Optimal debt with unobservable investments: Web-Appendix

Paul Povel*

Michael Raith**

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* Carlson School of Management, University of Minnesota, Minneapolis, MN 55455; phone: (612) 624 0266; email: povel@umn.edu.

** William E. Simon Graduate School of Business Administration, University of Rochester, Rochester, NY 14627, USA; phone: (585) 275 8380; email: raith@simon.rochester.edu
Appendix B: \( \pi \) as a function of \( k \)

In this appendix we discuss how our results change if we allow \( \pi \) to depend on \( k \). Define \( \pi(k) \) with \( \pi(0) \geq 0, \pi' > 0, \pi'' < 0, \text{ and } \lim_{k \to \infty} y'(k) + \pi'(k) < 1 \). A small investment is now associated with a small \( \pi \), which in turn may affect \( E \)'s incentives to repay his debt. We show that a debt contract, though not necessarily a simple debt contract, remains optimal.

B.1 Optimality of a debt contract

Proposition 2 holds as stated, with \( \pi \) replaced by \( \pi(k^*(W,T,\beta)) \). The only part of the proof affected by this change is step 3. By construction, switching from \((W,T,\beta)\) to \((W,T^1,\beta^1)\) (with \( \pi \) replaced by \( \pi(k^*(W,T,\beta)) \)) leaves \( E \)'s payoff unchanged for all \((R,\hat{R})\), as long as \( k^*(W,T^1,\beta^1) = k^*(W,T,\beta) \). However, if \( \pi \) is not constant, then switching to \((W,T^1,\beta^1)\) is not payoff-neutral for \( E \) if \( k \neq k^*(W,T,\beta) \), and \( k^*(W,T^1,\beta^1) \) may differ from \( k^*(W,T,\beta) \). We can show that \((W,T^1,\beta^1)\) nevertheless Pareto-dominates \((W,T,\beta)\).

Let \( u^1(R,\hat{R},k) = R - T^1(\hat{R}) + \beta(\hat{R})\pi(k) \), and define \( u^0(R,\hat{R},k) \) analogously for contract \((W,T,\beta)\).

For \( \hat{R} \in \rho_\alpha \), we have

\[
\begin{align*}
u^1(R,\hat{R},k) &= R - \hat{R} + \left[\beta(\hat{R}) + \frac{\hat{R} - T(\hat{R})}{\pi(k^*)}\right]\pi(k) - [\hat{R} - T(\hat{R})] \left(1 - \frac{\pi(k)}{\pi(k^*)}\right).
\end{align*}
\]

Similarly, for \( \hat{R} \in \rho_\beta \),

\[
\begin{align*}
u^1(R,\hat{R},k) &= R - T(\hat{R}) - [1 - \beta(\hat{R})]\pi(k^*) + \pi(k) - [\pi(k^*) - \pi(k)][1 - \beta(\hat{R})].
\end{align*}
\]

For \( \hat{R} \notin \rho \), we have \( u^1(R,\hat{R},k) = u^0(R,\hat{R},k) \). If \( E \) chooses \( k < k^* \), then \( \pi(k) < \pi(k^*) \) and hence \( u^1(R,\hat{R},k) < u^0(R,\hat{R},k) \) for all \( \hat{R} \in \rho \). By definition of \( k^* \), we then have

\[
\begin{align*}
\mathbb{E}_\theta[\max_{\hat{R}} u^1(R(k^*,\theta),\hat{R},k^*)] - k^* &= \mathbb{E}_\theta[\max_{\hat{R}} u^0(R(k^*,\theta),\hat{R},k^*)] - k^* \\
&\geq \mathbb{E}_\theta[\max_{\hat{R}} u^0(R(k,\theta),\hat{R},k)] - k \geq \mathbb{E}_\theta[\max_{\hat{R}} u^1(R(k,\theta),\hat{R},k)] - k.
\end{align*}
\]
(Notice that we are not assuming truth-telling for any \( k \neq k^* \) under either contract.) This means that \( E \) would never choose \( k < k^* \) under \((W, T^1, \beta^1)\); hence \( k^*(W, T^1, \beta^1) \geq k^*(W, T, \beta) \), possibly with strict inequality since for \( k > k^* \) the second inequality above is reversed. The contract \((W, T^1, \beta^1)\) is incentive compatible if for all \( R \) and \( \hat{R} < R \),

\[
    u^1(R, R, k) - u^1(R, \hat{R}, k) = \hat{R} - R + [\beta^1(R) - \beta^1(\hat{R})] \pi(k) \geq 0. \tag{B1}
\]

The term \( \beta^1(R) - \beta^1(\hat{R}) \) must be nonnegative. Suppose not: then \( \beta^1(R) < \beta^1(\hat{R}) \) would imply \( \beta^1(R) < 1 \) and therefore \( T^1(R) = R \), as well as \( T^1(R) < T^1(\hat{R}) < R \), a contradiction. Hence, since (B1) holds for \( k = k^* \), it must also hold for any larger \( k \).

Step 4 of Proposition 2 can be applied to show that \((W, T^1, \beta^1)\) must satisfy (7) and (8). \( I \)'s expected stage-4 payoff can then be written as

\[
    \int_{0}^{D/y(k)} y(k)\theta f(\theta)d\theta + [1 - F(D/(y/k))] \tag{B2}
\]

which is increasing in \( y(k) \). Thus, if \( I \)'s payoff is higher with \((W, T^1, \beta^1)\) than with \((W, T, \beta)\) for \( k = k^* \), the same must be true for any larger \( k \). As before, \( E \) can appropriate this increase by designing a new contract \((W, T^2, \beta^2)\).

**B.2 Investment incentives and the optimal contract**

Suppose \( E \) and \( I \) write a simple debt contract \((W, T, \bar{\beta})\), where \( \bar{\beta}(R) = 1 - (D - R)/\pi(k_0) \) and \( E \) and \( I \) expect \( E \) to choose \( k_0 \). Clearly, we can restrict our attention to contracts that set \( W = k_0 \). Define \( u(R, \hat{R}, k) = R - T(\hat{R}) + \beta(\hat{R})\pi(k) \). If \( E \) invests \( k \), for \( \hat{R} \geq D \) we have \( u(R, \hat{R}, k) = R - D + \pi(k) \), and for \( \hat{R} < D \)

\[
    u(R, \hat{R}, k) = R - \hat{R} + \left(1 - \frac{D - \hat{R}}{\pi(k_0)} \right) \pi(k) = R - D + \pi(k) + (D - \hat{R}) \left(1 - \frac{\pi(k)}{\pi(k_0)} \right). \tag{B3}
\]

Since \( \pi' > 0 \), inspection of (B3) shows that the contract induces truth-telling if \( k \geq k_0 \). If \( k < k_0 \), however, \( E \) would announce \( \hat{R} = 0 \) and not make any payment to \( I \). For the contract to be feasible, therefore, \( E \) must not have an incentive to choose any \( k < k_0 \).
If in stage 2 \( E \) invests \( k_0 \), he subsequently has an incentive to report his funds truthfully, and thus his expected payoff as of stage 2 is
\[
y(k_0) - D + \pi(k_0)
\] (recall that \( W = k_0 \)). If he invests \( k < k_0 \), he will not repay anything in stage 4, and thus his expected payoff in stage 2 is
\[
y(k) + \pi(k) + k_0 - k - \frac{D}{\pi(k_0)} \pi(k),
\] which coincides with (B4) for \( k = k_0 \). Under our assumptions, (B5) has a unique maximum in \( k \) for given \( k_0 \); denote it by \( \kappa(k_0) \). I would not agree to lend \( k_0 \) if he expected \( E \) subsequently to choose \( k < k_0 \); thus a simple debt contract is feasible only if \( \kappa(k_0) \geq k_0 \). Define the first-best investment as \( k^{FB} = \arg \max y(k) + \pi(k) - k \). Since \( k^{FB} \) maximizes the first four terms in (B5), it follows that \( \kappa(k^{FB}) < k^{FB} \). This means that no simple debt contract can induce \( E \) to choose \( k^{FB} \); and by continuity, the same holds for all \( k \in [\bar{k}, k^{FB}] \) for some \( \bar{k} < k^{FB} \).

Denote by \( k^{SB} \) the solution to \( \max_k y(k) - D(k) + \pi(k) \), where \( D(k) \) solves (A4). If \( \bar{k} \geq k^{SB} \), then the results of Section 5 continue to hold: A simple debt contract with \( W = k^{SB} \) and \( \bar{\beta} = 1 - (D - R)/\pi(k^{SB}) \) induces the choice of \( k^{SB} < k^{FB} \), and is optimal.

If \( \bar{k} < k^{SB} \), it may be optimal to write a non-simple debt contract, such as of the form described in Propositions 4 and 5, to induce \( E \) to choose \( k > \bar{k} \). As in Sections 6 and 7, however, both the benefit and the cost of using a non-simple contract are of first-order magnitude. If the cost of liquidating with higher probability exceeds the benefit of investing \( k > \bar{k} \) even at the margin, then the optimal contract is again a simple debt contract, with \( W = \bar{k} \) and \( \bar{\beta} = 1 - (D - R)/\pi(\bar{k}) \).

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1 To illustrate, let \( y(k) = \sqrt{k}, \pi(k) = \alpha y(k) \) for \( \alpha > 0 \), and assume that \( \theta \) is uniformly distributed over \([0,2]\). Then \( \bar{k} \geq k^{SB} \), and a simple debt contract is optimal if and only if \( \alpha \geq 3/2 \). If \( \alpha \) is much smaller than \( 3/2 \), then a contract of the form \((19)\) is preferred to a simple debt contract (but it is not necessarily the optimal contract); whereas if \( \alpha \) is not much smaller than \( 3/2 \), a simple debt contract where \( E \) invests \( \bar{k} < k^{SB} \) is preferred.