From (2.24) and (2.8) we have, for $t_0>T$,

$$-J_1(t_0)\pi^2 \leqslant \liminf_{t\to+\infty} t^{-1}\ln[f(t)-U(T)] \leqslant \limsup_{t\to+\infty} t^{-1}\ln[f(t)-U(T)] \leqslant -J_0(t_0)\pi^2. \tag{2.30}$$

However, from Lemma 17 and (2.13), both $J_0(t_0)$ and $J_1(t_0)$ converge to $K(-U(T))$ as $t_0\to+\infty$. We conclude that $\lim_{t\to+\infty} t^{-1}\ln[f(t)-U(T)]$ exists and equals $-K(-U(T))\pi^2$.

PROOF OF THEOREM 1. The proof is contained in Lemmas 5, 6, 9, 10, 15, and 19.

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REFERENCES

1 J. Eisenfeld and V. C. Mow, Mathematical analysis of stress relaxation in articular cartilage during compression, *Math. Biosci.*, to appear.
2 A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice-Hall, Englewood Cliffs, N.J., 1964.
3 H. Lipshitz, V. C. Mow, P. A. Torzilli, J. Eisenfeld, and M. G. Glimchen, On the stress relaxation of articular cartilage during confined compression, in *Proceedings of the VII International Congress of Rheology* (C. Klason and J. Jubat, Eds.), 1976, pp. 198–200.
4 J. M. Mansour and V. C. Mow, The permeability of articular cartilage under compressive strain and at high pressures, *J. Bone and Joint Surg.* 58-A: 509, 516 (1976).
5 J. M. Mansour and V. C. Mow, On the natural lubrication of synovial joints: normal and degenerate, *J. Lub. Technology* 99: 163–173 (1977).
6 V. C. Mow, W. M. Lai, and I. Redler, Some surface characteristics of articular cartilage—I. A scanning electron microcropy study and a theoretical model for the dynamic interaction of synovial fluid and articular cartilage, *J. Biomechanics* 7: 449–456 (1974).
7 V. C. Mow, W. M. Lai, I. Redler, and J. Eisenfeld, Some surface characteristics of articular cartilage—II. On the stability of articular surface and a possible bio-mechanical factor in etiology of chondrodegeneration, *J. Biomechanics* 7: 457–468 (1974).
8 L. Sokoloff, *The Biology of Degenerative Joint Disease*, Univ. of Chicago Press, Chicago, 1969.

Optimal Tactics for Close Support Operations. II. Degraded Communications with Linear Time-Varying Dynamics*

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ABSTRACT

The optimal decision rules for force commitments are obtained in mathematical studies of C^3 (command, control, and communication). Recursive equations for the solution of the C^3 problem are derived for a perturbation model with linear time-varying dynamics. Air and ground commanders are assumed to have perfect intelligence with degraded communication between them. Numerical results are given for several amphibious assaults.

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INTRODUCTION

The primary output of mathematical models of C^3 (command, control, and communication) is optimal decision rules for force commitments to be employed by subordinate headquarters in coordinating their activities to achieve objectives laid down by superordinate headquarters. The decision rules make use only of information available to each subordinate commander, and that information is determined by intelligence and communication networks.

Consider a C^3 problem in which there are two subordinate commanders, both striving to coordinate their decisions to attain the tactical objective set down by superordinate headquarters. In an amphibious campaign, the blue naval force lands ground troops and provides close support. The objective is to move inland a certain distance in a specified time. It is desired to attain the objective at minimum expected cost.

Some general concepts of C^3 are discussed in Refs. [1], [2], and [3]. Recursive equations for the solution of the C^3 problem are derived for the time-invariant linear dynamics with quadratic costs in Ref. [4]. The blue naval air and ground commanders are assumed to have perfect intelligence with degraded communication. In this model, each day blue air knows red air strength and blue ground knows red ground strength from intelligence activities; but due to communication outages, they cannot communicate with each other. Other communication and intelligence patterns are considered in other papers in this series. In this paper, equations are derived for linear time-varying dynamics. This is an important generalization of the previous results in that the dynamics now vary as the campaign progresses. The general equations are derived for a perturbation model followed by the recursive equation solution. The recursive equations are shown to be similar to those derived in Ref. [4]. Numerical results are given for campaigns with time-invariant and time-varying dynamics.

C^3 MODEL

The C^3 model to be derived is a perturbation model with linear time-varying dynamics and quadratic costs. It is assumed that the scenario for the main forces has already been planned. Thus only perturbations about the planned scenario are considered. For example, assume that the main objective is to move inland a distance of 200 miles in 21 days. The perturbed objective might be to move inland an additional 21 miles. The red air and ground strengths, p and q, are the perturbation strength increments (or decrements) about the main strengths in the 21 day campaign.

Let N be the duration of the campaign, and let the distance to the perturbed objective be s_0. Consider K days remaining with the front line at

the perturbed position s. The new perturbed position of the front line is

$$S = S(s,p,q,\alpha,\beta), \qquad (1)$$

where

S = new perturbed position increment of the front line about the planned position with K-1 days remaining,
s = current perturbed position increment of the front line about the planned position with K days remaining,
p,q = red air and ground perturbed strength increments (or decrements) about the main strengths, respectively, with K days remaining,
α,β = blue naval air and ground perturbed strength increments (or decrements) about the main strengths, respectively, with K days remaining.

The daily cost increment (or decrement) is given by

$$C = C(s,p,q,\alpha,\beta,K). \qquad (2)$$

An additional cost is assessed if the front line at the end of the campaign is at some perturbed position increment s other than s_0. This terminal cost is

$$\phi = \phi(s). \qquad (3)$$

The red air and ground commanders make the decision to employ strength increments p and q respectively each day. The decision making of the enemy is simplified by assuming that p and q are random variables with joint probability density function

$$w = w(p,q). \qquad (4)$$

Furthermore, assuming that p and q are independent random variables, the joint probability function can be expressed as the product

$$w(p,q) = P(p)Q(q). \qquad (5)$$

The minimum expected cost is defined by

$g_K(s)$ = the expected cost increment (or decrement) of a campaign beginning with the front line at s, of duration K, and employing an optimal sequence of decisions, (6)

$$K = 0,1,2,\ldots,N, \qquad \text{all } s.$$

Using Bellman's principle of optimality [5], the functions $g_{K+1}(s)$ and $g_K(s)$

are related by the recurrence equation

$$g_{K+1}(s)=\min_{\alpha,\beta}\int\int\{C(s,p,q,\alpha,\beta,K)$$
$$+g_K[S(s,p,q,\alpha,\beta)]\}P(p)Q(q)\,dp\,dq,$$
$$K=0,1,2,\ldots,N-1. \tag{7}$$

All integrals on p and q are from $-\infty$ to ∞. When no time remains,

$$g_0(s)=\phi(s). \tag{8}$$

The conditions for obtaining the minimum are

$$0=\frac{\partial}{\partial\alpha}\int B_{K+1}(s,p,q,\alpha,\beta)Q(q)\,dq, \tag{9}$$

$$0=\frac{\partial}{\partial\beta}\int B_{K+1}(s,p,q,\alpha,\beta)P(p)\,dp, \tag{10}$$

where

$$B_{K+1}(s,p,q,\alpha,\beta)=C(s,p,q,\alpha,\beta,K)+g_K[S(s,p,q,\alpha,\beta)]. \tag{11}$$

Equations (9) and (10) can then be written

$$0=\int\{C_\alpha(s,p,q,\alpha,\beta,K)+g'_K[S(s,p,q,\alpha,\beta)]$$
$$\times S_\alpha(s,p,q,\alpha,\beta)\}Q(q)\,dq, \tag{12}$$

$$0=\int\{C_\beta(s,p,q,\alpha,\beta,K)+g'_K[S(s,p,q,\alpha,\beta)]$$
$$\times S_\beta(s,p,q,\alpha,\beta)\}P(p)\,dp, \tag{13}$$

$K=0,1,2,\ldots,N-1$, where

$$\alpha=\alpha(K+1,s,p), \tag{14}$$

$$\beta=\beta(K+1,s,q), \tag{15}$$

since blue air knows only red air strength, and blue ground knows only red ground strength.

Equations (12) and (13) are the dynamic headquarters-by-headquarters optimality conditions. They state that at every decision making opportunity, each headquarters is to make the decision which reduces the marginal conditional expected cost of the remainder of the process to zero.

For linear time-varying dynamics with quadratic costs, the new perturbed position of the front line is assumed to be a function of the old perturbed position plus a linear combination of the strengths:

$$S=s+C_1\alpha+C_2\beta-C_3p-C_4q, \tag{16}$$

where

$$C_1=C_1(K),\quad C_2=C_2(K),\quad C_3=C_3(K),\quad C_4=C_4(K).$$

The perturbed daily cost is assumed to be proportional to the losses, which in turn are proportional to the strengths utilized. The ground losses are reduced by the air strength for close support missions. Thus the daily cost is assumed to be

$$C=C_5\alpha+(C_6-C_7\alpha)\beta+\tfrac{1}{2}C_8\alpha^2+\tfrac{1}{2}C_9\beta^2, \tag{17}$$

where

$$C_5=C_5(K),\quad C_6=C_6(K),$$
$$C_7=C_7(K),\quad C_8=C_8(K),\quad C_9=C_9(K).$$

The terminal cost is assumed to be

$$\phi(s)=\lambda(s-s_0)^2. \tag{18}$$

From general control theoretical considerations, the minimum expected cost has the form

$$g_K(s)=\rho_K+\sigma_Ks+\tau_Ks^2, \tag{19}$$

where the coefficients, ρ_K, σ_K, and τ_K, are computed for K stages remaining. Differentiating Eqs. (16), (17), and (19) and substituting into Eqs. (12) and

(13), the following equations are obtained:

$$0=\int\{(C_5-C_7\beta)+C_8\alpha$$
$$+[\sigma_K+2\tau_K(s+C_1\alpha+C_2\beta-C_3p-C_4q)]C_1\}Q(q)\,dq, \qquad (20)$$

$$0=\int\{(C_6-C_7\alpha)+C_9\beta$$
$$+[\sigma_K+2\tau_K(s+C_1\alpha+C_2\beta-C_3p-C_4q)]C_2\}P(p)\,dp, \qquad (21)$$

which are a system of Fredholm integral equations for the functions α and β.

RECURSIVE EQUATIONS

The recursive equations for the solution of Eqs. (6) and (7) are derived as follows. Making use of the equations

$$\int Q(q)\,dq=1, \qquad (22)$$

$$\int qQ(q)\,dq=\bar{q}, \qquad (23)$$

and similar relations for p, Eqs. (20) and (21) can be written as the linear Fredholm integral equations

$$\alpha(p)=\frac{1}{2C_1^2\tau_K+C_8}\Big[(C_7-2C_1C_2\tau_K)\int\beta(q)Q(q)\,dq$$
$$+2C_1C_4\tau_K\bar{q}-C_5-C_1\sigma_K-2C_1\tau_Ks+2\tau_KC_1C_3p\Big], \qquad (24)$$

$$\beta(q)=\frac{1}{2C_2^2\tau_K+C_9}\Big[(C_7-2C_1C_2\tau_K)\int\alpha(p)P(p)\,dp$$
$$+2C_2C_3\tau_K\bar{p}-C_6-C_2\sigma_K-2C_2\tau_Ks+2\tau_KC_2C_4q\Big]. \qquad (25)$$

We shall solve these integral equations observing that they have degenerate

kernels. Define two new parameters

$$v_1=\int\beta(q)Q(q)\,dq, \qquad (26)$$

and

$$v_2=\int\alpha(p)P(p)\,dp. \qquad (27)$$

Substituting $\beta(q)$ into Eq. (26),

$$v_1=\frac{1}{2C_2^2\tau_K+C_9}[(C_7-2C_1C_2\tau_K)v_2+2C_2C_3\tau_K\bar{p}$$
$$-C_6-C_2\sigma_K-2C_2\tau_Ks+2\tau_KC_2C_4\bar{q}]. \qquad (28)$$

Substituting $\alpha(p)$ into Eq. (27),

$$v_2=\frac{1}{2C_1^2\tau_K+C_8}[(C_7-2C_1C_2\tau_K)v_1+2C_1C_4\tau_K\bar{q}$$
$$-C_5-C_1\sigma_K-2C_1\tau_Ks+2\tau_KC_1C_3\bar{p}]. \qquad (29)$$

To simplify the notation, express Eqs. (28) and (29) in the form

$$v_1=B_1(B_2v_2-B_3s+B_4), \qquad (30)$$

$$v_2=B_5(B_2v_1-B_6s+B_7). \qquad (31)$$

Substituting v_2 into Eq. (30) and v_1 into Eq. (31), it can be shown that

$$v_1=D_1+D_2s, \qquad (32)$$

$$v_2=D_3+D_4s, \qquad (33)$$

where the D_i's are functions of the B_j's. Then substituting Eq. (32) into Eq. (24) and Eq. (33) into Eq. (25), it can be shown that the optimal decision functions are

$$\alpha(K+1,s,p)=u_K+v_Ks+w_Kp, \qquad (34)$$

$$\beta(K+1,s,q)=x_K+y_Ks+z_Kq, \qquad (35)$$

where

$$u_K = B_5 B_2 D_1 + B_5 B_7', \tag{36}$$

$$v_K = B_5 B_2 D_2 - B_5 B_6, \tag{37}$$

$$w_K = 2\tau_K C_1 C_3 B_5, \tag{38}$$

$$x_K = B_1 B_2 D_3 + B_1 B_4', \tag{39}$$

$$y_K = B_1 B_2 D_4 - B_1 B_3, \tag{40}$$

$$z_K = 2\tau_K C_2 C_4 B_1, \tag{41}$$

$$B_1 = \frac{1}{2C_2^2 \tau_K + C_9}, \tag{42}$$

$$B_2 = C_7 - 2C_1 C_2 \tau_K, \tag{43}$$

$$B_3 = 2C_2 \tau_K, \tag{44}$$

$$B_4 = 2C_2 C_3 \tau_K \bar{p} - C_6 - C_2 \sigma_K + 2\tau_K C_2 C_4 \bar{q}, \tag{45}$$

$$B_4' = 2C_2 C_3 \tau_K \bar{p} - C_6 - C_2 \sigma_K, \tag{46}$$

$$B_5 = \frac{1}{2C_1^2 \tau_K + C_8}, \tag{47}$$

$$B_6 = 2C_1 \tau_K, \tag{48}$$

$$B_7 = 2C_1 C_4 \tau_K \bar{q} - C_5 - C_1 \sigma_K + 2\tau_K C_1 C_3 \bar{p}, \tag{49}$$

$$B_7' = 2C_1 C_4 \tau_K \bar{q} - C_5 - C_1 \sigma_K, \tag{50}$$

$$D_1 = \frac{B_1}{1 - B_1 B_2^2 B_5}(B_2 B_5 B_7 + B_4), \tag{51}$$

$$D_2 = \frac{-B_1}{1 - B_1 B_2^2 B_5}(B_2 B_5 B_6 + B_3), \tag{52}$$

$$D_3 = \frac{B_5}{1 - B_1 B_2^2 B_5}(B_2 B_1 B_4 + B_7), \tag{53}$$

$$D_4 = \frac{-B_5}{1 - B_1 B_2^2 B_5}(B_2 B_1 B_3 + B_6). \tag{54}$$

Note that S can be expressed in terms of these coefficients and s, p, and q. Substituting the optimal α and β into Eq. (16), the new position of the front line becomes

$$\begin{aligned} S &= s + C_1(u_K + v_K s + w_K p) + C_2(x_K + y_K s + z_K q) - C_3 p - C_4 q \\ &= C_1 u_K + C_2 x_K + (1 + C_1 v_K + C_2 y_K)s + (C_1 w_K - C_3)p + (C_2 z_K - C_4)q. \end{aligned} \tag{55}$$

The expected cost, $g_{K+1}(s)$, is obtained by substituting Eqs. (16), (17), and (18) into Eq. (7):

$$\begin{aligned} g_{K+1}(s) = \min_{\alpha,\beta} \iint \Big\{ & C_5\alpha + (C_6 - C_7\alpha)\beta + \tfrac{1}{2}C_8\alpha^2 + \tfrac{1}{2}C_9\beta^2 \\ & + \rho_K + \sigma_K(s + C_1\alpha + C_2\beta - C_3 p - C_4 q) \\ & + \tau_K(s + C_1\alpha + C_2\beta - C_3 p - C_4 q)^2 \Big\} P(p)Q(q)\,dp\,dq. \end{aligned} \tag{56}$$

Substituting Eqs. (34) and (35) into Eq. (56),

$$\begin{aligned} g_{K+1}(s) = \iint \Big\{ & C_5(u_K + v_K s + w_K p) \\ & + [C_6 - C_7(u_K + v_K s + w_K p)](x_K + y_K s + z_K q) \\ & + \tfrac{1}{2}C_8(u_K + v_K s + w_K p)^2 + \tfrac{1}{2}C_9(x_K + y_K s + z_K q)^2 + \rho_K \\ & + \sigma_K[s + C_1(u_K + v_K s + w_K p) + C_2(x_K + y_K s + z_K q) - C_3 p - C_4 q] \\ & + \tau_K[s + C_1(u_K + v_K s + w_K p) + C_2(x_K + y_K s + z_K q) - C_3 p - C_4 q]^2 \Big\} \\ & \times P(p)Q(q)\,dp\,dq. \end{aligned}$$

(5

To simplify the notation, express Eq. (57) in the form

$$
\begin{aligned}
g_{K+1}(s) &= \rho_{K+1} + \sigma_{K+1}s + \tau_{K+1}s^2 \\
&= \int\int \{ C_5 u_K + C_5 v_K s + C_5 w_K p + C_6 x_K \\
&\quad + C_6 z_K q - C_7(u_K + w_K p)(x_K + z_K q) \\
&\quad + [C_6 y_K - C_7(u_K + w_K p) y_K - C_7(x_K + z_K q) v_K] s - C_7 v_K y_K s^2 \\
&\quad + \tfrac{1}{2} C_8 [u_K^2 + 2u_K w_K p + w_K^2 p^2 + 2(u_K v_K + v_K w_K p)s + v_K^2 s^2] \\
&\quad + \tfrac{1}{2} C_9 [x_K^2 + 2x_K z_K q + z_K^2 q^2 + 2(x_K y_K + z_K q)s + y_K^2 s^2] \\
&\quad + \rho_K + \sigma_K(A_1 + A_2 s + A_3 p + A_4 q) \\
&\quad + \tau_K [(A_1 + A_3 p + A_4 q)^2 + 2(A_1 + A_3 p + A_4 q) A_2 s + A_2^2 s^2] \} \\
&\quad \times P(p) Q(q)\, dp\, dq. \qquad (58)
\end{aligned}
$$

Integrating and setting the constant terms and the coefficients of s and s^2 equal, it can be shown that

$$
\begin{aligned}
\rho_{K+1} &= C_5 u_K + C_5 w_K \bar{p} + C_6 x_K + C_6 z_K \bar{q} \\
&\quad - C_7(u_K + w_K \bar{p})(x_K + z_K \bar{q}) + \tfrac{1}{2} C_8 (u_K^2 + 2u_K w_K \bar{p} + w_K^2 \overline{p^2}) \\
&\quad + \tfrac{1}{2} C_9 (x_K^2 + 2x_K z_K \bar{q} + z_K^2 \overline{q^2}) + \rho_K + \sigma_K (A_1 + A_3 \bar{p} + A_4 \bar{q}) \\
&\quad + \tau_K [A_1^2 + 2(A_1 A_3 \bar{p} + A_1 A_4 \bar{q} + A_3 A_4 \overline{pq}) + A_3^2 \overline{p^2} + A_4^2 \overline{q^2}], \qquad (59)
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{K+1} &= C_5 v_K + C_6 y_K - C_7(u_K + w_K \bar{p}) y_K - C_7(x_K + z_K \bar{q}) v_K \\
&\quad + C_8(u_K v_K + v_K w_K \bar{p}) + C_9(x_K y_K + y_K z_K \bar{q}) \\
&\quad + \sigma_K A_2 + 2\tau_K A_2 (A_1 + A_3 \bar{p} + A_4 \bar{q}), \qquad (60)
\end{aligned}
$$

$$
\tau_{K+1} = -C_7 v_K y_K + \tfrac{1}{2} C_8 v_K^2 + \tfrac{1}{2} C_9 y_K^2 + \tau_K A_2^2, \qquad (61)
$$

where

$$A_1 = C_1 u_K + C_2 x_K, \qquad (62)$$

$$A_2 = 1 + C_1 v_K + C_2 y_K, \qquad (63)$$

$$A_3 = C_1 w_K - C_3, \qquad (64)$$

$$A_4 = C_2 z_K - C_4. \qquad (65)$$

The recursive relations are given by Eqs. (59), (60), and (61), supplemented by Eqs. (36) to (41) and (42) through (54). These equations are similar to those derived in Ref. [4]; however, they are now considered to be perturbation equations.

NUMERICAL RESULTS

Numerical results were obtained using the recursive equations derived in the previous paragraphs. At each stage the coefficients in the equations for the optimal blue air strength, $\alpha(p)$, and the optimal blue ground strength, $\beta(q)$, are computed. The new position of the front line is computed in Eq. (55) using the optimal decisions.

Consider a campaign with the following duration, N, and additional distance to be covered, s_0:

$$N = 21 \text{ days},$$

$$s_0 = 21 \text{ miles}.$$

The average red strengths are assumed to be

$$\bar{p} = 1 \quad \text{and} \quad \bar{q} = 1$$

The second moments of p and q are given by

$$\overline{p^2} = \bar{p}^2 + \sigma_p^2; \qquad \overline{q^2} = \bar{q}^2 + \sigma_q^2$$

Assume the coefficients in Eqs. (15), (16), and (17) are constants with the exception of C_5:

$$C_1=0.1,\quad C_6=0.1,\quad \sigma_p=0.5,$$
$$C_2=1,\quad C_7=0.01,\quad \sigma_q=0.5,$$
$$C_3=0.1,\quad C_8=0.02,\quad \lambda=1.0$$
$$C_4=1,\quad C_9=0.02,$$

The coefficient C_5 in the cost equation is proportional to the red antiair strength and may vary during the course of the campaign as the red antiair fortifications are destroyed. Then for different values of C_5 the optimal air and ground decisions, $\alpha(p)$ and $\beta(q)$, for $K=21$, 11, and 1 stage to go are as follows:

Case I, C_5 Constant
$C_5=0.005$.

$K=21$:

$$\alpha(p)=1.358-2.573\times10^{-2}s+3.376\times10^{-4}p,$$

$$\beta(q)=1.928-4.503\times10^{-2}s+3.267\times10^{-2}q.$$

$K=11$:

$$\alpha(p)=1.848-4.911\times10^{-2}s+6.748\times10^{-4}p,$$

$$\beta(q)=2.755-8.594\times10^{-2}s+6.325\times10^{-2}q.$$

$K=1$:

$$\alpha(p)=11.56-0.5369s+0.5p,$$

$$\beta(q)=19.71-0.9396s+0.99q.$$

Case II, Time-Varying C_5
$C_5(K)=0.005+0.0005K$.

$K=21$:

$$\alpha(p)=0.7773-2.573\times10^{-2}s+3.376\times10^{-4}p,$$

$$\beta(q)=1.787-4.503\times10^{-2}s+3.267\times10^{-2}q.$$

$K=11$:

$$\alpha(p)=1.546-4.911\times10^{-2}s+6.748\times10^{-4}p,$$

$$\beta(q)=2.685-8.594\times10^{-2}s+6.325\times10^{-2}q.$$

$K=1$:

$$\alpha(p)=11.54-0.5369s+0.5p,$$

$$\beta(q)=19.71-0.9396s+0.99q.$$

In case I, the coefficient C_5 is constant. Considering the campaign as a whole with 21 days and 21 miles to go, and starting with the front line at $s=0$, if the red air and ground strengths are constant and equal to their average values of 1 (i.e., $p=\bar{p}=1$ and $q=\bar{q}=1$), then the optimal blue air and ground strengths are also constant and given by

$$\alpha(\bar{p})=1.359 \quad\text{and}\quad \beta(\bar{q})=1.961.$$

The daily cost given by Eq. (17) is

$$C=C_5\alpha+(C_6-C_7\alpha)\beta+\tfrac{1}{2}C_8\alpha^2+\tfrac{1}{2}C_9\beta^2$$
$$=0.00679+0.169+0.018+0.038$$
$$=0.233$$

In case II, $C_5(K)$ decreases as the campaign progresses. If the red air and ground strengths are equal to their average values, then $\alpha(\bar{p})=0.7776$ and $\beta(\bar{q})=1.819$ at the beginning of the campaign, and they gradually increase to $\alpha(\bar{p})=1.444$ and $\beta(\bar{q})=2.15$ at the end of the campaign. The front line moves forward at the increment of 0.8 miles per day at the beginning of the campaign, gradually increasing to 1.2 miles per day at the end of the campaign. The front line moves forward at a lower rate at the beginning of the campaign where the coefficients of the daily cost are higher, and moves forward at a higher rate near the end of the campaign where the coefficients are lower.

It should be noted that the optimal decisions are the perturbed strength increments about the main strengths, and the positions are the perturbed position increments of the front line about the planned position. In the examples given, $\alpha(\bar{p})$, $\beta(\bar{q})$, and the minimum expected cost increments are all positive.

In this paper the air and ground commanders were assumed to have partial intelligence with degraded communication between them. Future papers will consider different combinations of intelligence and communication capabilities.

REFERENCES

1. H. Kagiwada, and R. Kalaba, *C^3 Modeling*, USNPS Workship, 1978.
2. V. V. Druzhinin, and D. S. Kontorov, *Concept, Algorithm, Decision*, translated and published by USAF, 1972.
3. R. Thrall, C. Tsokos, and J. Turner, *Decision Information for Tactical Command and Control*, R. M. Thrall and Associates, Houston, 1977.
4. J. Hess, H. Kagiwada, R. Kalaba, K. Spingarn, and C. Tsokos, Optimal tactics for close support operations, I. Degraded communications, *J. Optimization Theory Appl.* (in press).
5. R. Bellman, and R. Kalaba, *Dynamic Programming and Modern Control Theory*, Academic, New York, 1966.

Application of Invariant Imbedding to Linear Multipoint Boundary Value Problems with General Boundary Conditions

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ABSTRACT

Two extensions of the usual application of invariant imbedding to the solution of linear boundary value problems are presented. The invariant imbedding formulation of a linear two point boundary value problem in which functional relationships are given between the variables at either one or both of the boundary points is presented. Also, extension of invariant imbedding to linear multipoint boundary value problems is given. Using these extensions singly or in combination, a general multipoint boundary value of linear ordinary differential equations can be solved. In addition, the problems of infinite initial conditions and/or indeterminate initial derivatives are resolved. Numerical examples demonstrate the feasibility and accuracy of the method.

INTRODUCTION

In recent years a technique known as invariant imbedding has been applied to a growing number of problems [1–4]. Invariant imbedding is a powerful tool that finds frequent application in the numerical solution of two point boundary value problems (TPBVP) of differential equations. While invariant imbedding has some interesting mathematical and physical implications, one of its most useful properties is its ability to convert two point boundary value problems into initial value problems.

At present, a substantial body of literature is being produced bearing the name invariant imbedding. Analysis of that literature shows that at least three distinct approaches to the problem are being outlined all under the

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