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Risk and the Gain from Information*

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1. INTRODUCTION

This paper is an exploration of how much truth there is in the following intuitive statement.

> As the economic environment gets riskier people will value their information sources more.

This is appealing but it is not universally true. An individual who is a risk seeker would not agree. Suppose that a gambler is haunted by a demon who tells him the outcome of a game of chance before he wagers. The demon's messages have positive and negative aspects to the risk seeker: They allow him to have higher income but they eliminate some of the thrill of the game.

John Gould [4] recognized that further restrictions must be placed on the tastes of the decision maker before the above intuitive statement is correct. Gould provides both a numerical counter example to this statement and a positive result stating one situation in which the statement is correct. Later Jean-Jacques Laffont [5] modified the technical definition of the phrase "increase in risk" and found that a counterintuitive response may occur. In this paper Gould's positive result will be significantly improved and Laffont's result will be shown to be inconsequential.

2. RISK AND THE GAIN FROM INFORMATION

Let α be the state of nature and X be the action taken by a decision-maker. The probability distribution of α is $F(\alpha)$, a monotonic function mapping the unit interval into the unit interval. The decision-maker receives

* Peter Diamond suggested that additional progress might be made on this topic and has made several crucial comments about the content of the paper.

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a payoff, $h(\alpha, X)$, which depends on the state of nature and action. If the decision-maker has no information upon which to base his action, he selects X^* to maximize expected payoff,

$$\int h(\alpha, X)\, dF(\alpha). \tag{1}$$

If the decision-maker has an information source which tells him the true value of α before the action is fixed then he selects X^*_α to maximize $h(\alpha, X)$ for each α. The gain from using the information is defined by

$$\begin{aligned} G &= \int h(\alpha, X^*_\alpha)\, dF(\alpha) - \int h(\alpha, X^*)\, dF(\alpha) \\ &= \int \max_X\, [h(\alpha, X)]\, dF(\alpha) - \max_X \left[\int h(\alpha, X)\, dF(\alpha) \right]. \end{aligned} \tag{2}$$

The gain from information is the difference between the expected payoff when the information us used optimally and the expected payoff when the action is selected optimally without information.[1]

Variation in the riskiness of the environment is modelled by shifts in the probability distribution. A standard approach associated with Rothschild and Stiglitz [9] is to say that probability distribution $F_1(\alpha)$ is riskier than $F_2(\alpha)$ if and only if both distributions have the same mean but $F_2(\alpha)$ has more weight near the mean. Technically this translates into

$$\int_0^1 F_1(\alpha)\, d\alpha = \int_0^1 F_2(\alpha)\, d\alpha,$$

and (3)

$$\int_0^y F_1(\alpha)\, d\alpha \geqslant \int_0^y F_2(\alpha)\, d\alpha, \qquad \text{for all } y \in [0, 1].$$

The intuitive statement found in the introduction can be formalized using the measure G and the Rothschild–Stiglitz measure of risk. Gould found that when the payoff is linear in the state, $h(\alpha, X) = \alpha a(X) + b(X)$, then the gain from information increases when the environment becomes Rothschild-Stiglitz riskier. This result is improved upon by the following theorem.

[1] Arrow [1] and LaValle [6] have pointed out that the difference between maximum expected utilities with and without information cannot be interpreted as the demand price for information. Results similar to those that follow can be established for a measure of value that represents the willingness-to-pay. More general increments in information may be defined (Marschak and Miyasawa [7]), but do not lead to definitive results.

THEOREM 1. *Let $h(\alpha, X)$ be strictly concave in X and concave in α and let $h_{\alpha\alpha}(\alpha, X_\alpha^*)\, h_{XX}(\alpha, X_\alpha^*) \leqslant h_{\alpha X}(\alpha, X_\alpha^*)^2$ for all α. A Rothschild–Stiglitz [9] increase in risk cannot decrease the gain from information. If h is strictly concave in α or the above inequality is strict then the gain from information will increase.*

Proof. The gain from information may be expressed as

$$G = -\int [h(\alpha, X^*) - h(\alpha, X_\alpha^*)]\, dF(\alpha). \tag{4}$$

The gain from information depends on the curvature of $W(\alpha) = h(\alpha, X^*) - h(\alpha, X_\alpha^*)$ as a function of α. Differentiate W twice to get

$$\begin{aligned} W_{\alpha\alpha}(\alpha) = h_{\alpha\alpha}(\alpha, X^*) &- h_{\alpha\alpha}(\alpha, X_\alpha^*) \\ &- 2h_{\alpha X}(\alpha, X_\alpha^*) \frac{dX_\alpha^*}{d\alpha} - h_{XX}(\alpha, X_\alpha^*) \left(\frac{dX_\alpha^*}{d\alpha}\right)^2 \\ &- h_X(\alpha, X_\alpha^*) \left(\frac{d^2 X_\alpha^*}{d\alpha^2}\right). \end{aligned} \tag{5}$$

The decision rule X_α^* must satisfy the first order condition

$$0 = h_X(\alpha, X_\alpha^*) \tag{6}$$

for all α. Differentiating this with respect to α gives

$$\frac{dX_\alpha^*}{d\alpha} = -h_{\alpha X}(\alpha, X_\alpha^*)/h_{XX}(\alpha, X_\alpha^*). \tag{7}$$

Substituting (6) and (7) into (5) gives

$$\begin{aligned} W_{\alpha\alpha}(\alpha) = h_{\alpha\alpha}(\alpha, X^*) - [&h_{\alpha\alpha}(\alpha, X_\alpha^*)\, h_{XX}(\alpha, X_\alpha^*) \\ &- h_{\alpha X}(\alpha, X_\alpha^*)^2]/h_{XX}(\alpha, X_\alpha^*). \end{aligned} \tag{8}$$

The concavity assumptions and the assumption that $h_{\alpha\alpha} h_{XX} < h_{\alpha X}^2$ imply that the first term of $W_{\alpha\alpha}$ is negative and the second positive so that their difference is always negative. $W(\alpha)$ is therefore a strictly concave function. Rothschild and Stiglitz's [9, Theorem 2] implies that a mean preserving increase in risk lowers the expected value of $W(\alpha)$, and from Eq. (4) it must increase the value of G. Q.E.D.

Theorem 1 is a generalization of Gould's Theorem 3; when h is linear in α it is concave in α and yet $h_{\alpha\alpha} h_{XX} = 0 \leqslant h_{\alpha X}^2$. Notice that $h(\alpha, X)$ is *not* concave in (α, X). There are diminishing returns in each component separately but "increasing returns to scale" for both components together.

Notice that complete concavity in (α, X) is not required for second order conditions. Even when h is concave in (α, X), Eq. (8) shows that W may still be concave as long as $h_{\alpha\alpha} h_{XX}$ is not too much larger than $h^2_{\alpha X}$. Therefore, concavity of $h(\alpha, X)$ in (α, X) does not imply that the gain from information decreases with Rothschild–Stiglitz risk. This comes close to contradicting a result derived by Laffont (5) and reproduced here.

THEOREM 2 (Laffont). *If h is concave in (α, X) (strictly in X) and increasing in α then the gain from information* decreases *as r increases where r is an index of type*-1 *compensated increase in risk satisfying the stochastic dominance property. Moreover, as r increases, risk increases in a Diamond–Stiglitz* [3] *sense.*

For r to represent a type-1 compensated change in risk the expected utility without information must be independent of r, or

$$\int h(\alpha, X^*(r))\, dF_r(\alpha, r) = 0, \qquad \text{for all } r. \tag{9}$$

Stochastic dominance means that

$$\int_0^y F_r(\alpha, r)\, d\alpha \geqslant 0, \qquad \text{for all } y \in [0, 1] \text{ and all } r, \tag{10}$$

and

$$F(\alpha, r_1) \neq F(\alpha, r_2), \qquad \text{for some } \alpha \text{ if } r_1 \neq r_2. \tag{11}$$

Laffont's theorem is correct as stated but if we strengthen the assumption about h by requiring strict concavity with respect to α then conditions (9), (10) and (11) cannot be satisfied simultaneously.

THEOREM 3. *If $h(\alpha, X)$ is strictly concave in α and increasing in α, a type*-1 *compensated increase in risk which satisfies the stochastic dominance property is impossible.*

Proof. Stochastic dominance implies that the mean of α cannot increase:

$$\frac{\partial}{\partial r} E\{\alpha\} = \int_0^1 \alpha dF_r(\alpha, r) = \alpha F_r(\alpha, r) \Big|_0^1 - \int_0^1 F_r(\alpha, r)\, d\alpha$$

$$= -\int_0^1 F_r(\alpha, r)\, d\alpha \leqslant 0. \tag{12}$$

The decision maker is strictly risk averse so the increased uncertainty without increased mean α must lower expected utility, contrary to type-1 compensation. Q.E.D.

Laffont shows that type-1 compensated increases in r satisfying stochastic dominance imply higher risk in the Diamond–Stiglitz sense. A measure of risk was defined by Diamond and Stiglitz [3] by assuming that increases in R hold maximum expected utility constant but spread the probability distribution of utility. Technically this means that

$$\int_0^y h_\alpha(\alpha, X^*(R))\, F_R(\alpha, R)\, d\alpha \geqslant 0, \qquad \text{for all } y \in [0, 1], \tag{13}$$

$$\int_0^1 h_\alpha(\alpha, X^*(R))\, F_R(\alpha, R)\, d\alpha = 0. \tag{14}$$

Condition (14) is equivalent to type-1 compensation. Inequality (13) is *not* equivalent to stochastic dominance. When R is substituted for r as a definition of risk, a result can be established which is intuitively pleasing and which almost reverses Laffont's conclusion.

THEOREM 4. *If $h(\alpha, X)$ is increasing in α, strictly concave in X and such that the Arrow* [2]–Pratt [8] *measure of absolute risk aversion is independent of X, then the value of information increases with Diamond–Stiglitz risk, R.*

Proof. Recall that X_α^* maximizes $h(\alpha, X)$. The envelope theorem states that

$$\frac{d}{d\alpha} h(\alpha, X_\alpha^*) = h_\alpha(\alpha, X_\alpha^*). \tag{15}$$

Gould shows that

$$\frac{d^2}{d\alpha^2} h(\alpha, X_\alpha^*) = \frac{h_{\alpha\alpha}(\alpha, X_\alpha^*)\, h_{XX}(\alpha, X_\alpha^*) - h_{\alpha X}(\alpha, X_\alpha^*)^2}{h_{XX}(\alpha, X_\alpha^*)}. \tag{16}$$

The measure of absolute risk aversion for $h(\alpha, X_\alpha^*)$ is

$$-\frac{\dfrac{d^2 h}{d\alpha^2}(\alpha, X_\alpha^*)}{\dfrac{dh}{d\alpha}(\alpha, X_\alpha^*)} = \frac{-h_{\alpha\alpha}(\alpha, X_\alpha^*)}{h_\alpha(\alpha, X_\alpha^*)} + \frac{h_{\alpha X}(\alpha, X_\alpha^*)^2}{h_\alpha(\alpha, X_\alpha^*) \cdot h_{XX}(\alpha, X_\alpha^*)}. \tag{17}$$

By assumption

$$-\frac{h_{\alpha\alpha}(\alpha, X_\alpha^*)}{h_\alpha(\alpha, X_\alpha^*)} \equiv \frac{-h_{\alpha\alpha}(\alpha, X^*(R))}{h_\alpha(\alpha, X^*(R))}, \tag{18}$$

$h_\alpha > 0$, and $h_{XX} < 0$, therefore

$$\frac{\dfrac{d^2h}{d\alpha^2}(\alpha, X_\alpha^*)}{\dfrac{dh}{d\alpha}(\alpha, X_\alpha^*)} \leqslant \frac{h_{\alpha\alpha}(\alpha, X^*(R))}{h_\alpha(\alpha, X^*(R))}, \qquad \text{for all } \alpha. \tag{19}$$

A decision-maker with perfect knowledge of α is less risk averse than on without information. By Theorem 3 of Diamond and Stiglitz an increase in I must increase the expected utility of a less risk averse utility. Since the valu of information is the difference in maximum expected utility betwee informed and uninformed situations, the value of information increases.

Q.E.D

Notice that Theorem 4 says nothing about the concavity of h in α, an nothing about the concavity in α and X together. In particular the result i true if $h_{\alpha X}$ is large or small. The stringent assumption of Theorem 4 is tha utility be expressed in the form

$$h(\alpha, X) = c(\alpha)\, a(X) + b(X). \tag{20}$$

Without this assumption it is possible for X_α^* to differ from $X^*(R)$ for som α by enough to reverse the inequality (19).

EXAMPLE 1. *Equipment Breakdown.* A firm has K machines which ar used with variable amounts of labor, X, to produce goods. Equipment break down at random and only αK machines will be operational. If the productio technology is Cobb–Douglas then the profit of the firm may be written

$$h = (\alpha K)^a X^b - rK - wX, \tag{21}$$

where r is the rental rate for machines and w is the wage rate. Assuming th the elasticities of output with respect to inputs, a and b, are positi fractions, the profit function is strictly concave in α and strictly concave i X. It is easy to verify that the sign of $h_{\alpha\alpha}h_{XX} - h_{\alpha X}^2$ is given by $1 - a -$ Using Theorem 1 a Rothschild–Stiglitz increase in the riskiness of machin will increase the gain from information when the Cobb–Douglas technolog exhibits constant or increasing returns to scale, $a + b > 1$. When there a decreasing returns to scale $h(\alpha, X_\alpha^*)$ is not convex in α and hence t response cannot be signed without additional knowledge about t probability distribution. Comparing Eq. (20) to Eq. (21), it is clear that t measure of absolute risk aversion is independent of X and therefo Theorem 4 implies that the gain from information increases with Diamond Stiglitz risk, regardless of returns to scale.

EXAMPLE 2. *Production to Meet Uncertain Demand.* A firm has contracted to supply a random quantity of goods, α, using capital, X, and labor, L, in the production. Capital is to be selected first and labor is to vary ex post to satisfy demand and minimize expected factor cost,

$$\int (rX + wL(\alpha, X))\, dF(\alpha), \tag{22}$$

where $L(\alpha, X)$ is the labor needed to produce α if the capital input is X. Here, we have

$$h = -(rX + wL(\alpha, X)). \tag{23}$$

If the technology is Cobb–Douglas, $\alpha = L^a X^b$, then h is concave in X if a and b are positive and h is concave in α if a lies between zero and one. Finally, $h_{\alpha\alpha} h_{XX} \leqslant h_{\alpha X}^2$ when $a + b \geqslant 1$. Therefore, when production is based on an increasing returns to scale Cobb–Douglas technology with diminish marginal product of labor, an increase in the Rothschild–Stiglitz uncertainty about demand will increase the gain from information. Notice that when there are decreasing returns to scale the result does not hold.

With Cobb–Douglas technology the payoff function

$$h = -(rX + w\alpha^{1/a} X^{-b/a}) \tag{24}$$

has the form of Eq. (20). Let us make the following change of variable: $\alpha = d - \hat{\alpha}$, where d is the maximum possible demand. If a is between zero and one and b is positive then a Diamond–Stiglitz increase in the riskiness of $\hat{\alpha}$ will increase the gain from information (irrespective of returns to scale).

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