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Direct Transformation of Variational Problems into Cauchy Systems. I. Scalar-Quadratic Case[1]

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Abstract. This series of papers addresses three interrelated problems: the solution of a variational minimization problem, the solution of integral equations, and the solution of an initial-valued system of integro-differential equations. It will be shown that a large class of minimization problems requires the solution of linear Fredholm integral equations. It has also been shown that the solution of a linear Fredholm integral equation is identical to the solution of a Cauchy system. In this paper, we bypass the Fredholm integral equations and show that the minimization problem directly implies a solution of a Cauchy system. This first paper in the series looks only at quadratic functionals and scalar functions.

Key Words. Variational problems, integral equations, parametric imbedding.

1. Introduction

Many interesting optimization problems result in variational problems of finding a function $z(t)$, $0 \leq t \leq 1$, that minimizes the functional

$$W[z] = \lambda \int_0^1 \int_0^1 z(s)k(t, s)z(t)\, dt\, ds + \int_0^1 z(t)^2\, dt + 2\int_0^1 f(t)z(t)\, dt.$$

By standard variational techniques, it can be shown that the optimal function $u(t)$ satisfies a linear Fredholm integral equation (Ref. 1). Recent work in the study of integral equations (Ref. 2) has shown that solutions of integral equations are equivalent to solutions of particular initial-valued

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systems of integro-differential equations (Cauchy systems). It appears that the three problems—variational, integral equations, and Cauchy systems—are equivalent to each other. An important missing link in the analysis has been the demonstration that the variation problem leads directly to a Cauchy system without passing through an integral equation. These papers provide that link. It would seem that the historical emphasis put on the integral equations is but an accident. The Cauchy systems can just as easily be developed and provide interesting insights and computational aid in studying such variational problems.

The direct reduction of the simplest problem in the calculus of variations to a Cauchy system is given in Ref. 3; the train of thought is quite different from that employed here.

2. Derivation

Suppose that we desire to find a scalar function $z(t)$, $0 \leq t \leq 1$, which minimizes the quadratic functional

$$W[z, \lambda] = \lambda \int_0^1 \int_0^1 z(s)k(t, s)z(t)\, ds\, dt + \int_0^1 z(t)^2\, dt + 2\int_0^1 f(t)z(t)\, dt, \tag{1}$$

where $k(t, s)$ is a symmetric, positive definite kernel, $f(t)$ is a given function, and λ is a sufficiently small nonnegative scalar parameter. The standard variational approach to this problem results in a linear Fredholm integral equation which the optimal function must satisfy.

Proposition 2.1. The function $u(t)$ which minimizes $W[u, \lambda]$ must satisfy the linear Fredholm integral equation

$$u(t)+f(t)+\lambda \int_0^1 k(t, s)u(s)\, ds = 0, \qquad 0 \leq t \leq 1. \tag{2}$$

Proof. Denote the optimal function by $u(t)$. The arbitrary admissible functions may be written as

$$w(t) = u(t) + \epsilon\eta(t),$$

where ϵ is an arbitrary scalar and $\eta(t)$ is an arbitrary function. The value of the function for $w(t)$ may be expressed as

$$W[w; \lambda] = W[u, \lambda] + \epsilon C[u, \eta; \lambda] + \epsilon^2 D[\eta; \lambda],$$

where

$$C[u, \eta; \lambda] = 2\lambda \int_0^1 \int_0^1 u(s)k(t, s)\eta(t)\, ds\, dt + 2\int_0^1 u(t)\eta(t)\, dt + 2\int_0^1 f(t)\eta(t)\, dt,$$

$$D[\eta, \lambda] = \lambda \int_0^1 \eta(s)k(t, s)\eta(t)\, ds\, dt + \int_0^1 \eta(t)^2\, dt.$$

For ϵ sufficiently small, the term in ϵ^2 may be ignored. If u is in fact the minimizing solution, then for all ϵ

$$C[u, \eta; \lambda] = 0. \tag{3}$$

We apply to Eq. (3) the fundamental lemma of the calculus of variations (Ref. 1):

$$\int_0^1 g(t)h(t)\, dt = 0$$

for arbitrary $h(t)$ implies that

$$g(t) = 0$$

for all $0 \leq t \leq 1$. Since $\eta(t)$ is arbitrary, we get Eq. (2). □

Another approach to the minimization problem is to ask how the optimal solution changes as λ varies. This is referred to as *parametric imbedding*. The standard technique would begin with the linear Fredholm integral equation and convert it into a system of initial-valued integro-differential equations. The basic result given in Ref. 2 is the following proposition.

Proposition 2.2. The function $u(t, \lambda)$ which satisfies the linear Fredholm integral equation (2) is the solution $u(t, \lambda)$ to the following initial-valued integro-differential equations in u, augmented by a resolvent kernel $K(t, s, \lambda)$, and conversely:

$$0 = u_\lambda(t, \lambda) + \psi(t, \lambda) + \int_0^1 K(t, s, \lambda)\psi(s, \lambda)\, ds, \tag{4}$$

$$K_\lambda(t, s, \lambda) = \int_0^1 K(t, s', \lambda)K(s', s, \lambda)\, ds', \tag{5}$$

$$u(t, 0) = -f(t), \tag{6}$$

$$K(t, s, 0) = -k(t, s), \tag{7}$$

$$\psi(t, s) = \int_0^1 k(t, s)u(s, \lambda)\, d\lambda, \tag{8}$$

$$0 \le t \le 1, \qquad 0 \le s \le 1. \tag{9}$$

The Cauchy system (4)–(9) has proved to be very useful for the computation of the solution of Fredholm integral equations, as well as for the study of the sensitivity of the solution to changes in the parameter λ.

We will now show that the Cauchy system (4)–(9) may be derived directly from the original minimization problem, without ever writing down the Fredholm integral equation (2). In fact, the solution of the minimization problem could have proceeded even if historically the Fredholm integral equation had never been discovered. The Cauchy system is perfectly adequate for describing the optimal function $u(t)$.

Proposition 2.3. The function $u(t, \lambda)$ which minimizes $W[u; \lambda]$ satisfies the Cauchy system (4)–(9).

Proof. Suppose that $u(t, \lambda)$ is the minimizing function for the parameter value λ; and suppose that $u(t, \lambda + d\lambda)$ is the minimizing function for the parameter value $\lambda + d\lambda$; then, the first variations must satisfy the following inequalities for arbitrary ϵ, η, $\bar{\epsilon}$, $\bar{\eta}$:

$$\epsilon C[u(t, \lambda), \eta; \lambda] \ge 0, \tag{10}$$

$$\bar{\epsilon} C[u(t, \lambda + d\lambda), \bar{\eta}; \lambda + d\lambda] \ge 0. \tag{11}$$

Since ϵ, η, $\bar{\epsilon}$, $\bar{\eta}$ are all arbitrary, pick

$$\bar{\eta} = \eta \qquad \text{and} \qquad \epsilon = -\bar{\epsilon},$$

so that (10) and (11) are

$$-\bar{\epsilon} C[u(t, \lambda), \bar{\eta}; \lambda] \ge 0,$$

$$\bar{\epsilon} C[u(t, \lambda + d\lambda), \bar{\eta}; \lambda + d\lambda \ge 0.$$

Add these two inequalities to get

$$\begin{aligned} \bar{\epsilon}[2\lambda \int_0^1 \int_0^1 & (u(s, \lambda + d\lambda) - u(s, \lambda))k(t, s)\bar{\eta}(t)\, ds\, dt \\ & + 2\int_0^1 (u(t, \lambda + d\lambda) - u(t, \lambda))\bar{\eta}(t)\, dt \\ & + 2\, d\lambda \int_0^1 \int_0^1 u(s, \lambda + d\lambda)k(t, s)\bar{\eta}(t)\, ds\, dt] \ge 0. \end{aligned} \tag{12}$$

Since $\bar{\epsilon}$ has arbitrary sign, the term in square brackets in (12) must be zero. Since $\bar{\eta}(t)$ is arbitrary, apply the fundamental lemma of the calculus of variations to the bracketed term to get

$$2\lambda \int_0^1 k(t,s)(u(s,\lambda+d\lambda)-u(s,\lambda))\,ds+2(u(t,\lambda+d\lambda)-u(t,\lambda))$$
$$+2d\lambda \int_0^1 u(s,\lambda+d\lambda)k(t,s)\,ds=0. \tag{13}$$

Divide (13) by 2 $d\lambda$ and take the limit as $d\lambda \to 0$, to obtain

$$0=u_\lambda(t,\lambda)+\lambda \int_0^1 k(t,s)u_\lambda(s,\lambda)\,ds+\int_0^1 k(t,s)u(s,\lambda)\,ds. \tag{14}$$

This is a Fredholm integral equation with kernel $k(t,s)$. Corresponding to $k(t,s)$ is a resolvent kernel $K(t,s,\lambda)$ such that solutions of Fredholm integral equations

$$0=v(t)+\psi(t,\lambda)+\lambda \int_0^1 k(t,s)v(s)\,ds \tag{15}$$

can be expressed as

$$0=v(t)+\psi(t,\lambda)+\lambda \int_0^1 K(t,s,\lambda)\psi(x,\lambda)\,ds.$$

Using the resolvent kernel to express the solution of (14), where

$$\psi(t,\lambda)=\int_0^1 k(t,s)u(s,\lambda)\,ds$$

we get

$$0=u_\lambda(t,\lambda)+\psi(t,\lambda)+\lambda \int_0^1 K(t,s,\lambda)\psi(s,\lambda)\,ds.$$

This is nothing other than Eq. (4) of the Cauchy system. The resolvent kernel must itself satisfy a Fredholm integral equation

$$0=K(t,s,\lambda)+k(t,s)+\lambda \int_0^1 k(t,s')K(s',s,\lambda)\,ds'. \tag{16}$$

Differentiate (16) with respect to λ to get

$$0=K_\lambda(t,s,\lambda)+\int_0^1 k(t,s')K(s',s,\lambda)\,ds'+\lambda \int_0^1 k(t,s')K_\lambda(s',s,\lambda)\,ds'. \tag{17}$$

From Eq. (16), we can reexpress the second term of (17):

$$0 = K_\lambda(t, s, \lambda) - (1/\lambda)[K(t, s, \lambda) + k(t, s)] + \lambda \int_0^1 k(t, s')K_\lambda(s', s, \lambda)\, ds'. \quad (18)$$

However, Eq. (18) is a Fredholm integral equation with the same kernel as Eq. (15). Its solution may be expressed using the same resolvent kernel $K(t, s, \lambda)$,

$$0 = K_\lambda(t, s, \lambda) - (1/\lambda)[K(t, s, \lambda) - k(t, s)] + \lambda \int_0^1 K(t, s, \lambda) - (1/\lambda)[K(s', s, \lambda) + k(s', s)]\, ds'. \quad (19)$$

Using Eq. (16) again, several terms in Eq. (19) cancel out, leaving

$$K_\lambda(t, s, \lambda) = \int_0^1 K(t, s', \lambda)K(s', s, \lambda)\, ds',$$

which is just Eq. (5) of the Cauchy system. To get the initial conditions (7), set

$$\lambda = 0$$

in Eq. (16). To get the initial condition (6), set $\lambda = 0$ in $W[z, \lambda]$. The minimizing function $u(t, 0)$ of

$$\int_0^1 u(t, 0)^2\, dt + 2\int_0^1 f(t)u(t, 0)\, dt$$

can easily be shown to be

$$u(t, 0) = f(t). \qquad \square$$

3. Discussion

The main purpose of this paper has been to reduce the quadratic variational problem in Eq. (1) to the Cauchy system in relations (4)–(9). In particular, we have been able to do this without making any use of the Euler equation, which takes the form of the Fredholm integral equation (2).

Subsequent papers in this series will be devoted to systems and to the treatment of nonquadratic variational problems. Application to team decision theory and other areas will be presented.

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