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Quantifying the Allais paradox

Risk aversion and eccentricity in weighted linear utility

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Weighted linear utility explains Allais' paradox, yet preserves much of the analytic structure of standard expected utility. This paper identifies the two crucial parameters of weighted linear utility – risk aversion and eccentricity. Eccentricity is a measure of how much the decision maker systematically distorts the probabilities in choosing between lotteries.

1. Introduction

Recently economists have explored the potential gains from using weighted linear utility (WLU) as an alternative to expected utility. This model of preferences captures behavior that is paradoxical in the expected utility paradigm, yet is simple enough to be analytically useful in applications. One of the earliest paradoxes is the Allais paradox [Fishburn (1988, p. 36)]. Empirical studies by Chew and Waller (1986), Conlisk (1989) and Camerer (1989) show that WLU performs better than expected utility in explaining this paradox.

In this paper we develop measures of the two parameters that define WLU and provide intuition for the relationship between these parameters and observed behavior. One of the parameters is much like the familiar risk aversion parameter. The second parameter, eccentricity, measures the novel contribution of WLU to decisions under uncertainty. In essence, eccentricity measures the degree to which the decision maker systematically distorts his perception of probabilities leading to the 'fanning out effect' [Machina (1987)].

The Allais paradox is illustrated in the following four lotteries:

$$\begin{aligned} &L_1 = \langle (\$0, \$1 \text{ Million}, \$3 \text{ Million}), (0, 1.0, 0) \rangle \\ &\text{versus} \\ &L_2 = \langle (\$0, \$1 \text{ Million}, \$3 \text{ Million}), (0.2, 0, 0.8) \rangle, \end{aligned} \tag{1}$$

$$\begin{aligned} &L_3 = \langle (\$0, \$1 \text{ Million}, \$3 \text{ Million}), (0.80, 0.20, 0) \rangle \\ &\text{versus} \\ &L_4 = \langle (\$0, \$1 \text{ Million}, \$3 \text{ Million}), (0.84, 0, 0.16) \rangle. \end{aligned} \tag{2}$$

Commonly preferences exhibit the ordering: L_1 preferred to L_2 and L_4 preferred to L_3. In the first case the individual acts risk averse, preferring the certainty of \$1 million to a gamble with a much greater expected monetary outcome. In the second comparison the person accepts greater risk, thinking perhaps that since it is unlikely to get either positive outcome one might as well go for the biggest 'pot'. It is well known that such choices cannot correspond to maximizing expected utility. We show below that eccentricity measures the distortion in the perceptions of probabilities that drives such choices.

Chew and MacCrimmon (1979) and Fishburn (1983) have shown that the Allais paradox may be explained while preserving a simple structure for the utility function,

$$V(\langle X, P\rangle) = \frac{\Sigma_i U(x_i) p_i}{\Sigma_j W(x_j) p_j} = \sum_i U(x_i) \frac{P_i}{\Sigma_j W(x_j) p_j} = \sum_i U(x_i) \hat{p}_i. \tag{3}$$

This *weighted linear utility* is non-linear in the probabilities and thus is not an expected utility. The numerator function U is called a *valuation* function and the denominator function W is called a *weighting* function. If the weighting function is a constant, the weighted linear utility reduces to expected utility. *WLU* is the transitive special case of Fishburn's (1988) skew symmetric bilinear utility and is one of the preference structures in Becker and Sarin's (1987) family of lottery dependent utilities.

In fact, *WLU* is the mathematically simplest alternative to expected utility that is capable of explaining the Allais paradox while at the same time avoiding such problems as intransitivities or violations of stochastic dominance principles [see Machine (1983, pp. 95–103)]. The preferences of eq. (3) look similar to expected utility except that the probabilities are transformed, as though they were replaced by perceived probabilities, $\hat{p}_i$. The objective of this paper is to understand these systematic distortions.

2. Risk aversion and eccentricity

How do we go about specifying the preferences of a *WLU* maximizer? We must be careful in referring independently to U and W in the utility function since the valuation and weighting functions are not uniquely defined. Fishburn (1983) has shown that if valuation function U and weighting function W represent a persons' preferences, then so do any linear transformations of these, $U^* = aU + bW$, $W^* = cU + dW$, as long as determinant, $ad-bc$, is positive. For example, substituting $-U$ for W and W for U is one legitimate representation of exactly the same preferences. We will follow the strategy of Pratt (1964) who showed that specifying a von Neumann–Morgenstern utility function of wealth was equivalent to specifying a risk aversion function of wealth. The question is what are the functional equivalents of risk aversion for *WLU*?

Suppose an individual with initial wealth x is faced with a stochastic additional source of wealth, ϵ, which has expected value μ and small variance σ^2. In *WLU*, the risk premium π that makes the decision maker indifferent between paying the premium or bearing the entire risk is defined implicitly by:

$$U(x + \mu - \pi)/W(x + \mu - \pi) = \mathrm{E}[U(x + \epsilon)]/\mathrm{E}[W(x + \epsilon)]. \tag{4}$$

Using Taylor series approximations, π is expressed as

$$\pi = R/(2/\sigma^2 + E), \tag{5}$$

where

$$R = R(x+\mu) = [(UW'' - U''W)/(U'W - W'U)], \tag{6}$$

and

$$E = \mathrm{E}(x+\mu) = [(U'W'' - U''W')/(U'W - W'U)]. \tag{7}$$

The parameter R, first identified by Chew (1983), is the *measure of absolute risk aversion* for *WLU*. For a constant weighting function, $W(x) = \text{constant}$, R reduces to the Arrow–Pratt measure of absolute risk aversion, $-U''/U'$. The single differential eq. (6) fails to provide enough information to find both the valuation function U and the weighting function W. Differential eq. (7) is required to complete the system.

We call the parameter E the *measure of absolute eccentricity*. Eccentricity E has no counterpart in expected utility theory. For a constant weighting function, $W(x) = \text{constant}$, eccentricity E equals zero. That is, you are not eccentric if and only if you are an expected utility maximizer. Together risk aversion and eccentricity determine *WLU* [integrate the differential eqs. (6) and (7) to recover U and W from R and E] and provide insight into the behavior of decision makers und uncertainty [as seen in (5)]. It is easy to check that R and E are independent of any equivalent representations of the U and W functions $U^* = aU + bW$, $W' = cU + dW$, $ad > bc$. In fact, we cannot make independent restrictions on the functions U and W; all we can do is impose structure on the risk aversion and eccentricity functions.

Risk aversion is a concept with which decision scientists are quite comfortable, but eccentricity is simultaneously a new concept and one whose predictions are more subtle than those of risk aversion. We will next interpret the measure of eccentricity in light of Allais' paradox.

3. Interpreting eccentricity

Eccentricity leads to choices that look as though probabilities have been distorted. The simplest way to see this is from the eq. (5) describing the risk premium. In the denominator the term $2/\sigma^2$ is twice the precision of the distribution of the random variable. Expected utility maximizers have a risk premium inversely proportional to the precision, but this is not true for *WLU* maximizers. In particular the eccentric decision maker acts as though the precision is $1/\sigma^{+} + E/2$.

Whether this distortion has a significant impact on observed behavior depends on the situation. If the precision is very large, eccentricity causes only a slight relative distortion. This is why many people choose lottery L_1 over lottery L_2 in the Allais alternative (1). On the other hand, if the precision is small then the distortion induced by eccentricity can have a large relative impact, leading to the choice of an apparently very risky lottery, as in L_4 over L_3 in choice (2). The power of the *WLU* model with eccentricity comes from this ability to provide differential predictions depending on the initial probability distribution.

To study the utility for lotteries of the form $\langle X, P \rangle$, we fix X and consider utility as a function of only P. For simplicity suppose there are three possible outcomes, $X = (x_1, x_2, x_3)$ where $x_1 < x_2 < x_3$. This allows the representation of preferences in a three dimensional simplex as in fig. 1 where the probabilities are plotted on the axes. They are labelled p_i and $\hat{p}_i$ because we want to look at the relationship between the true probabilities (p_i) and the perceived probabilities,

$$\hat{p}_i = p_i / \Sigma_j W(x_j) p_j. \tag{8}$$

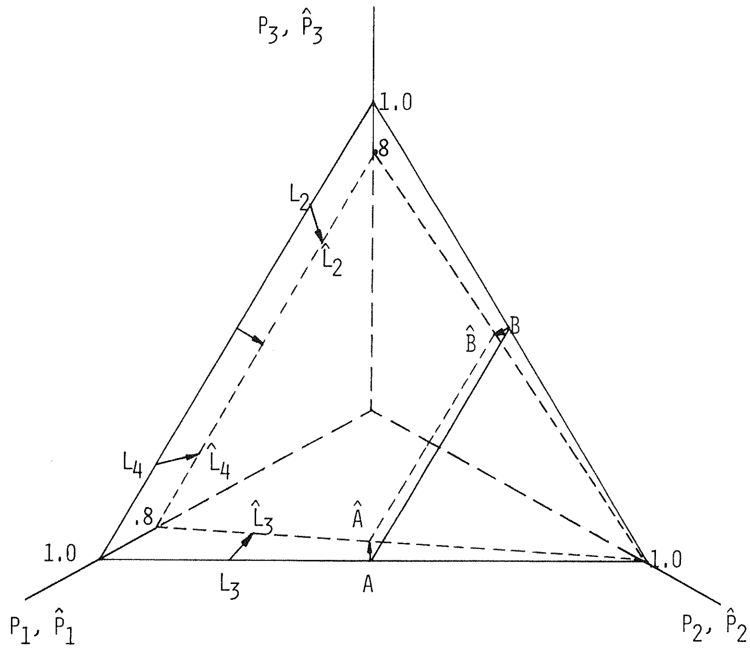

Fig. 1

Fig. 1 is drawn under the specific assumption that $W(x_1) = 1.25 = W(x_3)$ and $W(x_2) = 1.0$, so perceived probabilities are

$$\hat{p}_1 = p_1/(1 + 0.25(p_1 + p_3)), \tag{9}$$

$$\hat{p}_2 = (1 - p_1 - p_3)/(1 + 0.25(p_1 + p_3)), \tag{10}$$

$$\hat{p}_3 = p_3/(1 + 0.25(p_1 + p_3)). \tag{11}$$

As long as p_2 is held constant the distortion in probabilities is a constant multiple of the actual probabilities.

The unit simplex in P-space of fig. 1 corresponds to actual lotteries, while the simplex with dotted edges is the perceived probability simplex in $\hat{p}$-space. Notice that the transformations (9)–(11) lead the decision maker to act as though the probabilities along constant p_2 cross section are smaller then they are actually. Moreover, points closer to the $p_2 = 0$ cross section are distorted relatively more then those near $p_2 = 1$ which is not distorted at all. This leads to indifference curves in the unit simplex that fan out rather than the parallel indifference curves of expected utility [see Machina (1987)].

The eccentric's distorted probabilities easily explain the Allais paradox. Lottery L_4 appears to the eccentric decision maker to have a much lower risk (recall that precision is increased due to eccentricity) and larger mean than L_3. The lottery L_3 is closer to the $p_2 = 1$ corner than is L_4, so an eccentric decision maker does not distort these probabilities so much. The net impact of the distortion is to make $\hat{L}_4$ relatively more attractive to an eccentric than $\hat{L}_3$, leading a risk averter to choose a riskier lottery, precisely the Allais paradoxical behavior.

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