sigma^2} \sum_{t=1}^T x_{1,t}^2 < 0$$

$$f_{xy} = -\frac{1}{\sigma^2} \sum_{t=1}^T x_{2,t} x_{1,t}$$

$$f_{yy} = -\frac{1}{\sigma^2} \sum_{t=1}^T x_{2,t}^2 < 0$$

$$|H| = \begin{vmatrix} -\frac{1}{\sigma^2} \sum_{t=1}^T (x_{1,t}^2) & -\frac{1}{\sigma^2} \sum_{t=1}^T (x_{2,t} x_{1,t}) \\ -\frac{1}{\sigma^2} \sum_{t=1}^T (x_{1,t} x_{2,t}) & -\frac{1}{\sigma^2} \sum_{t=1}^T (x_{2,t}^2) \end{vmatrix} = \left(-\frac{1}{\sigma^2}\right)^2 |X' X|,$$

$$|H_1| = -\frac{\sum_{t=1}^T (x_{1,t}^2)}{\sigma^2} < 0,$$

$$|H_2| = \frac{\sum_{t=1}^T (x_{1,t}^2) \sum_{t=1}^T (x_{2,t}^2) - \left(\sum_{t=1}^T x_{2,t} x_{1,t}\right)^2}{\sigma^4} ? \quad (\text{to be determined ...})$$

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10.3 Sign of a quadratic form: Eigenvalue tests

- From Chapter 5: Orthogonally diagonalize a system of equations:

$$\mathbf{A} \mathbf{w} = \mathbf{y}$$

- Pre-multiply both sides by \mathbf{X}^{-1} :

$$\mathbf{X}^{-1} \mathbf{A} \mathbf{w} = \mathbf{X}^{-1} \mathbf{y} = \mathbf{v}$$

$$\mathbf{X}^{-1} \mathbf{A} (\mathbf{X} \mathbf{X}^{-1}) \mathbf{w} = \mathbf{v} \quad (\text{Let } \mathbf{v} = \mathbf{X}^{-1} \mathbf{w})$$

$$\Rightarrow \mathbf{\Lambda} \mathbf{v} = \mathbf{v}$$

where $\mathbf{\Lambda}$ is the eigenvalue (diagonal) matrix.

- Let's re-write the quadratic form:

$$d^2 z = q = f_{xx} dx^2 + 2 f_{xy} dxdy + f_{yy} dy^2$$

$$q = \begin{bmatrix} dx & dy \end{bmatrix} \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} = \mathbf{u}' \mathbf{H} \mathbf{u}$$

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10.3 Sign of a quadratic form: Eigenvalue tests

- Quadratic form:

$$q = u' \mathbf{H} u \quad (\text{note: the Hessian, } \mathbf{H}, \text{ is a symmetric matrix})$$

- Let $u = Ty$, where \mathbf{T} is the matrix of eigenvectors of \mathbf{H} , such that $\mathbf{T}' \mathbf{T} = \mathbf{I}$

- Then,

$$q = y' \mathbf{T}' \mathbf{H} \mathbf{T} y = y' \mathbf{\Lambda} y \quad (\mathbf{T}' \mathbf{H} \mathbf{T} = \mathbf{\Lambda})$$

$$q = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \dots + \lambda_i y_i^2 + \dots + \lambda_n y_n^2$$

\Rightarrow The sign(q) depends on the λ_i 's only.

- We say:

q is positive definite iff $\lambda_i > 0$ for all i .

q is positive semi-definite iff $\lambda_i \geq 0$ for all i .

q is negative semi-definite iff $\lambda_i \leq 0$ for all i .

q is negative definite iff $\lambda_i < 0$ for all i .

q is indefinite if some $\lambda_i > 0$ and some $\lambda_i < 0$.

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10.3 Sign of a quadratic form: Eigenvalue tests

- Example (continuation of 10.2):

$$z = [x \ y] \begin{bmatrix} 1 & 1/2 \\ 1/2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + 3$$

F.o.c.

$$\begin{cases} f_x = 2x + y = 0 \\ f_y = x + 4y = 0 \end{cases}$$

$$y^* = 0, x^* = 0, z^* = 3$$

Calculate matrix of second derivatives

$$|H| = \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix}, \quad \lambda_1 = 1.58582; \quad \lambda_2 = 4.4142$$

$\Rightarrow \lambda_1$ and λ_2 are positive, q is positive definite

$\Rightarrow z^*$ is minimum

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10.4 n-variable soc principal minors test for max or min

1-variable test of soc, principal minor test

$$\max : |H_1| < 0, \quad : ($$

$$\min : |H_1| > 0, \quad :)$$

2-variable test of soc

$$\max : |H_1| < 0, \quad |H_2| > 0 \quad : ($$

$$\min : |H_1| > 0, \quad |H_2| > 0 \quad :)$$

n-variable case soc,

$$\max : |H_1| < 0, |H_2| > 0, |H_3| < 0, \dots, (-1)^n |H_n| > 0 \quad : ($$

$$\min : |H_1| > 0, |H_2| > 0, |H_3| > 0, \dots, |H_n| > 0 \quad :)$$

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10.4 N-variable case: Example with n=3

$$1) \quad z = x^2 + 3y^2 - 3xy + 4yw + 6w^2$$

$$2) \quad dz = f_x dx + f_y dy + f_z dw = 0$$

$$3) \quad f_x = 2x - 3y + 0w = 0$$

$$4) \quad f_y = -3x + 6y + 4w = 0$$

$$5) \quad f_w = 0x + 4y + 12w = 0$$

$$6) \quad \begin{bmatrix} 2 & -3 & 0 \\ -3 & 6 & 4 \\ 0 & 4 & 12 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solve for x^*, y^*, w^*

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10.4 N-variable case: Example with n=3

1)
$$\begin{aligned} z &= x^2 + 3y^2 - 3xy + 4yw + 6w^2 \\ f_x &= 2x - 3y + 0w = 0, & f_{xx} &= 2, f_{xy} &= -3, f_{xw} &= 0 \\ f_y &= -3x + 6y + 4w = 0, & f_{yx} &= -3, f_{yy} &= 6, f_{yw} &= 4 \\ f_w &= 0x + 4y + 12w = 0, & f_{wx} &= 0, f_{wy} &= 4, f_{ww} &= 12 \end{aligned}$$

$$|H| = \begin{vmatrix} 2 & -3 & 0 \\ -3 & 6 & 4 \\ 0 & 4 & 12 \end{vmatrix}$$

Principal minors

$$|H_1| = 2 > 0, \quad |H_2| = 3 > 0, \quad |H_3| = 4 > 0,$$

$d^2z > 0$ (positive definite) \rightarrow minimum

Eigenvalue check: $\lambda = (0.04876, 5.7940, 14.1572)$.

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10.5 Economic Applications: Example 1

Multi-product firm.

Assume a competitive firm has revenue and cost functions below.

Determine Q_1^* , Q_2^* .

1 - 2) $P_{10} = 12, \quad P_{20} = 18,$

3 - 4) $R = P_{10}Q_1 + P_{20}Q_2, \quad C = 2Q_1^2 + Q_1Q_2 + 2Q_2^2$

5) $\pi = R - C = P_{10}Q_1 + P_{20}Q_2 - (2Q_1^2 + Q_1Q_2 + 2Q_2^2)$

6 - 7) $\frac{\partial \pi}{\partial Q_1} = P_{10} - 4Q_1 - Q_2 = 0, \quad \frac{\partial \pi}{\partial Q_2} = P_{20} - Q_1 - 4Q_2$

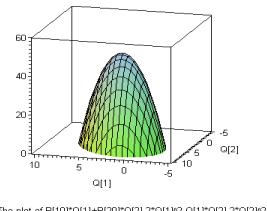
8 - 9) $\begin{bmatrix} -4 & -1 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} -P_{10} \\ -P_{20} \end{bmatrix}, \quad \frac{1}{15} \begin{bmatrix} -4 & 1 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} -P_{10} \\ -P_{20} \end{bmatrix} = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}$

10) $Q_1^* = \frac{4P_{10} - P_{20}}{15} = \frac{4(12) - 18}{15} = 2,$

11) $Q_2^* = \frac{4P_{20} - P_{10}}{15} = \frac{4(18) - 12}{15} = 4$

12) $\pi^* = 12(2) + 18(4) - 2(2)^2 - 2(4) - 2(4)^2 = 24 + 72 - 8 - 8 - 32 = 48$

13 - 14) $|H_1| = -4, \quad |H_2| = \begin{bmatrix} -4 & -1 \\ -1 & -4 \end{bmatrix} = 15 \text{ negative definite, max.}$



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10.5 Economic Applications: Example 2

- Assumptions behind *classical linear regression (CLM) model*:

(A1) $\mathbf{y} = \mathbf{X}\beta + \boldsymbol{\varepsilon}$ is correctly specified.

(A2) $E[\boldsymbol{\varepsilon} | \mathbf{X}] = 0$

(A3) $\text{Var}[\boldsymbol{\varepsilon} | \mathbf{X}] = \sigma^2 \mathbf{I}_T$

(A4) \mathbf{X} has full column rank – $\text{rank}(\mathbf{X}) = k$, where $T \geq k$.

$$\begin{aligned} \text{Objective function: } \min_{\beta} \{ \sum_{i=1}^T \varepsilon_i^2 \} &= \boldsymbol{\varepsilon}' \boldsymbol{\varepsilon} = (\mathbf{y} - \mathbf{X}\beta)' (\mathbf{y} - \mathbf{X}\beta) \\ &= (\mathbf{y}'\mathbf{y} - \beta'\mathbf{X}'\mathbf{y} - \mathbf{y}'\mathbf{X}\beta + \beta'\mathbf{X}'\mathbf{X}\beta) \\ &= (\mathbf{c} - \beta'\mathbf{d} - \mathbf{d}'\beta + \beta'\mathbf{A}\beta) \\ &= (\mathbf{c} - 2\beta'\mathbf{d} + \beta'\mathbf{A}\beta) \} \end{aligned}$$

- First derivative w.r.t. β : $\nabla \sum_{i=1}^T \varepsilon_i^2 = (-2\mathbf{d} + 2\mathbf{A}\beta)$ ($k \times 1$ vector)

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10.5 Economic Applications: Example 2

- Recall the rules for vector differentiation of linear functions and quadratic forms:

(1) **Linear function:** $\mathbf{y} = f(\mathbf{x}) = \mathbf{x}'\gamma + \omega$

where \mathbf{x} and β are k -dimensional vectors and ω is a constant. Then,

$$\nabla f(\mathbf{x}) = \gamma$$

(2) **Quadratic form:** $\mathbf{q} = f(\mathbf{x}) = \mathbf{x}'\mathbf{A}\mathbf{x}$

where \mathbf{x} is $k \times 1$ vector and \mathbf{A} is a $k \times k$ matrix, with a_{ji} elements. Then,

$$\nabla f(\mathbf{x}) = \mathbf{A}'\mathbf{x} + \mathbf{A}\mathbf{x} = (\mathbf{A}' + \mathbf{A})\mathbf{x}$$

If \mathbf{A} is symmetric, then $\nabla f(\mathbf{x}) = 2\mathbf{A}\mathbf{x}$

Now, we apply them to $S(\mathbf{x}; \theta) = \sum_{i=1}^T \varepsilon_i^2 = \boldsymbol{\varepsilon}' \boldsymbol{\varepsilon} = (\mathbf{y} - \mathbf{X}\beta)' (\mathbf{y} - \mathbf{X}\beta)$
 $= (\mathbf{y}'\mathbf{y} - \beta'\mathbf{X}'\mathbf{y} - \mathbf{y}'\mathbf{X}\beta + \beta'\mathbf{X}'\mathbf{X}\beta)$

10.5 Economic Applications: Example 2

- First derivative w.r.t. β : $\nabla S(\mathbf{x}; \theta) = (-2 \mathbf{d} + 2 \mathbf{A} \beta)$ ($k \times 1$ vector)

- F.o.c. (*normal equations*): $-2 (\mathbf{X}'\mathbf{y} - \mathbf{X}'\mathbf{X} \mathbf{b}) = \mathbf{0}$
 $\Rightarrow \mathbf{X}'\mathbf{X} \mathbf{b} = \mathbf{X}'\mathbf{y}$

- Assuming $(\mathbf{X}'\mathbf{X})$ is non-singular –i.e., invertible-, we solve for \mathbf{b} :
 $\Rightarrow \mathbf{b} = (\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}'\mathbf{y}$ ($a k \times 1$ vector)

- Q: Is \mathbf{b} a minimum? We need to check the s.o.c.

$$\frac{\partial}{\partial \beta} [-2(\mathbf{X}'\mathbf{y} - (\mathbf{X}'\mathbf{X})\beta)] = 2 \mathbf{X}'\mathbf{X}$$

- We have already seen that $\mathbf{X}'\mathbf{X}$ is a pd matrix (see next slide if you don't remember).

$$\Rightarrow \mathbf{X}'\mathbf{X} \text{ is pd} \Rightarrow \mathbf{b} \text{ is a min!}$$

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10.5 Economic Applications: Example 2

Definition: A matrix \mathbf{A} is *positive definite* (pd) if $\mathbf{z}'\mathbf{A}\mathbf{z} > 0$ for any \mathbf{z} .

In general, we need eigenvalues to check this (all should be positive).
For some matrices, it is easy to check. Let $\mathbf{A} = \mathbf{X}'\mathbf{X}$.

Then, $\mathbf{z}'\mathbf{A}\mathbf{z} = \mathbf{z}'\mathbf{X}'\mathbf{X}\mathbf{z} = \mathbf{v}'\mathbf{v} > 0$. $\Rightarrow \mathbf{X}'\mathbf{X}$ is pd $\Rightarrow \mathbf{b}$ is a min!

Since $\mathbf{X}'\mathbf{X}$ is pd, then $|\mathbf{X}'\mathbf{X}| > 0$. (Go back to check s.o.c. for MLE)

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