

CHAPTER VII

CURRENCY RISK MANAGEMENT: OPTIONS

In Chapter VI we introduced currency futures and forwards as currency risk management tools. This chapter introduces another risk management tool: currency option contracts. Options are contracts that help a firm to reduce the uncertainty created by having assets and liabilities denominated in foreign currency. An option contract provides a flexibility that a futures contract does not have. An option contract gives its holders a right, not obligation, as is the case of a futures contract.

I. Currency Options

1.A Options: Brief Review

In general, an option gives to the buyer the right, but not the obligation, to buy or sell an asset, whereas the option seller must respond accordingly. Many different types of option contracts exist in the financial world. The two major types of contracts traded on organized options exchanges are *calls* (which gives the holder the right to buy) and *puts* (which gives the holder the right to sell).

The complete definition of an option must clearly specify the *exercise* or *strike price*, X , (the price at which the right is "exercised"), the *expiration date*, and how the option can be exercised. A *European*-type option can only be exercised at expiration date. An *American*-type option can be exercised by the buyer at any time until the expiration date.

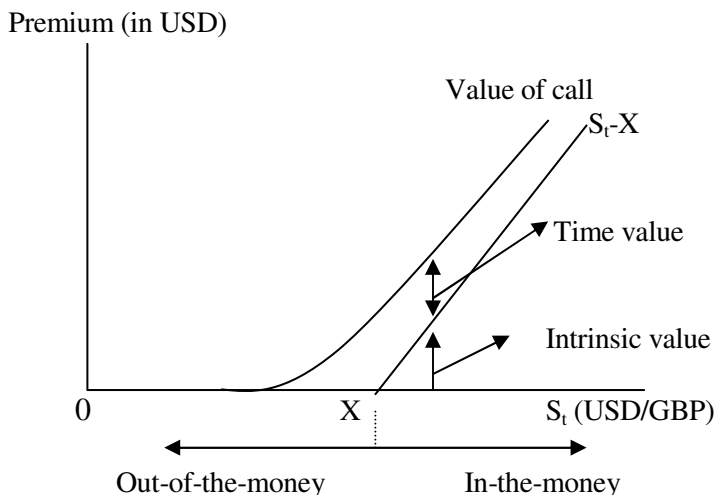
A call (put) option is said to be *in-the-money* if the price of the underlying asset exceeds (is below) the exercise price. On the other hand, if the price of the underlying asset is below (exceeds) the exercise, the call (put) option is said to be *out-of-the-money*. When the current price of the underlying asset is approximately equal to the exercise price, both the call and the put are *at-the-money*.

The option to buy or sell an asset has a price that must be paid at the time of contracting. As we will discuss below, the price of an option, the *premium*, fluctuates over time depending on the value of the underlying asset and other parameters. The *intrinsic value* of an option is defined as the maximum of zero and the value it would have if it were exercised immediately. For a call option, the intrinsic value is $\max(S-X,0)$, where S represents the value of the underlying asset. For a put option, it is $\max(X-S,0)$. Often it is optimal for the holder of an in-the-money American option to wait rather than exercise immediately. The option is said to have *time value*. The total value of an option can be thought of as the sum of its intrinsic value and its time value.

Exhibit VII.1 illustrates the concepts of time value and intrinsic value for a currency call option on the USD/GBP exchange rate.

Exhibit VII.1

Time Value and Intrinsic Value of an American Call Option on the USD/GBP exchange rate



Options are priced using variations of a complex formula called the Black-Scholes formula. An intuitive derivation of the Black-Scholes formula is presented in Appendix VII.

1.A.1 Currency Options

Markets in currency options have become essential for coping with the volatility of the U.S. dollar. Currency options are traded on markets throughout the world, including the U.S., London, Amsterdam, Hong Kong, Singapore, Sydney, Vancouver and Montreal. In all these markets three types of contracts are negotiated:

- i. On December 10, 1980 the Philadelphia Stock Exchange (PHLX) introduced option contracts on currencies. These contracts have enjoyed a spectacular growth. Like futures contracts, these are standardized contracts. In section I.B, below, we discuss PHLX contracts and other institutional issues.
- ii. OTC currency options are not tradeable and can only be exercised at maturity (i.e., they are European-type options). Commercial customers often turn to a bank when they need a large number of options of this type for a specific date. For example, on October 31, 1994, a Canadian exporter expects a payment of USD 10 million in three months, --for example, February 10, 1995. Listed options do not offer this specific date, and the amount involved (270 contracts) may be too large for the usual volume of transactions.
- iii. Other listed currency options are options on currency futures contracts. For example, since 1985, the CME trades options on its own currency futures.

1.A.2 Factors Affecting Currency Option Prices (premium)

Currency options are priced using a variation of the Black-Scholes formula for stock prices. The main revision comes from the fact that the opportunity cost to invest in a foreign currency is not the domestic risk-free rate, as for an ordinary asset, but rather the interest rate differential (domestic minus foreign). The intuition behind this modification is very simple: an investment in a foreign currency costs the domestic interest rate (to finance the purchase of currency) but earns the foreign interest rate. The Black-Scholes formula for European currency call options is given by:

$$C = e^{-i_f T} S N(d1) - X e^{-i_d T} N(d2),$$

where

$$d1 = [\ln(S/X) + (i_d - i_f + .5 \sigma^2) T]/(\sigma T^{1/2}),$$

$$d2 = [\ln(S/X) + (i_d - i_f - .5 \sigma^2) T]/(\sigma T^{1/2}).$$

The constants i_d , i_f , and σ are the continuously compounded riskless domestic interest rate per unit time, the continuously compounded riskless foreign interest rate per unit time and the standard deviation of the rate of return on the stock per unit time, respectively. $N(d1)$ and $N(d2)$ represent the cumulative normal distribution evaluated at points $d1$ and $d2$, respectively.

The well-known put-call parity relationship implies that the price of a European currency put option, P , is given by:

$$P = C - e^{-i_f T} S + X e^{-i_d T}.$$

Note that there are six factors affecting the premium of a currency option:

- i. the current exchange rate (S)
- ii. the strike price (X)
- iii. the time to expiration (T)
- iv. the volatility of the exchange rate (σ)
- v. the domestic interest rate (i_d)
- vi. the foreign interest rate (i_f)

Table VII.1 presents the effect on the price of an option of increasing one variable while keeping all others fixed.

TABLE VII.1

Effect on the Premium of a Currency Option of a Ceteris Paribus Increase in One Variable

Variable	European Call	European Put	American Call	American Put
S	+	-	+	-
X	-	+	-	+
T	?	?	+	+
σ	+	+	+	+
i_d	+	-	+	-
i_f	-	+	-	+

Table VII.1 is very easy to understand. For example, if a call currency option is exercised at some time in the future, the payoff from the call will be the amount by which the exchange rate exceeds the strike price (X). Call options on foreign currency become more valuable as the exchange rate increases and less valuable as the strike price increases. For a put currency option, when exercised, the payoff is the amount by which the strike price exceeds the exchange rate. Then, put options on foreign currency become less valuable as the exchange rate increases and become more valuable as the strike price increases. Similar arguments can be made for all the other variables.

In addition to the above factors, the right to exercise at any time before expiration embedded in American options has a price. Therefore, American options are always worth more than their European counterparts.

Example VII.1: Using the Black-Scholes formula to price currency options

It is September 10, 1999. Nairong Co. wants to price an European GBP put option. The GBP put option has a strike price of 1.62 USD/GBP and matures in 40 days. In addition, Nairong has the following information: the 40-day Euro-USD rate is 4.79%, the 40-day Euro-GBP rate is 5.83%, the annual USD/GBP volatility during the recent past was 8%, and today's exchange rate is 1.6186 USD/GBP.

Nairong Co. summarizes all the information:

$$S_t = 1.6186 \text{ USD/GBP}$$

$$X = 1.62 \text{ USD/GBP.}$$

$$T = 40/365 = .1096.$$

$$i_d = .0479.$$

$$i_f = .0583.$$

$$\sigma = .08.$$

Nairong obtains C, the call's premium, using the following steps:

(1) Calculate d1 and d2.

$$d1 = [\ln(1.6186/1.62) + (.0479 - .0583 + .5 \cdot .08^2) \times .1096] / (.08 \times .1096^{1/2}) = -0.062440,$$

$$d2 = [\ln(1.6186/1.62) - (.0479 - .0583 + .5 \cdot .08^2) \times .1096] / (.08 \times .1096^{1/2}) = -0.088923,$$

(2) Calculate N(d1) and N(d2).

Nairong uses the Normal Table and draws a normal distribution. First, recall that the normal distribution is symmetric around zero and, therefore, the Normal Table tells the area under the right tail of the normal curve. If d1 is negative (positive), Nairong will subtract (add) the .50% of the distribution under the left tail of the normal curve to the area under obtain from the Normal Table. In this way, Nairong obtains the cumulative area N(d1).

Second, Nairong looks for the cumulative normal distribution at $z = -0.06244$. The number obtained is .02489. But, since d1 (z) is negative, in order to obtain the whole area under the curve up to $z = -0.06244$, Nairong subtract .02489 to .50. Nairong uses a similar procedure to calculate d2. Then,

$$N(d1 = -0.06244) = .47511,$$

$$N(d2 = -0.088923) = .46457.$$

(3) Calculate C and P.

Replacing in the Black-Scholes formula, Nairong obtains:

$$C = e^{-0.05830 \times 1.096} 1.6186 .47511 - 1.62 e^{-0.04790 \times 1.096} .46457 = \text{USD } .015444.$$

Now, using the put-call parity equation, Nairong determines P:

$$P = .01544 - e^{-0.05830 \times 1.096} 1.6186 + 1.62 e^{-0.04790 \times 1.096} = \text{USD } .01867. \quad \square$$

1.A.3 Practical Considerations

The Black-Scholes model assumes a constant risk-free interest rate. This constant risk-free rate applies to all maturities. In practice, the risk-free rate has been usually approximated by the yield of a government bond maturing with the option. In the U.S., the government bond of choice used to be Treasury bills or bonds. Now, because of liquidity considerations, most traders use LIBOR.

The Black-Scholes model needs as an input the annualized variance, which is forward looking – i.e., the variance from today till maturity (T). Historical estimation of the variance is very common. It is usually better to work with “log returns” –i.e., $\log(1+\text{return})$. Remember to translate the estimated variance to an annual variance. For example, suppose you use monthly returns and your computed (monthly) variance is .0051. Then, the annualized variance is $.0051 \times 12 = 0.0612$, which corresponds to annual standard deviation of $\sqrt{0.0612} = .247386$ (or, 24.74%).

The cumulative normal distribution function is usually taken from a statistical Table, like in example VII.1. There are many numerical approximations. A simple and accurate approximation is given by Abramowitz & Stegun (1970), where they present the following procedure: for $x \leq 0$. For $x \geq 0$, use the procedure on $-x$ and the formula $N(x) = 1 - N(-x)$.

First, compute t , defined as $t = 1/(1 - 0.33267x)$.

Second, use t and x to compute

$$N(x) \approx (0.4361836 t + 0.1201676 t^2 + 0.937298 t^3) \cdot \exp(-x^2/2) / \sqrt{2\pi}.$$

For example, let $x = -1.64$. Then, $t = 1/(1 - 0.33267 \times (-1.64)) = 0.6470068$. Finally, we plug in x and t in:

$$N(x = -1.64) = (0.4361836 \times 0.6470068 - 0.1201676 \times 0.6470068^2 + 0.937298 \times 0.6470068^3) \cdot \exp(-1.64^2/2) / \sqrt{2\pi} = 0.050501730,$$

which is very close to the actual value of 0.050502583.

1.B Trading in Currency Options

The market for foreign currency options mainly consists of an interbank market centered in London, New York, and Tokyo, and exchange-based markets centered in Philadelphia (PHLX,

since 2007 part of NASDAQ), in New York, (the International Securities Exchange or ISE) and Chicago (CME Group). At the European Options Exchange (EOE) there are exchange-traded options for some currencies; however the only significant volume is for minor currency options. The CME offers options on currency futures.

The expiration dates of most foreign exchange options contracts are likewise set to correspond to the March, June, September, and December delivery dates on CME foreign exchange futures. PHLX foreign exchange options are opened with terms to maturity of one, three, six, and twelve months. As a consequence, options contracts expiring in March, June, September, December, and, in addition, the two nearby months not part of this cycle, are always trading. Exchange traded options have standardized strike price intervals, in addition to standardized expiration dates and currency amounts. PHLX options are on spot amounts of 10,000 units of foreign currency (100,000 for the MXN, 1,000,000 for the JPY). The exercise price of an option at the PHLX or CME is stated as the price in U.S. cents of a unit of foreign currency.

As with exchange-traded futures contracts, exchange-traded options are registered with a clearinghouse that guarantees both the long and short sides of puts and calls. For example, PHLX contracts are guaranteed by the Options Clearing Corporation (OCC).

OTC options, by contrast, can be tailor-made as to amount, maturity, and exercise price. The currency amount involved in OTC options is usually much larger than exchange-traded options (typically involving a minimum USD 1 million or so of foreign currency) and OTC options have been written on a variety of currencies.

1.B.1 Newspaper Quotes

Example VII.2: On September 10, 1999 the *Wall Street Journal* published the following option quotes for several currency options traded on Philadelphia Stock Exchange:

OPTIONS					
PHILADELPHIA EXCHANGE					
		Calls		Puts	
		Vol.	Last	Vol.	Last
Euro					105.54
10,000 Euro-cents per unit.					
102	Oct	...	0.01	3	0.38
104	Sep	3	0.74	90	0.15
104	Oct	3	1.90
104	Dec	...	0.01	25	1.70
106	Dec	8	1.85	12	2.83
108	Oct	75	0.43	...	0.01
112	Dec	1	0.08	1	7.81
Australian Dollar					65.37
10,000 Australian Dollars-cents per unit.					
64	Oct	...	0.01	20	0.31
65	Sep	20	0.30
66	Oct	30	0.42	...	0.01
British Pound					161.86
10,000 British Pounds-European Style.					
161	Sep	32	0.82
162	Oct	32	1.54	...	0.01
10,000 British Pounds-cents per unit.					
159	Oct	4	0.63
161	Sep	4	0.94	...	0.01

Recall that an option is usually defined by the underlying asset, the exercise price (X), and the expiration date (T). In the above WSJ example, the underlying asset is the foreign currency, exercise prices are given in the first column, and expiration dates are given in the second column.

Consider the (American) option contract on the Euro (EUR). On September 10, 1999, the spot price is 1.0554 USD/EUR. Ninety calls and one hundred thirty one puts are traded. The first call (put) contract represents the right to buy (sell) EUR 10,000 at USD 1.02 on October. The Oct 102 call is in-the-money since the spot rate exceeds the strike price. On the other hand, the Oct 102 put is out-of-the-money, since spot rate exceeds the strike price ($1.554 > 1.02$). Three puts were traded, the last contract was sold for a value of .38 cents per Euro. That is, the last put contract was bought by $\text{USD } 38.00 = \text{USD } .0038 \times 10,000$. No calls were traded. ¶

1.C Using Currency Options

Options contracts are routinely used to hedge an underlying position or to speculate on the future direction of the exchange rate. In this book we will emphasize hedging. A currency option can reduce currency risk.

Example VII.3: Iris Oil Inc., a Houston-based energy company, will transfer CAD 300 million to its USD account in 90 days. To reduce currency risk, Iris Oil decides to use a CAD option contract. Bank Two offers the following 90-day USD/CAD currency options:

<u>X</u>	<u>Calls</u>	<u>Puts</u>
.82 USD/CAD	----	0.21
.84 USD/CAD	1.58	0.68
.88 USD/CAD	0.23	----

Iris Oil decides to use the .84 USD/CAD put. The total premium cost is:
 $\text{USD } 0.0068 \times 300 \text{ M} = \text{USD } 2.04\text{M}.$

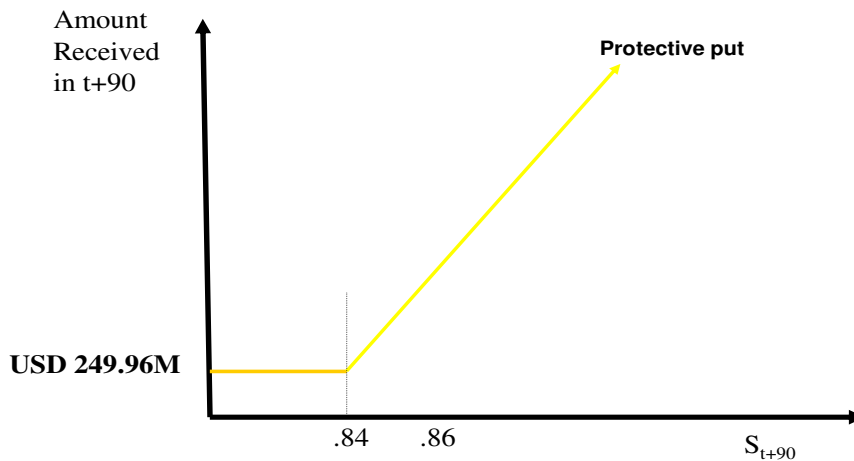
At $t+90$, there will be two situations: Option is ITM (exercised) or OTM (not exercised):

	<u>If $S_t < .84 \text{ USD/CAD}$</u>	<u>If $S_t > .84 \text{ USD/CAD}$</u>
Option CF:	$(.84 - S_t) \text{ CAD } 300\text{M}$	0
Plus	$S_t \text{ CAD } 300\text{M}$	$S_t \text{ CAD } 300\text{M}$
<u>Total</u>	<u>USD 252M</u>	<u>$S_t \text{ CAD } 300\text{M}$</u>

The cash flows in 90 days are:

USD 249,960,000	for all $S_t < .84 \text{ USD/CAD}$
$S_t \text{ CAD } 300\text{M} - \text{USD } 2.04\text{M}$	for all $S_t > .84 \text{ USD/CAD}$

Payoff Diagram for Iris Oil



There is still uncertainty regarding the final amount Iris Oil will transfer to its USD account. Iris Oil has set a floor, a worst case scenario: USD 249,960,000. There is an upside to the position. If S_t is higher than .84 USD/CAD, the option will not be exercised and Iris Oil will receive a higher amount than the floor: $S_t \text{ CAD } 300\text{M}$. ¶

II. Hedging with Currency Options

In Chapter VI we noted that hedging with futures contracts was very simple: one takes a position with a foreign exchange contract that is the reverse of the principal being hedged. We have been introduced to hedging with options in Chapter I. Hedging with options is also very simple. Suppose Cannigia Co., a U.S. firm, is long CHF for three months (say, Cannigia will receive a payment of CHF 20 million). Cannigia may buy today the right to sell those CHF at a given price in three months. That is, Cannigia would buy a CHF put. On the other hand, Balbo

Co., another U.S. firm, who is short CHF for one month (say, Balbo has to pay CHF 10 million), might want to buy today the right to buy those CHF at a given price in one month. That is, in this case, Balbo would buy a CHF call.

Example VII.4: It is September 12. A U.S. investor wants to hedge a GBP 1 million investment in British gilts (U.K. government bonds). On the PHLX he/she buys GBP puts for December at 160 ($X=160$ USD cents per GBP), at a premium of USD.05. $S_t=1.60$ USD/GBP. The GBP puts are in-the-money. On the PHLX, one contract covers GBP 10,000. Therefore, he/she should buy 100 contracts.

Suppose that GBP drops to 1.50 USD/GBP at expiration –i.e., below $X=1.60$ USD/GBP. Then, a profit is made on the put that exactly offset the currency loss on the portfolio. As shown below, the USD loss in the portfolio is USD 100,000.

Date	S_t	Long Position (“Underlying”)
September 12	1.60	1,600,000
Dec Expiration	1.50	1,500,000
Gain		-100,000

The gain in the hedging position (put position) is:

$$(1.60 - 1.50) \text{ USD/GBP} \times \text{GBP } 10,000 \times 100 = \text{USD } 100,000.$$

The premium keeps the investor from having a perfect hedge. The net profit on the put purchase equals the gain at expiration minus the premium. If we call V_0 , the number of GBP put options, and P_0 the premium, the net dollar profit on the put at the time of exercise t is:

Net dollar profit = $V_0(X-S_t) - V_0P_0$, that is, we have:

$$\text{GBP } 1,000,000 \times (1.60 - 1.50) \text{ USD/GBP} - 1,000,000 \times \text{USD } .05 = \text{USD } 50,000.$$

This profit does not cover the currency loss on the portfolio (equal to roughly USD 100,000) because the option premium cost USD 50,000. It would have been less profitable if the pound had dropped to only 1.59 USD/GBP: $1,000,000 \times (1.60 - 1.59) - 1,000,000 \times .05 = \text{USD } -40,000$. ¶

The advantage of buying options over buying futures is that options simply expire if the GBP appreciates rather than depreciates.

Example VII.5: Reconsider Example VII.4. If at expiration the GBP moves to USD 1.80, the option is not exercised, but a futures contract with a price of 1.60 USD/GBP will generate a loss of USD 200,000, nullifying the currency gain on the portfolio of assets. ¶

In Example VII.4, the U.S. investor is taking a position on currency options, which is exactly equal in size to the position in the underlying (cash) position. That is, the hedge ratio is equal to one. Similar to hedging with forward and futures currency contracts, if there is a perfect match between the maturities of the cash position and the option position, a hedge ratio equal to one is quite appropriate. We will see, below, that for several hedging situations a hedge ratio equal to one will not provide an optimal hedge.

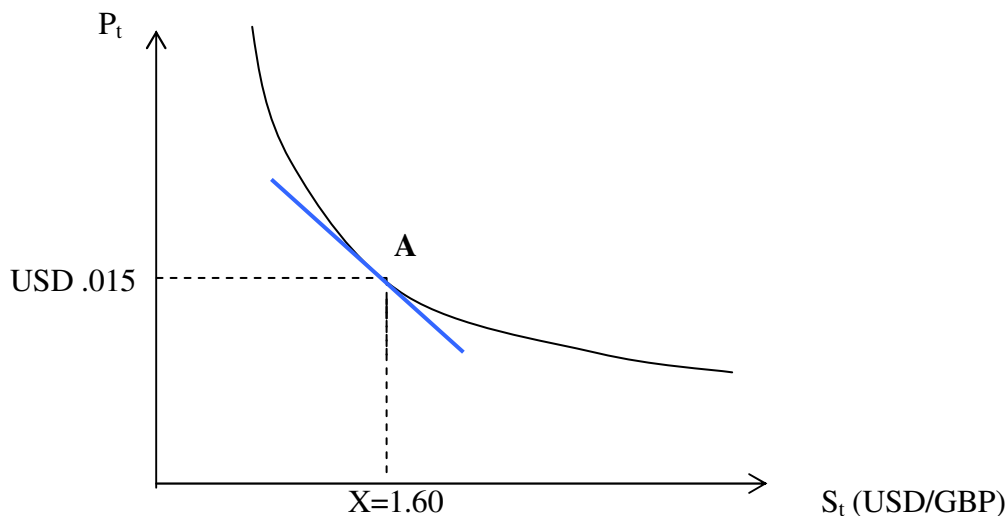
The approach to hedging used in this section uses options as insurance contracts, since they provide a floor or a cap. Options protect a portfolio in case of adverse currency movements, as do currency futures, and maintain its performance potential in case of favorable currency movements, while futures hedge in both directions. The price of this asymmetric advantage is the insurance cost implicit in the time value of the option. In this section, we treated the option premium as a sunk cost. Note, however, that options are usually resold on the market rather than left to expire, and when the option is resold part of the initial insurance premium is recovered.

2.A Dynamic Hedging with Options

Similarly to futures contracts, listed options are continually traded. An option position is usually closed by reselling the options in the market instead of exercising them. The profit on a position is the difference between the two market premiums and, therefore, completely dependent on market valuation. The modern approach to currency option hedging is similar to the dynamic approach to hedging with futures. Dynamic option hedging recognizes the fact that options are continuously traded, and is based on the relationship between changes in option premiums and changes in exchange rates. The objective of dynamic option hedging is to match changes in S_t with changes in currency option prices.

Recall the Black-Scholes formula. There is a relation between an option premium and exchange rates. Unlike the relation between $F_{t,T}$ and S_t , the option premium is non-linearly related to the underlying exchange rate. Graph VII.1 shows the relationship we usually observe between P_t (put premium) and S_t .

GRAPH VII.1
Value of a Pound Put in Relation to the Exchange Rate



Beginning with a specific S_t , say 1.60 USD/GBP, a put premium can go up or down in response to changes in the exchange rate. The slope of the curve at point A denotes the elasticity of the premium to any local movement in the USD/GBP exchange rate. In Graph VII.1, the premium is equal to 1.5 USD cents when $S_t=1.60$ USD/GBP, and the tangent at point A equals -0.5. This

slope is called *delta*, Δ . That is, delta represents the expected change in the option premium for a given small change in S_t .

Suppose that for every GBP of British gilts an investor buys one GBP put. One GBP put is defined here as a put option on one unit of British currency. One contract includes several unit puts, depending on the contract size (on the PHLX, one contract includes 10,000 GBP puts). The hedging position (the long put hedging the long position on British gilts) is hedged only at expiration. In the meantime, there is residual exposure, since for every 1 USD cent the GBP depreciates, each put goes up by only .5 USD cents.

A perfect options hedge involves a situation where every USD loss in the underlying position due to changes in the exchange rate is covered by a USD gain in the value of the options position. In this example, a perfect hedge would be achieved by buying two GBP puts for every GBP of British gilts in the underlying position. If the GBP depreciates by 1 USD cent, each put will go up by approximately .5 USD cents, offsetting the currency loss on the portfolio.

◆ **Delta and Small Changes in Exchange Rates**

The option delta measures the amount the option changes relative to a small change in the spot rate (for options on spot) or futures prices (for options on futures). The option price change is always less in absolute value than the price change in the spot or future. Therefore, an option delta has an absolute value between 0 and 1. For example, suppose a put option on the GBP has a delta value of -0.5. This delta value implies that if the spot decreases by USD .01, say from 1.60 to 1.59 USD/GBP, the put option value will increase by USD $-.01 \times (-0.5) = \text{USD } 0.005$. ◆

In general, if n GBP options are purchased, the gain on the hedging position is:

$$\text{Gain} = n \times (P_t - P_0),$$

where P_t is the put value at time t . For small movements in the exchange rate, we can write:

$$P_t - P_0 = \Delta(S_t - S_0) \tag{VII.1}$$

Following the logic we used to determine the optimal hedge ratio for futures contracts, the options hedge ratio is equal to $(-1/\Delta)$. For example, at point **A**, in Graph VII.1, the hedge ratio is $(-1/-0.5)=2$. Therefore, an optimal currency hedge is obtained by holding $n = -(V_0/\Delta)$ options. Note that the value of the hedging position changes by:

$$P_t - P_0 = -V_0 \times (S_t - S_0),$$

which exactly offsets the change in the value of the underlying cash position. This method is called *delta hedging*.

Example VII.6: Suppose we want to hedge GBP 1 million, and Δ is given by point **A**, in Graph VII.1, where $P_t = \text{USD } .015$ and $S_t = 1.60 \text{ USD/GBP}$. Then, $n = -(1,000,000/-0.5) = 2,000,000$. The number of contracts is:

$n/\text{size of contract} = 2,000,000/10,000 = 200$ contracts.

Now, suppose that the GBP depreciates by USD .01. Therefore, P_t increases approximately by USD .005, that is, $P_t = \text{USD } .02$. The change in the hedging (put) position is:

$$2,000,000 \times \text{USD } (.02 - .015) = \text{USD } 10,000.$$

The change in the underlying GBP position is: $1,000,000 \times (1.59 - 1.60) = \text{USD } -10,000$. ¶

We must note that Δ and the hedge ratio change with the exchange rate. Thus, every time exchange rates change, the number of options held must be adjusted.

Example VII.7: Go back to point A, in Graph VII.1 the hedging ratio was 2 and $n=2$ million. If the pound depreciates, options protect the portfolio but its Δ changes. Suppose S_t drops to 1.55 USD/GBP, the slope moves to -0.8. Therefore, the hedge ratio is 1.25 ($=-1/-0.8$) and $n=1,250,000$. Now, the number of contracts is:

$$n/\text{size of contract} = 1,250,000/10,000 = 125 \text{ contracts.}$$

To avoid over hedging, the investor sells 75 put contracts and realizes a profit. ¶

Exchange rates change continually, thus, hedge ratios should also be adjusted continually. A continuous adjustment, however, is expensive. In practice, a good hedge can be achieved only with periodical revisions in the option position, that is, when there is a significant swing in exchange rates (say 2% or more). Between revisions, options offer their usual asymmetric insurance within the general hedging strategy. This strategy may be seen as a mixed hedging-insurance strategy.

2.A.1 Using Delta to Measure the Exposure of a Trader

Delta is the first derivative of the premium with respect to the underlying spot price. That is, for European call options, $\Delta=N(d1)$, while for European put options, $\Delta=N(d1)-1$. Therefore, since $N(d1)$ represents an area under the Normal distribution, the delta of a call option is positive, while the delta of a put option is negative. The absolute value of delta is always between zero and one. For a deep in-the-money call option, Δ is close to 1. For an at-the-money call option, Δ is close to .5. For an out-of-the-money call option, Δ is close to zero. The same values hold for put options, for which the values of delta are negative. By definition, the delta of a long position in the underlying asset is equal to 1.

Example VII.8: Calculating delta.

Go back to Example VII.2, where we valued an European option. We have the following data:

$$S_t = 1.6186 \text{ USD/GBP,}$$

$$X = 1.62 \text{ USD/GBP,}$$

$$T = 40/365 = .1096.$$

$$i_d = .0479.$$

$$i_f = .0583.$$

$$\sigma = .08.$$

The option is an at-the-money option, $\Delta = N(d1) = .47511$, which is close to 0.5 as expected.

Now, if $S_t = 1.70$ USD/GBP (the option is deep in-the-money), $\Delta = N(d1) = .9633$, which is close to 1, as expected. On the other hand, if $S_t = 1.55$ USD/GBP (the option is deep out-of-the-money), $\Delta = N(d1) = .0448$, which is close to 0, as expected. ¶

Delta measures the sensitivity of the option value to changes in the price of the underlying asset. Measuring this exposure, Δ , is an important part of the risk management of options. Traders, for example, continuously take positions in different instruments, including options with different strike prices or maturities. It would be difficult -or too expensive- to have a zero net exposure in each traded option. Instead, traders use Δ to offset positions in different options.

The net delta of a position represents an equivalent long or short position in the underlying foreign currency (cash) position. Therefore, hedgers use delta to determine their actual foreign exchange exposure.

Example VII.9: Measuring exposure with puts and calls.

Suppose it is September 20. The spot exchange rate is 1.6186 USD/GBP. Suppose Mr. Krang is long five 162 Oct GBP European calls on spot GBP, with a $\Delta = 0.475$, and also long three 162 Oct European puts on spot GBP, with a $\Delta = -0.525$. Then, the *net* delta is:

$$5x(0.475) + 3x(-0.525) = 0.80.$$

Since a standardized GBP represents GBP 10,000, the net position is equivalent to a long position in spot GBP of $(0.80)x(\text{GBP } 10,000) = \text{GBP } 8,000$. ¶

Because the net delta of a position represents the equivalent long or short position in foreign currency, it is a measure of one's total portfolio risk. For that reason, the CME uses delta factors as a basis for margin requirements.

Traders try to attain a minimum exposure -hopefully, a *delta-neutral* position. An investor is said to be *delta-neutral*, when the investor's position has a net delta of zero. For small changes in the spot rate, the change in the value of the position will be zero. In this case, an investor is fully hedged for small changes in the spot rate.

Example VII.10: Measuring exposure with options on futures and futures

Suppose the delta on an American call on futures is .400. A trader is long 2 CME CHF futures contract and short 3 CME CHF call options on futures, the *net* delta of the position (per CHF) is:

$$2 + (-3)x(.400) = .800.$$

That is, the total net position to a long position is .800 CME CHF futures contracts. There are CHF 125,000 in a futures contract. Thus, the net position in futures is equivalent to a long position in futures of $(.800)x(\text{CHF } 125,000) = \text{CHF } 100,000$.

Note: We are using the fact that the delta of a futures contract when the underlying position is also in futures contracts is equal to one.

A trader can achieve a delta-neutral hedge. A CHF 250,000 long futures position hedged by 5 short CME CHF call options on futures produces a net delta of

$$2 + (-5) \times (.400) = 0. \quad \square$$

We should note that the delta value represents a slope -see point A in Graph VII.1. Therefore, a delta-neutral (perfectly hedged) position produces no profit if the change in the exchange rate is small. Then, there is a residual exposure through the speed at which Δ changes. This speed is measured by the *option gamma*.

2.A.2 Gamma

The option *gamma* measures how curved a graph relating the spot price to the option value is. Equivalently, gamma measures the amount the option delta changes relative to a small increase in the spot rate (for options on spot) or futures prices (for options on futures). It is a measure of the second derivative of the value of the option with respect to the spot rate (for options on spot) or futures prices (for options on futures), and then it is a valid measure for small changes in the price of the underlying asset.

Long option positions have the gamma positive. If the gamma is positive, then the delta will increase with an increase in the underlying price and decrease with a decrease in the underlying price.

Example VII.11: Consider the options of Example VII.9. Suppose the gamma of the GBP call option on the GBP is .065 (recall, $\Delta=.475$). If the spot rate increases by USD .01 (1 USD cent) from 1.6186 to 1.6286, the delta will increase from .475 to .540 ($=.475 + (1) \times .065$). If an investor is long a 162 Oct GBP call option, then a USD .01 increase in S_t means a change in the long position of GBP 650 ($= \text{GBP } 10,000 \times .065$). \square

If the original delta were zero while the gamma was positive, then a rising (falling) price would result in a positive (negative) delta position. This is a very good situation: an investor would like to have a positive delta (net long position) if prices are rising but a negative delta (net short position) if prices are falling.

Example VII.12: Consider an investor's position consists of a short spot CAD position of CAD 50,000, 1 long CAD call with $\Delta=.60$ and $\gamma=.15$, and a short CAD put with $\Delta=.40$ and $\gamma=.10$.

The net delta for the position is: $\Delta = -1 + .60 + (-1) \times (-.40) = 0$.

The net gamma is: $\gamma = .15 + (-1) \times (.10) = .05$

If the spot rate increases by USD .02, the net delta will change to .10 ($=0 + (2) \times .05$). That is, the investor has a net long position.

If the spot rate decreases by USD .02, the net delta will change to -.10 ($=0 + (-2) \times .05$). That is, the investor has a net short position. \square

2.B Hedging Strategies

Hedging strategies with options can be more sophisticated than those with futures: an investor can play with several maturities and exercise prices with options only. For example, hedgers can hedge with out-of-the-money, at-the-money, or in-the-money options. These different options have different prices and they also provide different coverage for the exposed cash flows of a given firm. Firms hedging with option face the same trade-off that buyers of insurance policies face when they select insurance coverage. Insurance with a high deductible is cheap, but the coverage starts at a high floor. On the other hand, insurance with a low deductible is more expensive, but the coverage starts at a low floor. This trade-off is also seen when hedgers use options to hedge. For example, out-of-the-money options are cheaper than at-the-money options, but they provide a lower degree of insurance protection to the company's cash flows.

Example VII.13: It is September 10, 1999. Suppose we want to hedge a long bond position worth EUR 1 million using Dec options (see Example VII.2). Suppose we can to use the options that are closer to $S_t = 1.0554$ USD/EUR. Thus, we can use the out-of-the-money Dec 104 or the in-the-money Dec 106.

(A) Out-of-the-money Dec 104 put.

Total cost = USD .0170 x 1,000,000 = USD 17,000

Floor = 1.04 USD/EUR x EUR 1,000,000 = USD 1,040,000.

(B) Out-of-the-money Dec 106 put.

Total cost = USD .0283 x 1,000,000 = USD 28,300

Floor = 1.06 USD/EUR x EUR 1,000,000 = USD 1,060,000

The typical trade-off is seen in this example. A higher minimum amount for the long gilt position (USD 1,060,000) is achieved by paying a higher premium (USD 28,300). ¶

Hedgers are aware that the hedge ratio changes with S_t . In a dynamic hedging situation, a popular strategy to adjust the hedge ratio is to keep a fixed number of options, but replace in-the-money option with cheap out-of-the-money options to maintain the same hedge ratio. For instance, in the Example VII.7, if the GBP later reverses its depreciation against the USD, the puts will become worthless; however, most of the profit will have been locked in and saved.

2.C Exotic Options

Exotic options are complex options that typically incorporate two or more option features. For example, a compound option (an option on an option) is considered an exotic option. It provides its holder with the right to buy another option. Two popular exotic options in currency markets are the knock-outs and the knock-ins. These two exotic options are just an example of barrier options.

2.C.1. Barrier Options: Knock-outs/Knock-ins

Barrier options are options where the payoff depends on whether the underlying asset's price reaches a certain level during a certain period of time. We will consider two types of barrier options: knock-out and knock-in options.

Knock-out: A standard option with an "insurance rider" in the form of a second, out-of-the-money strike price. This "out-strike" is effectively a stop-loss order: if the strike price is crossed by the cash-market spot price, the option contract ceases to exist. *Knock-ins* are just the reverse. The option contract does not exist unless and until the spot market price crosses the out-of-the-money "in-strike" price that triggers the contract.

Example VII.14: Suppose that the spot rate is 1.60 USD/GBP. Consider the following European option: a 1.65 USD/GBP March GBP call knock-out 1.75 USD/GBP. If in March the spot rate is 1.70 USD/GBP, the option will be exercised, and the writer of the option will lose USD .05 per GBP sold. But, if in March the spot rate is 1.75 USD/GBP or higher, the option is canceled. ¶

Think of knock-outs/knock-ins as insurance policies with higher deductibles and lower premiums. While nobody currently expects the USD/GBP to go to 2.00 or 1.00, knock-out or knock-in options -just in case- are a normally inexpensive form of risk protection. They also enable traders to maintain a position not just at a strike price but across a range of prices. You may think that the GBP when at 1.60 USD/GBP will go to 1.70 USD/GBP -but it might first dip to 1.50 USD/GBP. Knock and kick options make it possible to hold that position open despite fluctuations in the market.

III. Looking Ahead: Which hedging technique is better?

We have gone over two basic hedging tools: futures and options. Note that options give us a family of tools, since we have different strike prices to choose from. We have seen how futures and options help a firm or an investor to reduce currency risk. Option hedges tend to be more expensive than futures hedges. The choice of an instrument, however, will depend not as much on the cost of the hedge, but on the underlying situation that makes currency risk arise. For example, if a domestic firm acquires a foreign firm but the operation is subject to approval from foreign regulators, the best hedge is probably an option hedge, because it allows the domestic firm the freedom of not exercising the options contract if the acquisition does not go through.

Futures and options are instruments designed by financial markets. Firms can also limit their currency risk exposure by using other non-market techniques. Firms can organize the operation of the firm to reduce the overall currency risk faced by the firm. For example, a firm can achieve a very good match between payables and receivables denominated in foreign currency. Non-market techniques could have saved Laker Airways from bankruptcy.

These are the issues waiting for us in Chapter VIII.

Interesting readings

Parts of Chapter VII were based on the following books:

International Financial Markets, by J. Orlin Grabbe, published by McGraw-Hill.

International Financial Markets and The Firm, by Piet Sercu and Raman Uppal, published by South Western.

International Investments, by Bruno Solnik, published by Addison Wesley.

APPENDIX VII

OPTIONS: BLACK-SCHOLES AND THE DIFFERENT INSTRUMENTS

There are many option pricing models. The pioneering option pricing model of Black and Scholes (1973) has become the most popular model to price not just options, but derivatives in general. In the usual Black-Scholes environment the underlying asset return is assumed to follow a lognormal random walk. Let P be the underlying asset price that follows a geometric Brownian motion

$$dP = \mu P dt + \sigma P dz(\tau),$$

where $z(\tau)$ is a standard Gauss-Wiener process (think of it as a normal variable with zero mean and variance which equals the difference between the future time and current time), and μ and σ are the instantaneous mean and standard deviation of the underlying asset price, respectively.

Using equation using the standard method delivers

$$P(\tau) = P \exp [(\mu - \frac{1}{2}\sigma^2) \tau + \sigma z(\tau)].$$

Taking logarithm on both sides of the above equation yields:

$$\ln (P(\tau)/P) = (\mu - \frac{1}{2}\sigma^2) \tau + \sigma z(\tau),$$

which is normally distributed. This is why the Black-Scholes model is also called a lognormal model. The lognormal distribution is skewed to the left and the variables cannot be negative.

The Black-Scholes model requires many assumptions. Besides the lognormal assumption, the model requires:

- i. the short term interest rate is known and is constant.
- ii. the underlying asset pays no dividend
- iii. the option is European.
- iv. no transaction costs.
- v. no penalties for short selling.
- vi. it is possible to borrow any fraction of the price of the security to buy it or hold it. Borrowing cost is the short-term interest rate.
- vii. trading can be carried on continuous basis.

Of all the assumptions, the last one –continuous trading- is a big one. It makes possible the use of continuous time calculus. With these assumptions, Black and Scholes derived the price for a European call option, C , using a no-arbitrage argument. This formula is given by:

$$C = P N(d1) - X e^{-r\tau} N(d2),$$

where

$$d1 \equiv \frac{\ln(p/ X e^{r\tau})}{\sigma \sqrt{\tau}} + \frac{1}{2} \sigma \sqrt{\tau}$$

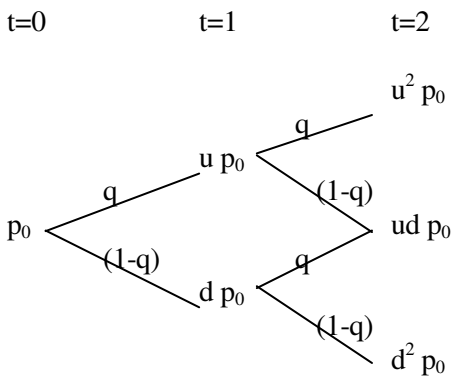
$$d2 \equiv \frac{\ln(p/ X e^{r\tau})}{\sigma \sqrt{\tau}} - \frac{1}{2} \sigma \sqrt{\tau}$$

where $N(z)$ is the standard normal distribution function.

A.VII.1 Binomial Model: Derivation of the Black Scholes formula

Assumptions of the model:

- i. Two long-lived assets: a risky stock and a riskless bond, with prices p_t and B_t .
- ii. We concentrate on the time interval: $t = 0, 1, \dots, T$.
- iii. The riskless bond earns a constant return, $R = (1+i_{rf})$ -where i_{rf} is the riskless rate.
- iv. The stock does not pay dividends until time T .
- v. The stock can move up by u or down by d . ($u > (1+i_{rf}) > 1 > d$).
- vi. The information in the economy is summarized by an event tree. For example:



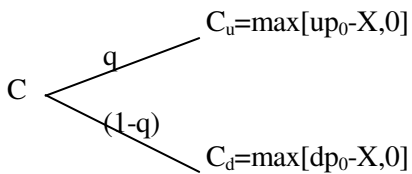
Note I: q represents the binomial probability for the stock to move up. Recall that the probability that a stock has n at time T upward moves is given by $q^n (1-q)^{T-n}$.

Note II: Following note I, we have that the number of complete realizations that have exactly n upward moves is:

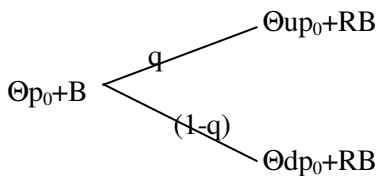
$$\binom{T}{n} \equiv \frac{T!}{n!(T-n)!}$$

A.VII.1.A One-period formulation

To see how to value a call on the stock, we start with the simplest situation: the expiration date, T , is the next period. To simplify notation, let now be time $t=0$ and $T=1$. The value of a call option, C , can take at time $T=1$ two values:



We form a portfolio containing Θ share of stock and the dollar amount B in the riskless bond. This will cost $\Theta p_0 + B$. At the end of the period the value of this portfolio will be



We can choose B and Θ in any way we wish. Suppose we choose them to equate the end-of-period values of the portfolio and the call for each possible outcome:

$$\begin{aligned}\Theta up_0 + RB &= C_u \\ \Theta dp_0 + RB &= C_d\end{aligned}$$

Solving these equations, we find

$$\Theta = [C_u - C_d]/[(u-d)p_0], \quad \text{and} \quad B = [uC_d - dC_u]/[(u-d)R]. \quad (\text{A.VII.1})$$

With Θ and B chosen in this way, we will call this the hedging portfolio.

If there are no arbitrage opportunities, the value of the call at time $T-1$ should be equal to the value of the hedging portfolio, $\Theta p_0 + B$.

$$\begin{aligned}C &= \Theta p_0 + B = \{[C_u - C_d]/[(u-d)p_0]\}p_0 + \{[uC_d - dC_u]/[(u-d)R]\} = \\ &= \{[(R-d)/(u-d)]C_u + [(u-R)/(u-d)]C_d\}/R.\end{aligned}$$

Let $\pi \equiv (R-d)/(u-d)$, and $(1-\pi) \equiv (u-R)/(u-d)$. We called π a risk-neutral probability.

Then,

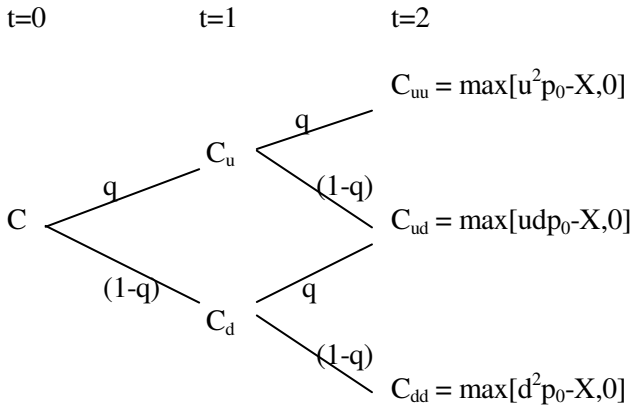
$$C = [\pi C_u + (1-\pi) C_d]/R. \quad (\text{A.VII.2})$$

Note III: the probability q does not appear in the formula. The only random variable on which C depends is p_0 . In equation (A.VII.2) preferences and tastes do not matter.

Investors' attitude toward risk and the characteristic of other assets do not influence directly C , but they might have an indirect effect through p_0 , u , d , and R .

A.VII.1.B Two-periods formulation

Now, we will consider the second easiest formulation, there are two periods remaining before T. Since now is time t=0, then T=2. That is,



At T-1, we know that

$$C_u = [\pi C_{uu} + (1-\pi) C_{ud}]/R.$$

$$C_d = [\pi C_{dd} + (1-\pi) C_{ud}]/R. \tag{A.VII.3}$$

Again, we select a portfolio with Θ in stock and B in the bond, whose end-of-period value will be C_u if the stock price goes up to up_0 and C_d if the stock price goes to dp_0 . To get these new values, we use (A.VII.1), with the new values of C_u and C_d .

Since Θ and B have the same functional form in each period, At time t=0 -i.e., T=2- the current value of the call will be again

$$C = \begin{cases} [\pi C_u + (1-\pi) C_d]/R & \text{if } C > p_0 - X, \\ p_0 - X & \text{otherwise.} \end{cases}$$

By substituting from equation (A.VII.3) into the above equations, we get

$$C = \frac{\{\pi^2 C_{uu} + 2\pi(1-\pi) C_{ud} + (1-\pi)^2 C_{dd}\}}{R^2} = \frac{\{\pi^2 \max[u^2 p_0 - X, 0] + 2\pi(1-\pi) \max[ud p_0 - X, 0] + (1-\pi)^2 \max[d^2 p_0 - X, 0]\}}{R^2}.$$

C now depends on p_0 , X, u, d, R and T=2.

A.VII.1.C T-period problem

By starting at expiration date and working backwards, we can write down the general valuation formula:

$$C = R^{-T} \sum_{n=0}^{n=T} \binom{T}{n} \pi^n (1 - \pi)^{T-n} \max[p_0 u^n d^{T-n} - X, 0] \quad (\text{A.VI.4}).$$

The more general formula for any t, is

$$C(p_t, t; X) = R^{-(T-t)} \sum_{n=t}^{n=T-t} \binom{T-t}{n} \pi^n (1 - \pi)^{T-t-n} \max[p_t u^n d^{T-t-n} - X, 0].$$

Let j be the minimum number of upward moves such that

$$p_t u^j d^{T-t-j} > X.$$

Using logs on both sides, we get

$$\ln(p_t) + j \ln(u) + (T-t-j) \ln(d) > \ln(X).$$

Solving for j, we could write j as the smallest non-negative integer such that:

$$j [\ln(u) - \ln(d)] + \ln(p_t) + (T-t) \ln(d) > \ln(X), \Rightarrow$$

$$j \geq \ln[X / (p_t d^{T-t})] / \ln(u/d).$$

Replacing in (A.VII.4), we obtain (recall we are at time t=0):

$$C = R^{-T} \sum_{n=j}^{n=T} \binom{T}{n} \pi^n (1 - \pi)^{T-n} [p_0 u^n d^{T-n} - X] \quad (\text{A.VI.5}).$$

Of course, if $j > T$, the call will finish out-of-the-money even if the stock moves upward every period, so $C=0$!

We break (A.VII.5) into two terms:

$$C = p_0 \sum_{n=j}^{n=T} \binom{T}{n} \pi^n (1 - \pi)^{T-n} \left(\frac{u^n d^{T-n}}{R^T} \right) - X R^{-T} \sum_{n=j}^{n=T} \binom{T}{n} \pi^n (1 - \pi)^{T-n} \quad (\text{A.VI.6}).$$

Digression: In a series of T-t independent trials of an experiment whose success rate is π and whose failure rate is $(1-\pi)$, the probability that there will be at least $j \geq 0$ successes is

$$\Phi(j; T-t, \pi) \equiv \sum_{n=j}^{n=T-t} \binom{T-t}{n} \pi^n (1 - \pi)^{T-t-n}.$$

This is called the complementary binomial distribution function. The binomial distribution function is equal to: $1-\pi$, and it gives the probability that the number of successes is strictly less than j.

Substituting into (A.VII.6) we obtain:

$$C = p_0 \Phi(j; T, \pi') - X R^{-T} \Phi(j; T, \pi),$$

where $\pi' \equiv u\pi/R$, and $(1-\pi') \equiv (d/R)(1-\pi)$,

π' is a probability, since $0 < \pi' < 1$. To see this, note that $\pi < (R/u)$ and

$$\pi^n (1-\pi)^{T-n} (u^n d^{T-n}/R^T) = (u\pi/R)^n (d(1-\pi)/R)^{T-n} = \pi'^n (1-\pi')^{T-n}.$$

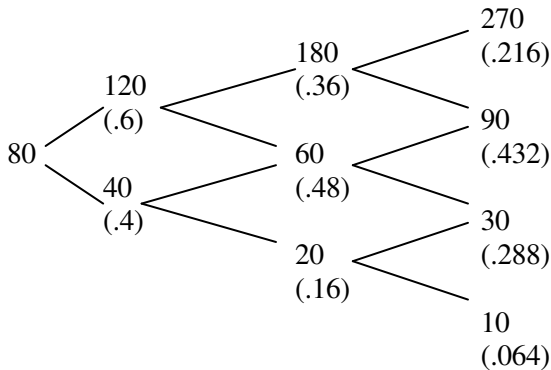
This is the binomial option pricing formula of Cox, Ross and Rubinstein (1979). All the features valid for the one period formula are also valid for any number of periods. In particular, the value of a call should be the expectation, in a risk-neutral world, of the discounted value of the payoff it will receive.

Example A.VII.1: Suppose you are given the following data: $p_0=80$, $T=3$, $X=80$, $u=1.5$, $d=0.5$, $i_{rf}=10\%$ ($R=1.1$). You want to price the call option, C .

In this case, $\pi = (R-d)/(u-d) = (1.1-0.5)/(1.5-0.5) = 0.6$.

$$R^{-1} = .909, R^{-2} = .826, R^{-3} = .751.$$

The paths the stock price may follow and their corresponding probabilities are:



Using the formula, the current value of the call would be

$$C = 0.751 [0.064 (0) + 0.288 (0) + .432 (90-10) + .216 (270-80)] = 34.065.$$

Recall that to form a riskless hedge, for each call we sell, we buy and subsequently keep adjusted a portfolio with Θ in stock and B in the bond.

The following diagram gives the paths C may follow and the corresponding value of π .

		107.272	190
	60.463	(1.00)	
	(.848)		10
34.065		5.454	
(.719)	2.974	(.167)	
	(.136))		0
		0	
		(0.00)	
			0 ¶

Example A.VII.2: Reconsider Example A.VII.1. Suppose that when $t=0$, the market price of the call is 36. The option is overpriced, therefore we could plan to sell it and assure ourselves of a profit equal to the mispricing differential.

For example, we could follow this steps under a udd price trajectory for the stock:

Step 1 ($t=0$): $p_0=80$. Sell the call for 36. Take 34.065 of this and invest it in a portfolio containing $\Theta=.719$ shares of stock by borrowing $.719(80)-34.065=23.455$. The remainder $36-34.065=1.935$ is invested in the bank.

Step 2 ($t=1$): $p_1=120$. The new $\Theta=.848$. Buy .129 ($=.848-.719$) more shares at 120 for a total of 15.480. Borrow to pay this amount. With $i_{rf}=10\%$, we already owed $23.455(1.1)=25.801$. Then, total current debt is 41.281 ($=25.801+15.480$).

Step 3 ($t=2$): $p_2=60$. The new $\Theta=.167$. Sell .681($=.848-.167$) shares at 60, taking in $.681(60)=40.860$. Use this to pay back part of your borrowing. Since we now owe $41.281(1.1)=45.409$, the repayment will reduce debt to 4.549 ($=45.409-40.860$).

Step 4 ($T=3$): $p_3=30$. The call we sold has expired worthless. We own .167 shares of stock selling at 30, with a total value of 5. Sell the stock and repay the $4.549(1.1)=5$. Go back to the bank and withdraw your original deposit, which has now grown to $1.935(1.1)^3=2.575$.

Suppose the price trajectory for the stock is udu.

Step 4' ($T=3$): $p_3=90$. The call we sold is in the money. Buy back the call, or buy one share of stock and incur a loss of 10 ($90-80$). Borrow to cover this, bringing your total debt to 15 ($=5+10$). We own .167 shares of stock selling at 90, with a total value of 15. Sell the stock and repay the 15. Go back to the bank and withdraw your original deposit, which has now grown to $1.935(1.1)^3=2.575$.

Conclusion: if we are correct in our original analysis, and if we faithfully adjust the hedge portfolio, then if an option is miss-priced, we can assure ourselves a riskless profit equal to the misspricing differential.¶

A.VII.2 The Black-Scholes formula

Basic intuition: As the number of trials -i.e., as the number of stock prices moves- increases to infinity, the central limit theorem shows that the binomial distribution converges to the standard normal distribution.

To increase the number of stock prices moves, we will allow more and more trading in the time interval $[0, T]$.

Let n be the number of periods of length h prior to expiration. Then, $h = T/n$.

As trading takes place more and more frequently, h gets closer and closer to zero. We must readjust R , u , and d in such a way that we obtain empirically realistic results as h becomes smaller or, equivalently as $n \rightarrow \infty$.

We will let R to continue mean the gross return on the bond over a fixed length of calendar time. We will use r to refer to the gross return over a period of length h . Then, for any choice of n , we will have

$$r^n = R^T, \Rightarrow r = R^{T/n}.$$

We also want to define u and d in terms of n . Recall that over each period, the stock price moves up or down for a gross return of u or d , respectively (with associated probabilities of q and $(1-q)$).

It will be easier to work with the natural logarithm of the one plus rate or return, $\ln(u)$ or $\ln(d)$. This gives the continuously compounded rate or return on the stock over each period. It is a r.v., which will be equal to $\ln(u)$ with probability q and $\ln(d)$ with probability $(1-q)$, in each period.

Example A.VII.3: Consider a typical price trajectory $uduud$. The final stock price will be $p^* = uduudp_0$. Then, $p^*/p_0 = u^3 d^2$, and $\ln(p^*/p_0) = 3 \ln(u) + 2 \ln(d)$. ¶

More generally, over n periods,

$$\ln(p^*/p_0) = j \ln(u) + (n-j) \ln(d) = j \ln(u/d) + n \ln(d),$$

where j is the (random) number of upward moves occurring during the n periods to expiration. The expected value of $\ln(p^*/p_0)$ is

$$E[\ln(p^*/p_0)] = \ln(u/d) E[j] + n \ln(d),$$

and its variance is

$$\text{var}[\ln(p^*/p_0)] = [\ln(u/d)]^2 \text{var}[j].$$

Since each of the n possible upward moves has probability q , then $E[j] = nq$, and $\text{var}(j) = nq(1-q)$. Replacing above, we have

$$E[\ln(p^*/p_0)] = [\ln(u/d) q + \ln(d)]n \equiv \mu_c n,$$

$$\text{var}[\ln(p^*/p_0)] = [\ln(u/d)]^2 q(1-q)n \equiv \sigma^2 n.$$

We would not reach a reasonable conclusion if either $\mu_c n$ or $\sigma^2 n$ went to zero as n became large. Since T is a fixed length of time, we must make the appropriate adjustments in u , d , and q ,

We would want at least the mean and variance of the continuously compounded rate of return of the assumed stock price to coincide with that of the actual stock price as $n \rightarrow \infty$. Let μT and $\sigma^2 T$ be the empirical counterparts of $\mu_c n$ and $\sigma^2 n$. Then we would like to choose u , d , and q , so that

$$[\ln(u/d) q + n \ln(d)]n \rightarrow \mu_c T,$$

$$[\ln(u/d)]^2 q(1-q)n \rightarrow \sigma^2 T.$$

Algebra shows we can accomplish this by letting

$$u = \exp(\sigma(T/n)^{1/2}),$$

$$d = \exp(-\sigma(T/n)^{1/2}),$$

$$q = 1/2 + 1/2(\mu/\sigma)(T/n)^{1/2}.$$

Clearly as $n \rightarrow \infty$, $\sigma^2 n \rightarrow \sigma^2 T$, while $\mu_c n = \mu T$ for all values of n .

Note: we are choosing r , u , d , and q in a way such that the multiplicative binomial distribution of stock prices goes to the lognormal distribution.

To apply the central limit theorem, we need to check the following condition:

$$\{q |\ln(u) - \mu_c|^3 + (1-q) |\ln(d) - \mu_c|^3\} / (\sigma^3 n^{1/2}) \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Intuition from this condition: higher-order properties such as skewness become less and less important relative to its standard deviation, as $n \rightarrow \infty$.

If the above condition holds, then

$$\text{Prob}[(\ln(p^*/p_0) - \mu_c n) / (\sigma n^{1/2}) \leq z] \rightarrow N(z),$$

where $N(z)$ is the standard normal distribution function. That is, as we trade more and more frequently, the probability that the standardized continuously compounded rate of return of the stock through T is not greater than the number z approaches the probability under a standard normal distribution.

Using this framework, we can obtain the limiting case for the binomial formula:

$$C(p, t; X) = P N(d1) - X R^{-(T-t)} N(d2), \quad (\text{A.VII.9})$$

where

$$d1 \equiv \frac{\ln(p/X e^{i_{rf}(T-t)})}{\sigma \sqrt{(T-t)}} + \frac{1}{2} \sigma \sqrt{(T-t)}$$

$$d2 \equiv \frac{\ln(p/X e^{i_{rf}(T-t)})}{\sigma \sqrt{(T-t)}} - \frac{1}{2} \sigma \sqrt{(T-t)}$$

This is the Black-Scholes formula. The two constants r and σ are the continuously compounded riskless interest rate per unit time and the standard deviation of the rate of return on the stock per unit time, respectively.

Let's rewrite Cox, Ross and Rubinstein (CRR) binomial option formula:

$$C = P_0 \Phi(j; T, \pi') - X R^{-T} \Phi(j; T, \pi).$$

To show that the two formulas converge, CRR show that as $n \rightarrow \infty$,

$$\Phi(j; T, \pi') \rightarrow N(d1) \quad \text{and} \quad \Phi(j; T, \pi) \rightarrow N(d1 - \sigma T^{1/2}).$$

Black and Scholes (1973) derived formula (A.VII.9) assuming continuous trading and a lognormal distribution for stock returns. Their solution relies on Ito's lemma.

CRR's approach contains continuous trading and the lognormal distribution as a limiting case.

Example A.VII.4: Approximating Black-Scholes with the binomial approximation.

Suppose we are given the inputs:

$$p_0 = 40.$$

$$X = 35 \text{ and } 45.$$

$$T = 1, 4, \text{ and } 7 \quad (\text{months}).$$

$$\sigma = .20 \text{ and } .40 \quad (\text{annual standard deviations}).$$

$$i_{rf} = .05 \quad (R = 1.05) \quad (\text{annual interest rates}).$$

Using this information we obtain u , d , and r using the relationships:

$$d = 1/u, \quad u = e^{\sigma(T/n)}, \quad r = R^{T/n}.$$

σ	X	n=5			n= ∞ (B-S)		
		1-mo	4-mo	7-mo	1-mo	4-mo	7-mo
0.2	35	5.14	5.77	6.45	5.15	5.76	6.40
	45	0.02	0.54	1.15	0.02	0.51	1.10
0.4	35	5.40	6.87	7.92	5.39	6.89	8.09
	45	0.46	1.99	3.30	0.42	2.10	3.34

The maximum difference is USD 0.11. For $n=50$, the maximum difference is less than \$0.03. For even moderate sizes of n , the binomial approximation works fairly well. ¶

A.VII.3 Approximating the cumulative normal distribution, $N(x)$

The standard normal Table reports the integral values of x given by

$$N(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

There is no closed form of the integral for all values of x . Therefore, approximations to the above integral are needed. The key to this task is to get an N th-order polynomial, which is an easy function to integrate, that accurately approximates $\exp(-x^2/2)$.

Using a N th-order Taylor series, it is easy to expand $\exp(-x^2/2)$:

$$\begin{aligned} e^{-\frac{x^2}{2}} &= 1 + \left(-\frac{x^2}{2}\right) + \frac{1}{2!} \left(-\frac{x^2}{2}\right)^2 + \frac{1}{3!} \left(-\frac{x^2}{2}\right)^3 + \frac{1}{4!} \left(-\frac{x^2}{2}\right)^4 + \frac{1}{5!} \left(-\frac{x^2}{2}\right)^5 + \dots \\ &= 1 - \frac{x^2}{2} + \frac{1}{2} \frac{x^4}{4} - \frac{1}{6} \frac{x^6}{8} + \frac{1}{24} \frac{x^8}{16} - \dots \end{aligned}$$

Then, integrating term by term we get

$$N(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \approx \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} \left(1 - \frac{x^2}{2} + \frac{1}{2} \frac{x^4}{4} - \frac{1}{6} \frac{x^6}{8} + \frac{1}{24} \frac{x^8}{16} - \dots \right) dx = \frac{1}{\sqrt{2\pi}} \left[x - \frac{x^3}{6} + \frac{1}{2} \frac{x^5}{20} - \frac{1}{6} \frac{x^7}{56} + \frac{1}{24} \frac{x^9}{144} - \dots \right]_{-\infty}^x$$

Suppose we want to calculate the integral from 0 to $x=1.64$, using a 3rd degree polynomial. Then,

$$N(x) = \int_0^{x=1.64} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \approx \frac{1}{\sqrt{2\pi}} \left[x - \frac{x^3}{6} + \frac{1}{2} \frac{x^5}{20} - \frac{1}{6} \frac{x^7}{56} \right]_0^{1.64} = \frac{1}{\sqrt{2\pi}} \left[1.64 - \frac{1.64^3}{6} + \frac{1}{2} \frac{1.64^5}{20} - \frac{1}{6} \frac{1.64^7}{56} \right] = \frac{1.1064686}{\sqrt{2\pi}} = 0.44141713$$

The value from the normal table is $N(x=1.64) = 0.44949742$. The approximation has a .008 error. The approximation becomes better if we use a higher order polynomial. For example, if we use a 7th degree polynomial, then $N(x=1.64) = 0.449488$.

Abramowitz and Stegun (1972) provide a polynomial approximation to the cumulative function of a standard normal distribution $N(x)$. This approximation is given by:

$$\begin{aligned} N(x) &= 1 - (a_1 y + a_2 y^2 + a_3 y^3) f(x) && \text{for } x \geq 0, \text{ and} \\ N(x) &= 1 - N(-x) && \text{for } x < 0, \end{aligned}$$

where

$$\begin{aligned} y &= 1/(1+a_0 x) \\ a_0 &= -.33267, \\ a_1 &= .4361836, \\ a_2 &= -.1201676, \\ a_3 &= .9372980, \text{ and} \end{aligned}$$

$f(x) = \exp(-x^2/2)/(2\pi)^{1/2}$ is the density function of a standard normal distribution.

The Abramowitz and Stegun approximation is normally accurate to four decimal places and are always accurate to 0.0002.

A.VII.4 The Different Instruments

i. Stock options.

Trading in listed options started with options on common stocks. Markets have developed through the world to the point where options on most active stocks are now traded. In the United States, the volume of shares traded through options is often larger than the volume of the actual stocks traded on the NYSE.

Example A.VII.5: The *Wall Street Journal* publishes listed option contracts. On November 1, 1994, the most active contracts were

Monday, October 31, 1994

MOST ACTIVE CONTRACTS

Option/Strike	Vol	Exch	Last	Net Chg.	a-Close	Open Int
PhysCp Dec 25	5,121	CB	1/8	...	245/8	65
Ph Mor Nov 60	4,913	AM	2	- 2 ¹ / ₄	61 ¹ / ₄	4,685
Novell Nov 17 ¹ / ₂	4,531	AM	13/8	+3/8	18 ¹ / ₂	6,945
I B M Nov 75	4,453	CB	1 ¹ / ₂	- 1/8	743/8	26,041
G M Dec 40	4,006	CB	1 ¹ / ₄	- 1/2	39 ¹ / ₂	6,414
Hewlet Nov 90 p	3,355	CB	1/2	- 5/8	97 ³ / ₄	3,625
G M Nov 45	3,057	CB	1/2	- 7/8	39 ¹ / ₂	15,209
Merck Jan 35	2,784	CB	2	- 3/8	35 ³ / ₄	26,046
Cypr Mn Dec 25 p	2,760	CB	1/2	+ 5/8	263/8	5
Compaq Nov 40	2,743	PC	1 ³ / ₄	- 1/2	407/8	8,996
I B M Nov 75 p	2,736	CB	15/8	+ 1/2	741/8	8,119

ii. Commodity options.

The most active trading takes place in gold and silver options, which many money managers use for their gold-linked assets. It should be stressed that some of the exchanges trade in bullion options, which require physical delivery of the metal, while others trade in gold futures contracts.

iii. Interest rate options.

Options are traded on U.S. debt instruments such as treasury bills, treasury bonds, and certificates of deposit (CDs). Similar options exist in a few countries with developed bond markets, including the U.K. and the Netherlands. Besides these exchange-traded options, some interest rate options are negotiated over the counter. These tend to be longer-term options on short-term interest rates. They often take the form of *caps* and *floors*.

iv. Stock Index options.

Options on stock indexes have developed in several countries. Stock index options are options on the level of the index. The contract size is equal to the index times a *multiplier* set by the exchange. For example, the Standard & Poor's 500 options traded on the CME have a contract multiplier of 500. In other words, the investment required to purchase one contract is equal to the premium multiplied by USD 500.

Example A.VII.6: On November 1, 1994 the *Wall Street Journal* reports a closing price for S&P 500 (SPX) 472.35. The closing premium for the December 450 call on the S&P 500 was 25₋. The investment required to purchase one contract is USD 12,812.50. ¶

All settlement procedures require cash, rather than physical delivery of an index. The option contracts on Indexes can be either American or European.

Example A.VII.7: On September 21, 1994, the FT-SE 100 closed at 3037.3. The same day, the *Financial Times* reported the following quotes for the option on the FT-SE 100 (American and European style options):

■ **FT-SE 100 INDEX OPTION (LIFFE) (*3436) £10 per full index point**

	2850		2900		2950		3000		3050		3100		3150		3200	
	C	P	C	P	C	P	C	P	C	P	C	P	C	P	C	P
Oct	196	12½	153½	19½	115½	31	82	47½	53½	70	33	99	18½	135	10½	176½
Nov	21½	25½	175½	35½	139½	48	107½	66½	78½	87	57	115	38½	147	25½	184
Dec	233½	33½	196½	62	160	65½	130½	87	100	107	78	134	56	162½	42	197½
Mar	251	50½	216½	65	185	83½	158	105½	127	124	101	148	79½	176½	62½	210½
Jan+		279½	97			219½	134			168	182			123½	238	

Calls 4,909 Puts 6,865

■ **EURO STYLE FT-SE 100 INDEX OPTION (LIFFE) £10 per full index point**

	2875		2925		2975		3025		3075		3125		3175		3225	
	C	P	C	P	C	P	C	P	C	P	C	P	C	P	C	P
Oct	175	17	133	24½	94½	36	67	58½	42½	83½	25	116	14	154	7	196½
Nov	196	29½	159	42	125	57½	95	77½	69½	101	48½	129½	32	162½	19½	200
Dec	210½	41½	174½	55½	142	72½	113½	93	88	116½	66	144	48	175	33½	210½
Mar		224½	83			164½	120½			114½	167			75½	225½	
Jun+		269	105			209½	141½			156½	184½			114	238	

Calls 8,603 Puts 3,521 * Underlying Index Value. Premiums shown are based on settlement prices.
+ Long dated expiry months

v. Currency options.

Currency options have already been discussed in this chapter.

Exercises:

1. Go back to Exercise VI.4. The investor considers buying currency puts on the yen instead of selling futures contracts. In Philadelphia, a JPY put with a strike price of 0.009 USD cents and 3-month maturity is worth 0.40 USD cents per 100 JPY (USD .00004 per JPY).

Assume that three months later, the portfolio is still worth JPY 200 million. Simulate various values of the spot JPY/USD exchange rate. Compare the results of the following two currency hedging strategies for your different values of the exchange rate three months later. In the first strategy, the investor sell JPY 200 million forward, and in the second strategy he buys yen puts for JPY 200 million

2. Consider the Euro contract on Example VII.2. Calculate the total price (premium) paid for the in-the-money December calls and puts. If you buy 2 Dec 106 EUR calls, what is the total price paid? Is the Dec 106 EUR call in-the-money or out-of-the money?

3. Go back to Example VII.2. Suppose you bought 5 Sep 161 GBP PHLX calls --with a strike price of 161 (1.61 USD/GBP). Broker's fees are USD 28 + USD 1.50 per contract sold.

i.- Ignoring broker's fees, how many USD would you have paid?

ii.- What approximate ceiling cost of GBP 156,250 would you have locked in?

iii.- Suppose you had held the five options for 20 days and then sold them for "2.95" (USD cents per GBP). What was the net dollar profit (including transaction costs), assuming a USD interest opportunity cost of 10 percent per annum (per 360 days) on the premium?

4. Go back to Graph VII.1 point A, suppose S_t drops to 1.57 USD/GBP and Δ changes to -.6. How many puts must the investor sell or buy to avoid over hedging at the same strike price? Answer the same question, but suppose S_t increases to 1.85 USD/GBP and Δ changes to -.31.

5. A U.S. bank buys a Nov USD put (JPY call!) with a strike price of 103 JPY/USD and writes a Nov USD call (JPY put) with a strike price of 102 JPY/USD. What happens if at expiration the spot rate is 104.50 JPY/USD?

6. Go back to equation (VII.1). Show that $n = - (V_0/\Delta)$ provides a good hedge. (Hint: show that for small movements in the exchange rate, the change in the hedge portfolio is zero when $n = - (V_0/\Delta)$.)

7. Suppose that the spot rate is .28 USD/PLN (PLN = Polish zloty). Consider the following European option: a .30 USD/PLN June PLN call knock-in .35 USD/PLN. You buy 10,000 of these European call knock-in options. You paid .01 USD per option.

a. What are the profits (losses) if in June the spot rate is .33 USD/PLN?

b. What are the profits (losses) if in June the spot rate is .37 USD/PLN?

8. You work for a Brazilian investment bank pricing OTC BRR/USD options. Suppose you are given the following data:

X= 1.50 BRR/USD.

S= 1.60 BRR/USD.

$T = 3$ years.

$i_{\text{USD}} = 5.5\%$ (annual risk-free rate)

$i_{\text{BRR}} = 14.7\%$ (annual risk-free rate)

$\sigma = 23\%$,

where σ is the annual volatility estimated using historical data for the past three years. Suppose you are long USD 10,000,000, determine the size of your options hedging position.

Calculate C (call premium) and P (put premium) using the Black-Scholes formula.

9. Reconsider Exercise VII.9. In the market, you observe that C is equal to BRR .4517, which is different from the theoretical price you have estimated. You feel that your estimate of σ , calculated using historical data, is not a good estimator of the future 3-year volatility. Given the observed C, you should be able to obtain an implied estimate for σ . You use the method of trial and error. What is this *implied volatility*?