Sale-Backs in Bankruptcy

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When bankrupt firms are sold, they are often repurchased by their former owner or manager. These insiders are by default better informed than outsiders about the true value of the firm or its assets, so other potential buyers must worry about overpaying if they win. The presence of insiders may thus have a chilling effect on the bidding. We ask how insiders should be treated in bankruptcy sales: Should they be allowed to submit bids? If so, under what conditions? We derive properties of an optimal sale procedure and show that it must be biased against insiders. Specifically, it should be harder for insiders to win with low bids than for outsiders. We show that the “market tests” that are routinely required in bankruptcy sales are suboptimal since they treat all potential buyers alike and forgo the benefits of biasing the procedure against insiders.

1. Introduction

A large body of work has studied how businesses are or should be reorganized in bankruptcy, focusing in particular on reorganization under Chapter 11 of the U.S. Bankruptcy Code. In fact, most of the academic work on corporate bankruptcy studies reorganizations, and very little of it studies the sale or liquidation of firms in bankruptcy. This contrasts with the practice of bankruptcy: only a small fraction of bankrupt firms are actually reorganized and most firms (in particular, small firms) are sold, either as going concerns or piecemeal. Some authors even claim that the “traditional” reorganizations for which Chapter 11 was designed have more or less disappeared: for most small firms, a sale or liquidation is the only option; and even for the largest firms, where a reorganization may be feasible, it turns out that either most are sold or they enter Chapter 11 with a prepackaged plan and nothing left to negotiate.1 Thus, most of the academic work analyzes problems that arise very rarely, and there is

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1. See Baird and Rasmussen (2003, 2004) and Baird (2004); but see also LoPucki (2004).

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almost no research on the type of bankruptcy (going-concern sales and piece-meal liquidations) that most bankrupt firms experience nowadays.

Of particular interest are bankruptcy sales in which the former owner-manager buys back the firm or its assets. There is some evidence suggesting that, in practice, this is quite common. Stromberg (2000) and Thorburn (2000) report that in their sample of Swedish bankruptcies, more than half of all going-concern sales are sale-backs to the former owners. A study of all corporate insolvencies started in the United Kingdom between January 2002 and June 2003 finds that 50% of the going-concern sales are sale-backs (see R3 2004). Similar numbers are reported for the Netherlands (see Knegt 1996). And Kaiser (1996) argues that bankruptcy sale-backs were strategically used by debtors in France until 1994, to wipe out senior debt. Given the international prevalence of sale-backs, we would expect them to be common in the United States too.

Sale-backs to former owners or managers (henceforth insiders) are relevant because this possibility chills the competition for the bankrupt firm (or its assets). By default, an insider is much better informed than third parties about the value of the bankrupt business or its assets (its order books, customer relationships, the skills and motivation of the staff, the state of the buildings and equipment, etc.). This creates a strategic problem for the outsiders: whoever becomes the new owner of the bankrupt firm will have won because she submitted the highest bid, and given that all other bidders had lower value estimates (their bids were lower), it is possible that the winning bid was too high, possibly higher than what the firm is actually worth. In other words, winning against better informed bidders may imply that the winner overpaid (this is known as the “winner’s curse”).

In response to the threat of overpaying, outsiders should submit low bids (much lower than their value estimates), or they may even prefer to abstain from bidding altogether (say, if they have to incur costs to prepare value estimates or a bid). In response, the insider can submit a low bid, with a reasonable chance of winning. This reduces the expected price that can be raised, and the recovery rates that creditors can expect.

An important question that we analyze in this article is whether insiders should be allowed to bid or whether bids from insiders should not be allowed because of their chilling effect on competition. We analyze under what conditions an insider should be allowed to participate and what sale procedure a bankruptcy trustee should use when a bid from an insider is allowed.

There is a second dimension on which insiders may be different from other bidders. The founder-manager of a bankrupt firm may add value to its future operations, making the firm without her less valuable to other bidders. We abstract from this possibility in the following. If the founder-manager adds value to the business, then outside bidders should try to hire her back, with appropriately designed performance compensation. For example, the founder-manager could be hired as a manager focusing on technical sides of the business, with an equity stake or stock options as compensation; and professional managers could run the nontechnical side of the business (in particular, the financial side, which the founder-manager may have neglected in the time leading
up to the firm’s bankruptcy). Furthermore, the insider’s relevance to the operations is not necessarily linked with her status as a better informed bidder: her relevance may be based on her ability to execute certain tasks, her personality, and so forth, without giving source to any informational advantages.

We first analyze how an insider should be treated, if the sale is structured as a standard auction, say, a sealed-bid auction (high price) or an ascending auction (open outcry). We show that if the trustee is restricted to using standard auctions, then it may be optimal to exclude an insider from bidding in a bankruptcy sale. This is the case if the bidder asymmetry is sufficiently strong, making it easy for the insider to win with a low bid. This result contrasts with the idea that adding a bidder is always beneficial for the seller because it increases the competition between the buyers (see Bulow and Klemperer 1996). Going from two to one bidder is a major reduction in that competition. Yet, it may be better for the estate to exclude insiders.

Next, we consider a situation in which the bankruptcy trustee can freely design the sale procedure. We find that in this case, it is optimal to always let an insider participate, no matter how much better informed she is compared with outsiders. In other words, the extreme bias of not allowing bids from insiders can only be optimal if the trustee is not allowed to fine-tune the bias. However, the bankruptcy trustee should structure the sale such that it is biased against the insider: she should be able to buy back the firm only with a high bid, but not with a low bid. This is beneficial since allowing her to participate strengthens competition between the buyers; and handicapping her (by discarding her low bids) makes it easier for outsiders to compete (it mitigates their concerns about the winner’s curse).

Our results have important practical implications. In the United States, bankruptcy sales have become common, and in most cases, the bankruptcy courts require evidence that a proposed purchase price is “top dollar.” Such evidence is provided by exposing the existing (stalking horse) bid to “overbids” by third parties. The stalking horse can then top these bids, outside bidders can in turn raise their bids, and so on. This structure is equivalent to an ascending auction, in which bids are raised until no one is willing to top the currently highest bid. We show that such a procedure is suboptimal in the presence of bidder asymmetries: it treats all bidders symmetrically, thereby forgoing the benefits of biasing the procedure against an insider. It may actually pay under these circumstances to exclude an insider from bidding, if the bankruptcy trustee feels that the informational advantage over outside bidders is too large. But again, a superior approach would be to let anybody participate in the bidding but to bias the sale procedure against an insider. Given that sale-backs seem to be quite frequent, there is much scope for improving the way bankruptcy sales are handled in practice: the courts should give up the strict requirement of overbids and allow the trustee to design the selling procedure, if the trustee suspects that bidders are asymmetrically informed.

The question how a bankruptcy sale should be structured is a relatively unexplored topic. The only article that asks how bankrupt firms should optimally be sold (once the decision to sell has been made) is Cornelli and Felli (2000).
They suggest to auction off a majority stake in the bankrupt firm while keeping a minority stake for creditors, which allows the creditors to retain some of the potential upside in the reorganized business.\textsuperscript{2} Roe (1983) suggests that auctions could be used to alleviate distributional conflicts in reorganizations: by selling a small fraction of the shares in the reorganized company to the public, the parties in a reorganization can solicit the market’s valuation of the remaining shares. However, Bhattacharyya and Singh (1999) show that different classes of claimants may disagree on what type of auction to use. (This issue has recently become more relevant, given the increased influence that senior lenders have on bankruptcy outcomes; see Baird and Rasmussen 2006.)

Most other studies on bankruptcy sales ask whether sales are superior to reorganizations (see Schwartz 2005). Some authors argue that a bankruptcy trustee should sell a bankrupt firm as soon as possible since the new owners can better decide what the most efficient use of the firm’s assets is, and this decision can be separated from the question how the sale proceeds should be distributed (Baird 1986, 1993; Jackson 1986; Bebchuk 1988; Aghion et al. 1992; Baird and Morrison 2001). However, some authors have voiced concerns with the idea of mandatory bankruptcy auctions, emphasizing the need for flexibility (Schwartz 1997), incentive problems (Easterbrook 1990; Bhattacharyya and Singh 1999; Povel 1999; Ayotte 2007), signaling problems (White 1994), or the possible lack of liquidity in the market for bankrupt firms or their assets (Aghion et al. 1992; Shleifer and Vishny 1992). Finally, some authors argue that sophisticated auction designs should be used as part of a reorganization procedure (Bebchuk 1988; Aghion et al. 1992; Adler and Ayres 2001). The empirical evidence suggests that liquidations may yield lower prices than asset sales by healthy firms but they do not lead to less efficient outcomes or lower recovery rates than reorganizations (Hotchkiss and Mooradian 1998; Maksimovic and Phillips 1998; Pulvino 1998; Thorburn 2000).

Importantly, our article is the first to ask how bidder asymmetries should affect the design of a bankruptcy sale. There is evidence that asymmetric information is a concern in practice. Gilson et al. (2000) find that bankrupt firms are hard to value, since they tend to lack tracking by analysts, and have little trading, and this makes it easier (and possibly more attractive) to misrepresent information. The evidence in Hotchkiss and Mooradian (1998) also suggests that informational issues may deter bidding by potentially less well-informed firms.

Focusing on procedure design issues in connection with sale-backs is important for two reasons (see also Adler and Triantis 2002). First, if the sale procedure is not designed optimally, the bids will be lower than necessary and the recovery rates will also be lower. This hurts the creditors once a firm is in bankruptcy, but since lenders anticipate this possible outcome, it hurts all firms that take on debt to finance their operations since lenders must require

\textsuperscript{2} Their model is a private value model, that is, each bidder has her own bidder-specific valuation that is not available to other bidders. The aim of retaining a stake is that the creditors can extract a (proportional) fraction of the winning bidder’s private value.
higher interest payments (and to some firms, debt financing may become unavailable altogether). Second, the possibility of a cheap sale-back creates the wrong incentives outside of bankruptcy. If an insider can cheaply repurchase the assets and continue working in the same line of business, then a default becomes less threatening, and all else equal, an insider will worry less about keeping her firm out of bankruptcy. Again, this cost is not ultimately borne by the lenders: they anticipate the moral hazard problems at the lending stage and incorporate possible losses by making their external funds more expensive.

Technically, our article is related to those of Povel and Singh (2004, 2006) who also analyze auctions in which one bidder is better informed. Povel and Singh (2004) derive the optimal auction for a bidding environment like ours and show that bidder asymmetry can actually be beneficial for a seller. Povel and Singh (2006) use a more complex valuation model, which allows for both common and private value components. The focus is on takeovers, on the implementation of the optimal sale procedure, and on the commitment problems that may arise (commitment problems seem to be less of an issue in bankruptcy sales). The more complex model yields results that are consistent with ours. In particular, it is optimal to make it harder for the insider to win with a low bid. Neither of these articles analyzes bidding equilibria in standard auctions in the presence of bidder asymmetries. They also do not analyze the possible optimality of disallowing sale-backs to insiders in such standard auctions.

The rest of the article is organized as follows: In Section 2, we present a simple model with asymmetrically informed bidders. In Section 3, we discuss standard auctions (first price, second price) and their equilibrium bidding strategies, and we show that an exclusive offer to an outside bidder may dominate a standard auction in which that bidder competes with the insider. In Section 4, we derive the key properties of an optimal selling procedure, and we show that standard auctions are not optimal. In Section 5, we discuss the implications of our results for bankruptcy law and practice, including the observation that the “market tests” that are routinely required in bankruptcy sales are not optimally designed. Section 6 concludes. Some of the proofs are in the Appendix.

2. The Model

A bankruptcy trustee needs to sell a firm whose operations and assets cannot be split up, say, because a piecemeal liquidation yields much lower prices than a going-concern sale. We assume that a sale is optimal; a restructuring of the operations and liabilities may have been attempted earlier, and if so, it failed. For example, the creditors may agree that there is no reason to keep the firm under its current management because managing the assets and operations does not require any specific experience, skills, or connections.

There are two potential bidders: the former manager-owner of the bankrupt firm (bidder 1 or the insider) and an outside bidder (bidder 2 or the outsider), for example, a competitor or a private equity fund. All players are risk neutral. Both bidders value the firm equally, but the value is unknown to them. That is, bidders have common values, which is an appropriate assumption if the
bidders would operate the firm in similar ways after winning and if each could realize the same synergies and investment opportunities. Assuming common values is consistent with the idea that managing the firm does not require specific skills or experience. It is also consistent with the idea that the firm has a “market value” that the sale procedure is supposed to elicit. The alternative would be to assume that bidders have private values, that is, one bidder’s valuation is completely independent of another bidder’s valuation. Povel and Singh (2006) allow for a combination of private and common values. Doing so makes the analysis less tractable, but it does not change the main result: that the optimal sale procedure should be biased against the insider, if she claims to have a low-value estimate.

The bidders cannot observe the true value of the firm. Instead, each of them privately observes a signal \( t_i (i = 1, 2) \), drawn independently from the same density function \( f \), with support \([t, \bar{t}]\) and c.d.f. \( F \). Denote the hazard rate\(^3\) by \( H(t_i) = f(t_i)/(1 - F(t_i)) \). The full-information value of the firm is a weighted average of the two signals:

\[
v(t_1, t_2) = \psi_1 t_1 + \psi_2 t_2, \quad \text{such that } \psi_1 \in \left[\frac{1}{2}, 1\right) \quad \text{and} \quad \psi_2 = 1 - \psi_1.
\]

This type of model (common values with independent signals) was introduced by Myerson (1981) and is also used in Bikhchandani and Riley (1991), Bulow and Klemperer (1996, 2002), Bulow et al. (1999), and Povel and Singh (2004, 2006). The underlying assumption is that the value of a firm depends on what potential buyers are willing to pay for it, and this in turn depends on the information that these potential buyers have about the firm. An alternative way to model common values is to assume that the true value is given exogenously but unknown, and the bidders receive noisy information about this unknown value. Both approaches capture the idea that bidders value the firm equally (in a pure common value setup) and that bidders receive informative but imperfect signals. The model we use has the advantage that it remains tractable if bidders’ signals are not equally informative.

We use a common value model for simplicity: allowing for private values complicates the description of the model and its analysis but the results are very similar (we discuss them below; for details, see Povel and Singh 2006). Private value components may be relevant in some cases (say, if some bidders expect to realize synergies that are not available to other bidders), but allowing for this possibility merely complicates the analysis without adding any significant new insights.

Also, the common value assumption is appropriate for bankruptcy sales. A bankrupt firm’s claimants may have considered various alternative restructuring plans but decided that there is no benefit from actively participating in the restructuring, preferring to let outsiders deal with the problems and take cash instead. In other words, a decision to sell suggests that all potential new

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\(^{3}\) The hazard rate \( H(t) \) measures the probability of the possible signals \( t_i \), conditional on not being \( below \ t \). It is used to update beliefs about bidders’ signals while bids are being raised.
owners would make very similar restructuring decisions, expecting to generate the same benefits (which are unknown at the time of the sale, though). An additional reason to use a common value model is the existence of an active market for firms’ assets and for entire firms (driven to a large degree by private equity funds). The bankruptcy trustee should expect the bankrupt firm to have a resale value after it is sold, and this resale value is common to all potential buyers.

We assume that the insider is better informed than the outsider. Technically, this follows from the assumption that \( \psi_1 \geq 1/2 \), which implies \( \psi_1 \geq \psi_2 \). The insider is better informed in the sense that conditional on her signal, her value estimate has a lower variance (this conditional variance is \( \psi_i^2 \) for bidder \( i \), and since \( \psi_1 \geq \psi_2 \), it is larger for bidder 2). Assuming that \( \psi_1 < 1 \) ensures that both signals are informative. The assumptions that the weights \( \psi_1 \) and \( \psi_2 \) add up to one and that the signals \( t_i \) are i.i.d. ensure that the expected value of the firm does not depend on \( \psi_1 \) and \( \psi_2 \): it is easy to show that \( E[v(t_1, t_2)] = E[t_i] \), irrespective of \( \psi_i \). This normalization allows us to examine the effects of bidder asymmetry while keeping the ex ante expected value of the bankrupt firm constant.

One important question is how the trustee can observe \( \psi_1 \), that is, find out about potential bidders and how well informed they are. The bankruptcy trustee’s job entails meeting with various stakeholders of the bankrupt firm, to gather information about its situation and prospects. During these meetings, the firm’s competitors, suppliers, customers, and employees may point to possible buyers and describe what bidding environment can be expected. (If the trustee is the debtor in possession, this information should already be available to her, of course.) If the trustee is told that there is a strong bidder for the assets and other bidders will shy from competing with her, this suggests a situation with bidder asymmetries. Notice that we only require that the trustee can gather information about the existence of bidder asymmetries (and the identity of the better informed bidder); we do not require that the trustee shares the information that is available to the bidders, which would be a much stronger requirement on her information-gathering capacity. The realizations of these signals are initially observable only to the bidders who receive them, but once a new owner takes over the firm, she will eventually learn the signals, too. The trustee could eventually observe the signal realizations, too, by being on the job long enough. But typically, the bankruptcy court and the claimants are interested in resolving the case quickly, so waiting is not an option for the trustee.

We abstract from transaction costs in preparing a bid or a value estimate, such that we can focus on the role of informational asymmetry. This assumption is mild since it should be costlier for outsiders to evaluate a firm than for an insider, so if we included an information-gathering stage before the bidding stage, we would expect the outsider to be less well informed (in equilibrium) than an insider. We also abstract from private benefits of control that an insider may enjoy: these would complicate the analysis without adding any interesting insights.4

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4. The effect would be that in equilibrium, minor control rents can have an extreme and implausible impact on the outcome (see Klemperer 1999, sec. 7.2).
We normalize the bankruptcy trustee’s valuation of the firm to zero: her only goal is to sell the bankrupt firm (this may be because the firm’s creditors do not want to participate in a reorganization) and to maximize the expected price. This does not mean that there is no alternative use for the firm’s assets but merely that a going-concern sale seems by far the most attractive use. Consistent with this and to simplify the analysis, we further assume that \( t \), the lower bound of the signals’ support, is sufficiently high: \( t H(t) \geq \psi_1 \). This ensures that it is always optimal to sell the bankrupt firm, that is, the bankruptcy will never set a reserve price in equilibrium. For tractability reasons, we also make a standard monotone hazard rate assumption: that \( H \) is increasing.

3. Standard Auctions

Many types of negotiation can be analyzed as competitive processes that resemble auctions. The literature has studied various types of standard auctions. We now study equilibria for some of these auctions in our model with asymmetrically informed bidders. Since some auction types are superficially different but strategically equivalent, we focus on two essential types of standard auctions: the first-price and the second-price sealed-bid auction (they are equivalent to the Dutch and the ascending or English auctions). We also consider an exclusive sale offer to one bidder at a fixed price. The standard auctions provide a useful benchmark for the analysis of the optimal procedure. We will show that they are generally suboptimal because treating bidders symmetrically is not optimal if the bidders themselves are asymmetrically informed. We discuss the exclusive sale offer since it illustrates how a key feature of the optimal procedure increases the expected price: the optimal procedure is biased against the insider, who has to submit a high bid if she wants to win. The exclusive sale offer is extremely biased against the insider: any bid from her is simply ignored. Finally, it is useful to study the standard auctions and the exclusive sale offer since they are limit cases of the optimal procedure: the standard auctions are optimal if the bidders are symmetrically informed (\( \psi_1 = 1/2 \)) and the exclusive sale offer is optimal if the insider is perfectly informed and the outsider not at all (\( \psi_1 \approx 1 \)).

**Proposition 1.**

(i) The bidding strategy \( b^{FP}(t_i) = E_{t_i < t_f}[t_j] \) \((i, j = 1, 2; i \neq j)\) is an equilibrium in a standard sealed-bid first-price auction.

(ii) The bidding strategy \( b^{SP}(t_i) = t_i \) \((i = 1, 2)\) is an equilibrium in a standard sealed-bid second-price auction.

(iii) The above equilibria of the standard first- and second-price auctions raise the same expected price; this expected price is constant in \( \psi_1 \).

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5. Setting a reserve price would require that the sale is called off for good if the bids are too low. Given that the firm must be sold (such that its bankruptcy can be concluded), this would not be a credible threat, as long as bidders offer at least something for the firm.
Proof. See the Appendix.

The equilibria identified above are standard. We have not explicitly analyzed the properties of the Dutch and the ascending auction; in our setup, they are equivalent to the first- and second-price auction, respectively. The proofs use standard methods, adapted to the setting with bidder asymmetries. The bids are higher in the second-price auction because the price paid by the winner is the second-highest price—that makes it optimal to submit higher bids than in a first-price auction. And the result that both types of auction yield the same expected price is a consequence of the standard “revenue equivalence theorem”: auctions that use the same allocation rule given the bidders’ signals (here, the bidder with the highest signal wins) and give a net payoff of zero to a bidder with the lowest possible signal (here, such a bidder does not win or pays a fair price) raise the same expected price (see Klemperer 1999 for an overview).

It can easily be shown that in a standard auction, a larger asymmetry (higher $\psi_1$) is beneficial for the insider, whose rents increase, and detrimental for the outsider (the expected payoffs for given signals are derived in the proof). However, the two effects cancel out, and the net effect on the expected price is nil. This is a consequence of our normalization (1): changes in $\psi_1$ do not affect the expected value of the firm for sale.\footnote{Using a different valuation setup, Cantillon (2000) shows that an increase in bidder asymmetry decreases the expected revenue generated by a standard auction.} This normalization is useful since we want to study how changes in $\psi_1$ affect the optimality of sale-backs and the bias in the optimal procedure. The simplicity of the benchmark case will make a comparison particularly easy when studying the features that make the optimal procedure more attractive.

Later in the article, we derive the optimal procedure and show that the key feature is that it is biased. Before doing so, we discuss an extreme form of bias, and we compare its outcome to that of a standard auction, to help understand how the optimal bias helps raise a higher price.

**Proposition 2.** Consider an exclusive sale offer to the outsider: to buy the firm at a price

$$R^{EO} = \psi_1 \int_t^T tf(t)dt + (1 - \psi_1)T. \tag{2}$$

If $\psi_1$ is sufficiently high, this exclusive offer raises a higher expected price than a standard auction in which both bidders can participate (analyzed in Proposition 1).

Proof. The exclusive offer is better than standard auction if and only if

$$\psi_1 \int_t^T t_1 f(t_1)dt_1 + (1 - \psi_1)T - 2 \int_t^T \int_{t_1}^{t_2} t_2 f(t_2)dt_2 f(t_1)dt_1 > 0. \tag{3}$$
The first two terms are $R^{EO}$, as defined in equation (2); the last term is the expected price from a standard auction, derived in the proof of Proposition 1. Since $E[t_1] > t$, the sum of the three terms is increasing in $w_1$. In the limit as $\psi_1$ approaches one, $R^{EO}$ approaches the unconditional expected value of the bankrupt firm, so it leaves the bidders without rent. In contrast, the bidders always earn rents when standard auctions are used: bidder $i$ expects a rent

$$
\int_t^\infty \left( \int_t^{t_i} (\psi_1 t_1 + (1 - \psi_2) t_2 - b^{SP}(t_j)) f(t_j) dt_j \right) f(t_i) dt_i 
= \int_t^\infty \left( \int_t^{t_i} (\psi_1 t_1 + (1 - \psi_2) t_2 - t_j) f(t_j) dt_j \right) f(t_i) dt_i 
= \psi_i \int_t^\infty \left( \int_t^{t_i} (t_i - t_j) f(t_j) dt_j \right) f(t_i) dt_i,
$$

which is strictly positive for $i = 1$ and weakly positive for $i = 2$. \hfill \blacksquare

The outsider would always accept the exclusive offer. The price $R^{EO}$ is equal to the outsider’s expected value, conditional on having observed the lowest possible signal, $t$, but not conditional on the insider’s signal (which is substituted in the payoff definition by its expected value). The price $R^{EO}$ thus earns her a strictly positive net expected payoff if $t_2 > t$ and zero if $t_2 = t$. This expected net payoff is referred to as a “rent.”

Since $R^{EO}$ is strictly increasing in $\psi_1$, the bankruptcy trustee can extract much of the rent for high values of $\psi_1$. The reason for this is simple. Consider the limit case, if $\psi_1 \approx 1$; the outsider does not have any information and is willing to pay the unconditional expected value. This is reflected in the changes of $R^{EO}$, which converges to $E[t_i]$ as $\psi_1$ increases. That is, in the limit as $\psi_1$ approaches one, the exclusive offer consists of a price equal to the unconditional expected value of the bankrupt firm. With a value of $\psi_1$ smaller than one, the outsider is somewhat informed about the value of the firm, and she will accept the exclusive sale offer only if $R^{EO}$ is smaller than $\psi_1 E[t_i] + (1 - \psi_1) t_2$. By setting $R^{EO}$ equal to $\psi_1 E[t_i] + (1 - \psi_1) t_1$, the bankruptcy trustee can sell the object with certainty. This comes at a cost, since the lower $\psi_1$, the larger the rent that the outsider expects to earn.

Nevertheless, the exclusive sale offer may be more profitable than selling through a standard auction. We show in the proof that the bidders’ rents are positive if the bankruptcy trustee uses a standard auction. Therefore, the expected price in a standard auction must be strictly less than the expected value of the firm. Since the expected price raised by a standard auction is constant in $\psi_1$, biasing against the insider is profitable for sufficiently high $\psi_1$, when the exclusive sale offer can extract almost all rents.

An example may help to illustrate the trade-off. Assume that the signals $t_i$ are uniformly distributed over the interval $[1, 2]$ (such that the assumption $t H(t) \geq \psi_1$ is satisfied). Then, equation (3) can be simplified to $\psi_1 > 2/3$. That is, if $1/2 \leq \psi_1 \leq 2/3$, the standard auction is preferred, whereas if
$2/3 < \psi_1 < 1$, then the trustee should not allow bids from the insider. It is readily verified that $R^{EO} = (\psi_1/2) + 1$ while the expected price paid in the standard auction is $4/3$. In the limit, as $\psi_1$ approaches one, $R^{EO}$ approaches the value of $3/2$, which is equal to the unconditional expected value. In other words, the bidders expect to pay a fair price and earn a net payoff of zero. The bidders’ net payoffs are the following: under the standard auction, bidder 1 expects to earn $\psi_1/6$ (increasing in $\psi_1$), whereas bidder 2 expects $(1 - \psi_1)/6$ (decreasing in $\psi_1$); if the exclusive offer is used, bidder 1 earns a net payoff of zero (since she is excluded), and bidder 2 earns a net payoff of $(1 - \psi_1)(t_2 - 1)$ (decreasing in $\psi_1$). The expected total net payoffs of the two bidders are $1/6$ under a standard auction and $(1 - \psi_1)/2$ (decreasing in $\psi_1$) if the exclusive offer is used.

The exclusive offer $R^{EO}$ may at first sight seem rather generous: after all, the outsider will never reject it, even if her valuation is the lowest possible. In other words, the exclusive offer is a very crude sale procedure. However, in our bankruptcy setting, it seems plausible. A more sophisticated sale procedure would use a reserve price, that is, a threat by the trustee to withdraw from the sale, thereby encouraging the outsider to reveal her valuation if it is not very low (and pay the corresponding price). In many cases, it may be hard for a bankruptcy trustee to withdraw from a sale that was agreed on (in principle) by the claimants. The only alternative may be a piecemeal liquidation. If that is a credible threat, it will make the exclusive offer more attractive, that is, it becomes even more likely that excluding the insider is optimal (thus reinforcing our result that excluding the insider may be preferred to letting her participate in a standard auction).

The result that excluding the better informed bidder may be beneficial extends to settings with more than two bidders. Consider the case of three bidders, one of whom is better informed and the other two have equally informative signals (the weights on the signals of the two less well-informed bidders are both equal to $(1 - \psi_1)/2$). We can describe equilibrium bids for an ascending (open outcry) auction and compare the expected price with the price raised by an ascending auction in which only the less well-informed bidders can participate. (The analysis of the three-bidder ascending auction is more complicated, since after the first bidder drops out, the remaining two can infer that bidder’s signal and will change their bidding strategy accordingly.) For the example used above (with signals distributed uniformly in the interval $[1,2]$), we find that the restricted auction generates a higher expected price if $\psi_1 > 1/2$. That is comparable to our result for the model with two bidders: in both cases, at the cutoff level of $\psi_1$, the signal $t_1$ is twice as informative as that of other bidders. (In the three-bidder case, compare $\psi_1 = 1/2$ with $\psi_2 = \psi_3 = (1 - \psi_1)/2 = 1/4$; in our original model with two bidders, compare the cutoff $\psi_1 = 2/3$ with $\psi_2 = (1 - \psi_1) = 1/3$.)

To sum up, letting the insider participate has both advantages and disadvantages. The advantage is that her inclusion increases competition; the disadvantage is that the outsider must fear the winner’s curse, which reduces competition. Thus, if the bankruptcy trustee is restricted to using standard auctions, she may benefit from excluding the insider. And consequently, in the
presence of such a restriction, a valid argument can be made for managers of bankrupt firms to be excluded from the liquidation procedure (as buyers). However, this argument holds only if standard auctions are mandated by law, that is, if all potential bidders must be treated equally. If instead the trustee is given the flexibility to optimally design a procedure in each bankruptcy case, she will not want to exclude the insider.

The optimal procedure (analyzed in the Section 4) combines the advantages of both the standard auctions (more competition with more bidders) and the exclusive sale offer to the outsider (avoidance of the winner’s curse). The insider is always allowed to participate, but a larger bidder asymmetry makes it less likely that the insider will win. This is consistent with the results in this section: for sufficiently high values of $\psi_1$, the exclusive sale offer dominates the standard auctions, whereas for lower values of $\psi_1$, the standard auctions extract higher prices. As we will see, the key to maximizing the expected price is optimally biasing the procedure against the insider and yet ensuring that the insider participates.

4. The Optimal Procedure

We derive the optimal selling procedure using the arguments developed in Bulow and Roberts (1989) or Bulow and Klemperer (1996). They show that there is a formal equivalence between the maximization problem of a seller in an auction and that of a monopolist supplying several markets, if the monopolist can charge different prices in different markets (so-called third-degree price discrimination is possible) and has limited capacity. The monopolist should supply more output in markets with high demand, where a higher price can be charged; similarly, the auctioneer should increase the probability of winning for a bidder with a higher willingness to pay. The focus is on marginal changes in both setups: marginal return in the monopolist’s case and marginal willingness to pay in the case of the auction.

The equivalence is convenient because it allows us to base our analysis on the idea of “marginal revenue,” which is well understood in the monopolist’s setup. It also shortens the presentation of our results considerably: the alternative is to use the mechanism design approach developed in Myerson (1981), which leads to exactly the same results but the proofs are much longer (details are available upon request).

Using arguments based on marginal revenue requires that we define it or, more precisely, the variables that allow us to calculate it. In the case of the monopolist, multiplying price and quantity yields revenue and taking derivatives with respect to quantity yields marginal revenue. In the auction setup, the equivalent to the monopolist’s price is a bidder’s valuation. The equivalent to quantity is the probability that the bidder’s valuation is higher than a given level. The monopolist must decide what quantity to offer in which market, trading off the marginal costs and benefits of moving units between markets. The auctioneer must decide how likely any bidder (the equivalent to a “market”) is to win, given the revealed valuations.
Thus, a bidder’s “price” is her valuation $v(t_1, t_2)$, and her “quantity” is the ex ante probability that her valuation is higher, $(1 - F(t_i))$. Multiplying the two and taking derivatives with respect to quantity yields the bidder’s marginal revenue:

$$\frac{\partial}{\partial (1 - F(t_i))} [v(t_1, t_2)(1 - F(t_i))] = \frac{\partial}{\partial t_i} [v(t_1, t_2)(1 - F(t_i))]$$

$$= v(t_1, t_2) - \frac{\psi_i}{H(t_i)}$$

(after replacing $\frac{\partial}{\partial t_i} v(t_1, t_2)$ by $\psi_i$ and $f(t_i)/(1 - F(t_i))$ by $H(t_i)$). The optimal selling procedure follows immediately from equation (4): the bidder with the higher marginal revenue should be the winner. With pure common values, the first terms in the bidders’ marginal revenues (cf. equation (4)) are equal, and the bankruptcy trustee compares only the second terms, the ratios $\psi_i/H(t_i)$. It immediately follows that bidder 1 should win if

$$H(t_1) \geq \frac{\psi_1}{1 - \psi_1} H(t_2),$$

and bidder 2 otherwise. A procedure is optimal if equation (5) is satisfied and each bidder with the lowest possible signal realization ($t_i$) earns her reservation payoff (zero in our model).\(^7\)

**Proposition 3.** The optimal procedure is biased: if $\psi_1 > 1/2$, the insider’s probability of winning is strictly smaller than 1/2, and the outsider’s probability of winning is strictly larger than 1/2.

**Proof.** If $\psi_1 = 1/2$, the fraction on the right-hand side of equation (5) vanishes, so the optimal allocation rule is symmetric: whoever has the highest signal wins. If $\psi_1 > 1/2$, equation (5) is violated if $t_1 = t_2$, and it is also violated if $t_1$ is somewhat larger than $t_2$.

The equilibrium strategies for standard auctions (cf. Proposition 1) are symmetric, and the rule that the highest bid wins then implies that the bidder with the higher signal wins. In other words, both bidders have equal chances of winning. This is different with an optimally designed selling procedure. An inspection of equation (5) shows that the insider’s chances of winning decrease in the extent of the bidder asymmetry: the higher the $\psi_1$, the higher the signal $t_1$ has to be to defeat a given signal $t_2$. If the bidders are symmetric, however, the optimal procedure treats them symmetrically, so standard auctions (like those described in Proposition 1) are optimal.

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7. For details of this argument, see Bulow and Roberts (1989) and Bulow and Klemperer (1996). The analysis is easily extended to the case of more than two bidders, which does not decrease the bias described in equation (5), but as bidders are added, it becomes less likely that any given bidder wins.
Under an optimal selling procedure, there is a cutoff signal for the insider, such that if her signal is below this cutoff, she will certainly not win. (This follows because the inequality in equation (5) is not satisfied for $t_1 = t_2 = t_\star$.) In contrast, the outsider has a chance of winning with any signal. This is an important characteristic of optimal selling procedures with asymmetrically informed bidders: in order to extract value from the better informed bidder, it is important to make it hard for her to win with a low signal. With monotonically increasing bidding strategies, this translates into accepting only high bids from the insider, who must then come forward with a high bid and thereby reveal a possibly high valuation that she would rather have kept secret.

Of course, biasing the allocation rule is not without problems. It is easy to show that the bidders may try to undermine the procedure if they did not win. For example, a losing insider may offer to pay more than the price that the winner is supposed to pay, and similarly, a losing outsider may offer to top a winning insider’s price. Similar problems arise in takeover situations (outside of bankruptcy, i.e., involving healthy firms), where the use of deal protection devices has become common (see Povel and Singh 2006). However, in a bankruptcy setting, commitment to the outcome (and rules) of a sale procedure seems to be less of a problem. The bankruptcy judge approves the sale and its rules, and a court order determines who won in an auction (additionally, bankruptcy sales are often held in the court room).

Proposition 4. Standard auctions (cf. the equilibrium strategies described in Proposition 1) are optimal if and only if $\psi_1 = 1/2$, that is, if the bidders are symmetric.

Proof. Proposition 3 shows that if $\psi_1 = 1/2$, the optimal allocation rule treats the bidders symmetrically, just like the standard auctions. From the revenue equivalence theorem (cf. Myerson 1981), a selling procedure is equivalent to the optimal procedure if the allocation rules are the same and a bidder’s payoff is zero if she observed the lowest possible signal. This is the case if $\psi_1 = 1/2$, so the standard auctions are optimal in this case. Proposition 1 shows that in equilibrium, the standard auctions generate the same expected price, constant in $\psi_1$. In contrast, the optimal procedure generates an expected price that is increasing in $\psi_1$ (for a proof of this claim, see Povel and Singh 2004). Thus, for all $\psi_1 > 1/2$, the standard auctions are strictly suboptimal.

The optimal sale procedure raises a higher expected price than a standard auction. The bankruptcy trustee can extract more rents by appropriately biasing the procedure, which does not happen with standard auctions: they treat the bidders symmetrically and thereby let them earn rents that are unnecessarily high. The key feature of the optimal procedure is thus its bias against the insider: she is less likely to win and she wins only if her signal (and bid) is sufficiently high.

To illustrate how biasing the procedure increases the expected price, consider our earlier question, whether the insider should be allowed at all to bid at
all. Recall that if $\psi_1 < 1$, accepting bids only from the outsider leaves some money on the table (see the discussion in Section 3). Allowing the insider to participate may help the seller recoup some of this money. The optimal procedure raises a higher expected price through cherry-picking: it sells the firm to the insider if and only if her signal is very high, forcing her to pay a high price. The outsider receives the firm otherwise. The outsider’s willingness to pay will be reduced if she hears that the insider was excluded since she can infer that the insider’s signal was low. But this loss is smaller than the gain from selling to the insider: the outsider would have earned a large profit by paying an average price for a high-value firm; and given that the insider needs a high bid to win the firm, she has little scope to shade her bid and earn a large rent.

Our analysis extends to a setup that allows for both private value and common value components, that is, a setup in which the bidders put more emphasis on their own signal than the other bidders do. Povel and Singh (2006) have analyzed such a model. The analysis is more complex, but the main result remains valid: in the presence of bidder asymmetries, the trustee should bias the selling procedure against the better informed bidder, by making it harder for her to win with a low bid. What complicates the analysis is that with private value components, the trustee must also worry about awarding the firm to the bidder with the highest valuation (this is not an issue in our model since the value is common to all bidders, given the signals). A strongly biased procedure helps in extracting a larger fraction of the value that the bidders expect to receive, but a less biased procedure creates more value, so the total value extracted may be higher (if the less biased procedure extracts a somewhat smaller fraction of a significantly larger value). This trade-off gives rise to preferential treatment of the better informed bidder if she is willing to pay a high price. But it does not change the result that it should be harder for her to win with a low bid.

5. Implications

One of the questions that we asked at the start of this article was whether a former owner or manager should be allowed to bid for the assets of a bankrupt firm and, if so, under what conditions. We find that insider bids should always be allowed if the right sale procedure can be chosen and that it may be better not to allow insider bids, depending on the circumstances, if a standard auction must be used. We now analyze the negotiating environment in which bankrupt firms are sold in the United States and in particular what constraints are imposed on bankruptcy sale procedures. We describe an important flaw in the rules designed to protect the interests of the claimants and that this rule can easily be amended.

There are three ways to sell bankrupt firms in the United States. First, firms can be liquidated under Chapter 7 of the Bankruptcy Code. This seems to be appropriate for firms that are not sold as going concerns but liquidated piece-meal (or if there are no significant assets to liquidate). Second, a Chapter 11 reorganization plan can have a sale of the firm as its main operating decision (otherwise focusing mostly on the distribution of the proceeds). A third
alternative is a so-called Section 363 sale of essentially all assets, outside the scope of a plan negotiated with the creditors.

Section 363 sales have their benefits: the negotiations include only the buyer and the seller, but none of the other parties involved in the bankruptcy. So the negotiations can be completed quickly and efficiently and with less uncertainty over the outcome (by not requiring the confirmation of a plan). Another benefit is the possibility to purchase assets “free and clear of liens.” However, Section 363 sales have one main (and unexpected) drawback. In an attempt to protect the estate (and the creditors) of a bankrupt firm, a market test of a negotiated deal is usually required. The trustee is trusted in negotiating a sale but only if the sale shows evidence of being fair and of raising a reasonable price. As a test, case law has developed a recommendation to expose an existing offer to overbids: if no bidder is willing to bid higher, then the offered price is deemed to be fair or top dollar.8

By requiring this exposure to overbids, the bankruptcy courts implicitly require the bankruptcy trustee to use a standard auction. Even if the trustee negotiates a sale with one or several parties and the negotiations are more complex than a standard auction, the overbid requirement is anticipated by all parties, so strategically, the negotiations represent cheap talk, and the parties’ real focus is on the bidding that follows. Consider the situation of the first bidder, commonly referred to as the stalking horse. She may consider submitting a low first bid. In fact, there is nothing that keeps her from doing so, whereas she may benefit if no overbid materializes. If other bidders overbid, the stalking horse typically has the right to match such overbids or to top them. The other bidders, in turn, can submit even higher bids. This process will stop only if no bidder is willing to raise the price further. This process is the same as that of an ascending auction, except that the first bid comes from a specific bidder.

Our analysis shows that in the presence of bidder asymmetries, standard auctions are suboptimal (see Proposition 4). Overbid requirements seem optimal given an existing bid for the firm’s assets. However, there is a tension between the ex-post benefits of allowing overbids and the damage that this causes to the initial bidding incentives. Once all bids are on the table, it is in the interest of the estate to allow overbids since this can only increase the price that will eventually be paid. Also, it may seem unfair to casual observers if overbids are not allowed. However, our analysis shows that it is important to bias the selling procedure, if it is to extract the highest possible price. The reason for this is that if the bidders anticipate that the selling procedure will treat them symmetrically, the problems that motivated our analysis (outsiders’ worries about the winner’s curse) are not resolved. Consequently, the bidders will not put their best offers on the table. The dissenting opinion in *Omnicare, Inc., v. NCS Healthcare, Inc.*, recognized this, arguing that

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8. The Supreme Court clearly stated that “the best way to determine value is exposure to a market,” including (but not limited to) an invitation to other potential buyers to outbid the existing offer. See *Bank of America National Trust & Savings Association v. 203 N. LaSalle Street Partnership*, 526 U.S. 434 (1999).
sometimes exclusive deals need to be followed through even if an unsolicited higher bid comes in from a third party, since otherwise willing bidders would not put their best bids on the table in the first place.9

The immediate policy implication is that the bankruptcy courts should put less trust in the overbid requirement and instead allow the trustee to choose the best method to sell a bankrupt firm’s assets.10 Our model suggests that if the trustee can identify a bidder who is better informed than others, then the trustee should make it harder for this better informed bidder to win with a low bid. The trustee may even prefer to exclude that bidder altogether (this may be necessary to prevent her from topping up a high bid from a less well-informed bidder).

There are two issues that need to be considered when delegating the design of a selling procedure to a trustee. First, a trustee must be able to identify a better informed bidder and how much better her information is. As we suggested above, the trustee may find out about bidder asymmetries while acquainting herself with the bankrupt firm in conversations with employees, customers, suppliers, and other stakeholders. But if it is hard to identify a better informed bidder, then it is best to ignore the issue and treat all bidders symmetrically. Second, in many cases, the bankruptcy trustee is the debtor in possession, that is, the owner-manager of the bankrupt firm. Incentive conflicts would arise if the owner-manager was allowed to design the selling procedure and then also to bid for the bankrupt firm. However, it seems that in practice this conflict of interest has been identified: such bids would constitute a federal crime (18 U.S.C. §154), and judges have tended to treat bids from insiders with suspicion.11 The appropriate course of action would be the same as in management buyouts: the manager-owner should be expected to resign from her management position before bidding, including the position of trustee.

6. Conclusion

The aim of this article was to analyze the desirability of a frequent outcome of bankruptcy sales: that a former owner or manager of a bankrupt firm buys back...
the assets and gets a fresh start. Our focus was on how this possible outcome affects bidding by other potential buyers, who have to worry more about the winner’s curse when competing with a much better informed rival. Bankruptcy trustees must take this effect into consideration when planning a sale of a bankrupt firm since the rational response to the winner’s curse is to submit lower bids or not to submit any bids at all.

In practice, bankruptcy judges and trustees worry explicitly about the degree of competition in the sale of a bankrupt firm, by requiring that existing bids are exposed to overbids by other parties. But as we show, these market tests are insufficient since they resemble a standard ascending auction. They treat all bidders symmetrically, which is suboptimal. If using a symmetric auction is a requirement, then it is optimal to exclude insiders from the bidding if their informational advantage over third parties is sufficiently large.

However, it would be better to give a bankruptcy trustee the freedom to design a sale procedure that treats bidders asymmetrically. Specifically, the procedure should be biased against insiders, making it harder for them to win with low bids. The benefit of an appropriately designed procedure is that it lets anybody participate in the bidding: intuitively, the more competition there is to buy the bankrupt firm, the higher the expected price. And the optimal bias induces generally higher bids: an insider cannot lowball since her low bids would be rejected, and the outside bidders worry less about the winner’s curse, so their bids will be higher, too.

The bankruptcy literature discusses other issues that arise in connection with financial distress, which we do not explicitly address in our model. However, these do not affect our analysis. For example, one question is how an improved sale procedure affects the debtor’s incentives, both in bankruptcy and outside. Outside of bankruptcy, a better designed sale procedure should improve a debtor’s incentives to perform well and stay out of bankruptcy since bankruptcy becomes a less attractive outcome. With an ill-designed procedure, a debtor can hope to purchase the assets back at a low price and to get a fresh start; with an optimally designed procedure, the option still exists but the debtor would have to pay a price that is closer to what the assets are actually worth. Inside bankruptcy, there are numerous incentive problems, but they are not special to the optimal sale procedure that we describe in this article. These incentive problems are well understood, and bankruptcy negotiations are structured explicitly to mitigate these incentive problems (see Povel 1999; Ayotte 2007).

In sum, improving the sale procedure should benefit the creditors of a bankrupt firm, who can expect higher recovery rates, and this in turn should benefit firms who are taking on debt to finance their operations since this debt becomes a cheaper source of funds. Our article shows how a sale should be conducted, once the parties in control have decided that a sale is the best way to proceed. We do not argue that sales are necessarily superior to reorganizations. Instead, we argue that under the current practice, sales may have yielded lower recovery rates than necessary and that it is possible to improve upon this subpar performance.
Appendix

Proof of Proposition 1.

(i) First-price equilibrium: The conjectured equilibrium bidding functions are symmetric and strictly increasing and therefore invertible. Define \( \beta_i \) such that \( \beta_i(b_i(t_i)) = t_i \). Abusing notation, let \( b_i \) be \( i \)'s optimal bid given a signal realization \( t_i \). Her expected payoff is

\[
\int_{t_i}^{b_i(t_i)} (\psi_i t_i + (1 - \psi_i)t_j - b_i)f(t_j)dt_j.
\]

Derive the first-order condition (with respect to \( b_i \)) and rearrange it as follows:

\[
-F(\beta_j(b_i)) + (\psi_i t_i + (1 - \psi_i)\beta_j(b_i) - b_i)f(\beta_j(b_i)) \frac{\partial \beta_j(b_i)}{\partial b_i} = 0. \quad (A1)
\]

Using the symmetry of the conjectured bidding functions, rewrite equation (A1) as

\[
-F(t_i) + (\psi_i t_i + (1 - \psi_i)t_i - b(t_i))f(t_i)\frac{1}{(\frac{\partial b(t_i)}{\partial t_i})} = 0. \quad (A2)
\]

Substituting the bidding strategy \( b^{FP} \) in equation (A2) shows that equation (A2) is satisfied, that is, the conjectured bidding function satisfies the first-order condition.

(ii) Second-price equilibrium: In the second-price auction, a bidder can only affect her probability of winning and not the price. Any deviation from the bidding strategy \( b^{SP}(t_i) = t_i \) will enable a bidder to win in states she makes a loss or lose in states that she is better off winning. The equilibrium payoff for bidder \( i \) is

\[
V^{SP}(t_i) = \int_{t_i}^{\psi_i t_i + (1 - \psi_i)t_i - b(t_i)f(t_j)dt_j = \psi_i \int_{t_i}^{t_i} f(t_j)dt_j, \quad (A3)
\]

which is nonnegative. Now consider two possible types of deviation.

\textit{Deviations} \( t_i^+ > t_i \). In many cases, this will not affect the insider’s payoff. The only changes arise if she now wins the auction but would have lost it by bidding \( t_i \). This happens if \( t_j \in (t_i, t_i^+] \). The integrand in equation (A3) is decreasing in \( t_i \) so since the integrand is zero in \( t_j = t_i \), it is negative for all \( t_j \in (t_i, t_i^+] \).

\textit{Deviations} \( t_i^- < t_i \). In many cases, this will not affect the insider’s payoff. The only changes arise if she now loses the auction but would have won it by bidding \( t_i \). This happens if \( t_j \in (t^-_i, t_i] \). The
The integrand in equation (A3) is zero in \( t_j = t_i \) and it is decreasing in \( t_j \), so the forgone payoff is strictly positive.

(iii) Same expected price: The expected price with the second-price auction is

\[
R^{\text{SP}} = 2 \int_{t_i}^{t_f} \int_{t_i}^{t_f} t_j f(t_j) dt_j f(t_i) dt_i,
\]

whereas that for the first-price auction is

\[
R^{\text{FP}} = 2 \int_{t_i}^{t_f} F(t_i) \left( \frac{1}{F(t_i)} \int_{t_i}^{t_f} t_j f(t_j) dt_j \right) f(t_i) dt_i
= 2 \int_{t_i}^{t_f} \int_{t_i}^{t_f} t_j f(t_j) dt_j f(t_i) dt_i.
\]

We have \( R^{\text{SP}} = R^{\text{FP}} \), constant in \( \psi_1 \).

References


