10. AGGARWAL, J. K., *Computation of Optimal Control for Time-Delay Systems*, IEEE Transactions on Automatic Control, Vol. AC-15, pp. 683–685, 1970.
11. WERNER, R. A., and CRUZ, J. B., *Feedback Control Which Preserves Optimality for Systems with Unknown Parameters*, IEEE Transactions on Automatic Control, Vol. AC-13, pp. 621–629, 1968.
12. SANNUTI, P., and KOKOTOVIC, P., *Near-Optimum Design of Linear Systems by a Singular Perturbation Method*, IEEE Transactions on Automatic Control, Vol. AC-14, pp. 15–21, 1969.
13. INOUE, K., AKASHI, H., OGINO, K., and SAWARAGI, Y., *Sensitivity Approaches to Optimization of Linear Systems with Time Delay*, Automatica, Vol. 7, pp. 671–679, 1971.
14. JAMSHIDI, M., and MALEK-ZAVAREI, M., *Suboptimal Design of Linear Control Systems with Time Delay*, IEE Proceedings, Vol. 119, pp. 1743–1746, 1972.
15. CHAN, H. C., and PERKINS, W. R., *Optimization of Time Delay Systems Using Parameter Imbedding*, Automatica, Vol. 9, pp. 257–261, 1973.
16. JAMSHIDI, M., *Sub-Optimal Control of Coupled Time-Delay Systems*, International Journal of Control, Vol. 17, pp. 995–1008, 1973.
17. SOLIMAN, M. A., and RAY, W. H., *Optimal Feedback Control for Linear Quadratic Systems Having Time Delays*, International Journal of Control, Vol. 15, pp. 609–627, 1972.
18. MCAULAY, R. J., *A Gradient Method for Systems with Time Delays and Its Application to Waveform Design*, IEEE Transactions on Automatic Control, Vol. AC-14, pp. 230–237, 1969.
19. GRACOVETSKY, S. A., and VIDYASAGAR, M., *Suboptimal Control of Neutral Systems*, International Journal of Control, Vol. 18, pp. 121–128, 1973.
20. KALMAN, R. E., *Contributions to the Theory of Optimal Control*, Boletin de la Sociedad Matematica Mexicana, Vol. 5, pp. 102–119, 1960.
21. MUELLER, T. E., *Optimal Control of Linear Systems with Time Lag*, University of Illinois, Urbana, Illinois, Coordinated Science Laboratory, Report No. R-254, 1965.

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Optimal Tactics for Close Support Operations: Part I, Degraded Communications[1]

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Abstract. A formidable problem in the mathematical studies of C^3 (command, control, and communication) is the determination of the optimal decision rules for force commitments to be employed by headquarters. Recursive equations are derived for an amphibious campaign with time-invariant linear dynamics and quadratic costs. Air and ground commanders are assumed to have perfect intelligence with degraded communication between them.

Key Words. Command, control, and communication; integral equations; optimal decision functions; minimum expected cost.

1. Introduction

Mathematical studies of C^3 (command, control, and communication) involve concepts from dynamic programming, team decision theory, and Fredholm integral equations (Refs. 1–2). The primary output of the mathematical models is optimal decision rules for force commitments to be employed by subordinate headquarters in coordinating their activities to achieve objectives laid down by superordinate headquarters. The decision rules make use only of information available to each subordinate commander, and that information is determined by intelligence and communication networks. Thus, the decision rules form decision aids to be used

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[1] The authors have benefitted greatly from discussions with Colonels R. Geesey, C. Albo, and R. Reeves, USAF.
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by seasoned commanders. The second type of output of these models is the assessment in military costs due to changes in the intelligence and communication networks. Some general equations were derived by Kagiwada and Kalaba in Ref. 3 using concepts discussed in Refs. 4–5. In this paper, the recursive solution of the equations is derived for a particular subset of the general equations.

Consider a C^3 problem in which there are two subordinate commanders, both striving to coordinate their decisions to attain the tactical objective set down by superordinate headquarters. In an amphibious campaign, the blue naval force lands ground troops and provides close support. The objective is to move inland a certain distance in a specified time. It is desired to attain the objective at minimum expected cost.

The recursive equations are derived for time-invariant, linear dynamics with quadratic costs. The blue naval air and ground commanders are assumed to have perfect intelligence with degraded communication. Numerical results are obtained which give the optimal decisions per day of the blue air and ground commanders as a function of the present position of the front line and the daily strengths of the red air and ground forces.

2. C^3 Model

For the case to be considered, the blue naval air commander has intelligence which provides him with the red air commander's strength p, for the coming day; but, because of the lack of communication, this information is not sent to the blue ground commander. The blue ground commander has intelligence which provides him with the red ground commander's strength q, for the coming day; but again, because of lack of communication, he does not communicate this to the blue naval air commander.

Let N be the duration of the campaign, and let the distance from the shore to the objective be s_0. Consider K days remaining with the front line at a distance s from the shore. The new position of the front line is assumed to be a function of the old position plus a linear combination of the strengths,

$$S = s + C_1\alpha + C_2\beta - C_3 p - C_4 q, \tag{1}$$

where S is the new position of the front line with $K-1$ days remaining; s is the current position of the front line with K days remaining; α, β are the blue naval air and ground strengths, respectively, with K days remaining; p, q are the red air and ground strengths respectively with K days; and C_1, C_2, C_3, C_4 are constants.

The daily cost is assumed to be proportional to the blue losses, which in turn are proportional to the strengths utilized. The ground losses are

reduced by the air strength for close support missions. Thus, the daily cost is assumed to be

$$C = C_5\alpha + (C_6 - C_7\alpha)\beta + \tfrac{1}{2}C_8\alpha^2 + \tfrac{1}{2}C_9\beta^2, \tag{2}$$

where C_5, C_6, C_7, C_8, C_9 are constants.

An additional cost is assessed if the front line at the end of the campaign is at some position s other than s_0. The terminal cost is assumed to be

$$\phi(s) = \lambda(s - s_0)^2. \tag{3}$$

The red air and ground commanders make the decision to employ the strengths p and q, respectively, each day. The decision-making of the enemy is simplified by assuming that p, q are random variables with joint probability density function.

$$w = w(p, q). \tag{4}$$

Furthermore, assuming that p, q are independent random variables, the joint probability function can be expressed as the product

$$w(p, q) = P(p)Q(q). \tag{5}$$

The minimum expected cost is defined by

$$g_K = g_K(s), \qquad K = 0, 1, 2, \ldots, N, \qquad \text{all } s, \tag{6}$$

where $g_K(s)$ is the expected cost of a campaign beginning with the front line at s, of duration K, and employing an optimal sequence of decisions. Using Bellman's principle of optimality (Ref. 1), the functions $g_{K+1}(s)$ and $g_K(s)$ are related by the recurrence equation

$$g_{K+1}(s) = \min_{\alpha,\beta} \iint [C_5\alpha + (C_6 - C_7\alpha)\beta + \tfrac{1}{2}C_8\alpha^2 + \tfrac{1}{2}C_9\beta^2 + g_K(S)]P(p)Q(q)\, dp\, dq, \qquad K = 0, 1, 2, \ldots, N. \tag{7}$$

All integrals on p, q are from 0 to ∞. From general control-theoretical considerations, the minimum expected cost has the form

$$g_K(s) = \rho_K + \sigma_K s + \tau_K s^2, \tag{8}$$

where the coefficients ρ_K, σ_K, τ_K are computed for K stages remaining.

The minimization in Eq. (7) is over

$$\alpha = \alpha(K, s, p) \qquad \text{and} \qquad \beta = \beta(K, s, q).$$

Then, differentiation leads to the conditions

$$0 = \int \{(C_5 - C_7\beta) + C_8\alpha + [\sigma_K + 2\tau_K(s + C_1\alpha + C_2\beta - C_3p - C_4q)]C_1\}Q(q)\,dq, \tag{9}$$

$$0 = \int \{(C_6 - C_7\alpha) + C_9\beta + [\sigma_K + 2\tau_K(s + C_1\alpha + C_2\beta - C_3p - C_4q)]C_2\}P(p)\,dp, \tag{10}$$

which are a system of Fredholm integral equations for the functions α, β. Equations (9)–(10) are the dynamic headquarters-by-headquarters optimality conditions. They state that, at every decision-making opportunity, each headquarters is to make the decision which reduces the marginal conditional expected cost of the remainder of the process to zero.

3. Recursive Equations

The recursive equations for the solution of Eqs. (6)–(7) are derived as follows. Making use of the equations

$$\int Q(q)\,dq = 1, \tag{11}$$

$$\int qQ(q)\,dq = \bar{q}, \tag{12}$$

and similar relations for p, Eqs. (9)–(10) can be written as the linear Fredholm integral equations

$$\alpha(p) = [1/(2C_1^2\tau_K + C_8)]\Big[(C_7 - 2C_1C_2\tau_K)\int \beta(q)Q(q)\,dq + 2C_1C_4\tau_K\bar{q} - C_5 - C_1\sigma_K - 2C_1\tau_K s + 2\tau_K C_1C_3p\Big], \tag{13}$$

$$\beta(q) = [1/(2C_2^2\tau_K + C_9)]\Big[(C_7 - 2C_1C_2\tau_K)\int \alpha(p)P(p)\,dp + 2C_2C_3\tau_K\bar{p} - C_6 - C_2\sigma_K - 2C_2\tau_K s + 2\tau_K C_2C_4q\Big]. \tag{14}$$

We shall solve these integral equations observing that they have degenerate kernels. Define two new parameters

$$v_1 = \int \beta(q)Q(q)\,dq, \tag{15}$$

$$v_2 = \int \alpha(p)P(p)\,dp. \tag{16}$$

Substituting $\beta(q)$ into Eq. (15), we have

$$v_1 = [1/(2C_2^2\tau_K + C_9)][(C_7 - 2C_1C_2\tau_K)v_2 + 2C_2C_3\tau_K\bar{p} - C_6 - C_2\sigma_K - 2C_2\tau_K s + 2\tau_K C_2C_4\bar{q}]. \tag{17}$$

Substituting $\alpha(p)$ into Eq. (16), we have

$$v_2 = [1/(2C_1^2\tau_K + C_8)][(C_7 - 2C_1C_2\tau_K)v_1 + 2C_1C_4\tau_K\bar{q} - C_5 - C_1\sigma_K - 2C_1\tau_K s + 2\tau_K C_1C_3\bar{p}]. \tag{18}$$

To simplify the notation, express Eqs. (17)–(18) in the form

$$v_1 = B_1(B_2v_2 - B_3s + B_4), \tag{19}$$

$$v_2 = B_5(B_2v_1 - B_6s + B_7). \tag{20}$$

Substituting v_2 into Eq. (19) and v_1 into Eq. (20), it can be shown that

$$v_1 = D_1 + D_2s, \tag{21}$$

$$v_2 = D_3 + D_4s, \tag{22}$$

where the D_i's are functions of the B_j's. Then, substituting Eq. (21) into Eq. (13) and Eq. (22) into Eq. (14), it can be shown that the optimal decision functions are

$$\alpha(p) = u_K + v_Ks + w_Kp, \tag{23}$$

$$\beta(q) = x_K + y_Ks + z_Kq, \tag{24}$$

where

$$u_K = B_5B_2D_1 + B_5B_7', \tag{25}$$

$$v_K = B_5B_2D_2 - B_5B_6, \tag{26}$$

$$w_K = 2\tau_K C_1C_3B_5, \tag{27}$$

$$x_K = B_1B_2D_3 + B_1B_4', \tag{28}$$

$$y_K = B_1B_2D_4 - B_1B_3, \tag{29}$$

$$z_K = 2\tau_K C_2C_4B_1, \tag{30}$$

$$B_1 = 1/(2C_2^2\tau_K + C_9), \quad (31)$$
$$B_2 = C_7 - 2C_1C_2\tau_K, \quad (32)$$
$$B_3 = 2C_2\tau_K, \quad (33)$$
$$B_4 = 2C_2C_3\tau_K\bar{p} - C_6 - C_2\sigma_K + 2\tau_K C_2 C_4\bar{q}, \quad (34)$$
$$B_4' = 2C_2C_3\tau_K\bar{p} - C_6 - C_2\sigma_K, \quad (35)$$
$$B_5 = 1/(2C_1^2\tau_K + C_8), \quad (36)$$
$$B_6 = 2C_1\tau_K, \quad (37)$$
$$B_7 = 2C_1C_4\tau_K\bar{q} - C_5 - C_1\sigma_K + 2\tau_K C_1 C_3\bar{p}, \quad (38)$$
$$B_7' = 2C_1C_4\tau_K\bar{q} - C_5 - C_1\sigma_K, \quad (39)$$
$$D_1 = [B_1/(1 - B_1B_2^2B_5)](B_2B_5B_7 + B_4), \quad (40)$$
$$D_2 = [-B_1/(1 - B_1B_2^2B_5)](B_2B_5B_6 + B_3), \quad (41)$$
$$D_3 = [B_5/(1 - B_1B_2^2B_5)](B_2B_1B_4 + B_7), \quad (42)$$
$$D_4 = [-B_5/(1 - B_1B_2^2B_5)](B_2B_1B_3 + B_6), \quad (43)$$

Note that S can be expressed in terms of these coefficients and s, p, q. Substituting the optimal α, β into Eq. (1), the new position of the front line becomes

$$\begin{aligned} S &= s + C_1(u_K + v_K s + w_K p) + C_2(x_K + y_K s + z_K q) - C_3 p - C_4 q \\ &= C_1 u_K + C_2 x_K + (1 + C_1 v_K + C_2 y_K)s + (C_1 w_K - C_3)p + (C_2 z_K - C_4)q. \end{aligned} \quad (44)$$

The expected cost $g_{K+1}(s)$ is obtained by substituting Eqs. (1) and (8) into Eq. (7):

$$\begin{aligned} g_{K+1}(s) = \min_{\alpha,\beta} \iint \{&C_5\alpha + (C_6 - C_7\alpha)\beta + \tfrac{1}{2}C_8\alpha^2 + \tfrac{1}{2}C_9\beta^2 \\ &+ \rho_K + \sigma_K(s + C_1\alpha + C_2\beta - C_3p - C_4q) \\ &+ \tau_K(s + C_1\alpha + C_2\beta - C_3p - C_4q)^2\}P(p)Q(q)\,dp\,dq. \end{aligned} \quad (45)$$

Substituting Eqs. (23)–(24) into Eq. (45), we have

$$\begin{aligned} g_{K+1}(s) = \iint \{&C_5(u_K + v_K s + w_K p) \\ &+ [C_6 - C_7(u_K + v_K s + w_K p)](x_K + y_K s + z_K q) \\ &+ \tfrac{1}{2}C_8(u_K + v_K s + w_K p)^2 + \tfrac{1}{2}C_9(x_K + y_K s + z_K q)^2 + \rho_K \\ &+ \sigma_K[s + C_1(u_K + v_K s + w_K p) + C_2(x_K + y_K s + z_K q) - C_3p - C_4q] \\ &+ \tau_K[s + C_1(u_K + v_K s + w_K p) + C_2(x_K + y_K s + z_K q) \\ &- C_3p - C_4q]^2\}P(p)Q(q)\,dp\,dq. \end{aligned} \quad (46)$$

To simplify the notation, express Eq. (46) in the form

$$\begin{aligned} g_{K+1}(s) &= \rho_{K+1} + \sigma_{K+1}s + \tau_{K+1}s^2 \\ &= \iint \{C_5u_K + C_5v_Ks + C_5w_Kp + C_6x_K + C_6z_Kq \\ &\qquad - C_7(u_K + w_Kp)(x_K + z_Kq) \\ &\quad + [C_6y_K - C_7(u_K + w_Kp)y_K - C_7(x_K + z_Kq)v_K]s - C_7v_Ky_Ks^2 \\ &\quad + \tfrac{1}{2}C_8[u_K^2 + 2u_Kw_Kp + w_K^2p^2 + 2(u_Kv_K + v_Kw_Kp)s + v_K^2s^2] \\ &\quad + \tfrac{1}{2}C_9[x_K^2 + 2x_Kz_Kq + z_K^2q^2 + 2(x_Ky_K + y_Kz_Kq)s + y_K^2s^2] \\ &\quad + \rho_K + \sigma_K(A_1 + A_2s + A_3p + A_4q) \\ &\quad + \tau_K[(A_1 + A_3p + A_4q)^2 + 2(A_1 + A_3p + A_4q)A_2s \\ &\quad + A_2^2s^2]\}P(p)Q(q)\,dp\,dq. \end{aligned} \quad (47)$$

Integrating and setting the coefficients of the constant terms, s, and s^2 equal, it can be shown that

$$\begin{aligned} \rho_{K+1} = {} & C_5u_K + C_5w_K\bar{p} + C_6x_K + C_6z_K\bar{q} - C_7(u_K + w_K\bar{p})(x_K + z_K\bar{q}) \\ &+ \tfrac{1}{2}C_8(u_K^2 + 2u_Kw_K\bar{p} + w_K^2\bar{p}^2) + \tfrac{1}{2}C_9(x_K^2 + 2x_Kz_K\bar{q} + z_K^2\bar{q}^2) \\ &+ \rho_K + \sigma_K(A_1 + A_3\bar{p} + A_4\bar{q}) \\ &+ \tau_K[A_1^2 + 2(A_1A_3\bar{p} + A_1A_4\bar{q} + A_3A_4\bar{p}\bar{q}) + A_3^2\bar{p}^2 + A_4^2\bar{q}^2], \end{aligned} \quad (48)$$

$$\begin{aligned} \sigma_{K+1} = {} & C_5v_K + C_6y_K - C_7(u_K + w_K\bar{p})y_K - C_7(x_K + z_K\bar{q})v_K \\ &+ C_8(u_Kv_K + v_Kw_K\bar{p}) + C_9(x_Ky_K + y_Kz_K\bar{q}) \\ &+ \sigma_KA_2 + 2\tau_KA_2(A_1 + A_3\bar{p} + A_4\bar{q}), \end{aligned} \quad (49)$$

$$\tau_{K+1} = -C_7v_Ky_K + \tfrac{1}{2}C_8v_K^2 + \tfrac{1}{2}C_9y_K^2 + \tau_KA_2^2, \quad (50)$$

where

$$A_1 = C_1u_K + C_2x_K, \quad (51)$$
$$A_2 = 1 + C_1v_K + C_2y_K, \quad (52)$$
$$A_3 = C_1w_K - C_3, \quad (53)$$
$$A_4 = C_2z_K - C_4. \quad (54)$$

The recursive relations are given by Eqs. (48)–(50), supplemented by Eqs. (25)–(30) and (31)–(43).

4. Numerical Results

Numerical results were obtained using the recursive equations. At each stage, the coefficients u_K, v_K, w_K for the optimal blue air strength $\alpha(p)$ and the coefficients x_K, y_K, z_K for the optimal blue ground strength $\beta(q)$ are computed in Eqs. (23)–(24) using Eqs. (25)–(43). These coefficients are functions of the coefficients ρ_K, σ_K, τ_K, of the expected cost given by Eq. (8) for K stages remaining. Once the coefficients in the equations for $\alpha(p)$, $\beta(q)$ have been computed, the coefficients in the equation for the expected cost for $K+1$ stages remaining can be computed using Eqs. (48)–(50). The new coefficients u_{K+1}, v_{K+1}, w_{K+1} for $\alpha(p)$ and x_{K+1}, y_{K+1}, z_{K+1} for $\beta(q)$ can then be computed and the process repeated until the coefficients have been computed for all N stages of the campaign. The coefficients are stored for subsequent processing. The process is initiated with no time remaining by the terminal cost

$$g_0(s) = \lambda(s - s_0)^2,$$

from which ρ_0, σ_0, τ_0 can be computed.

The equation for the new position of the front line is a function of s, the optimal decisions, and the red strengths p, q. The new position of the front line can thus be computed recursively using Eq. (44), starting with N stages remaining, after the coefficients u_K, v_K, w_K, x_K, y_K, z_K have been computed and stored as described above. The minimum expected cost can be computed at each stage using Eq. (8) with the position of the front line s just computed.

Let us consider a special campaign with the following duration N and distance to be covered s_0:

$$N = 21 \text{ days}, \qquad s_0 = 21 \text{ miles}.$$

The average red strengths are assumed to be

$$\bar{p} = 1, \qquad \bar{q} = 1.$$

The second moments of p and q are given by

$$\overline{p^2} = \bar{p}^2 + \sigma_p^2, \qquad \overline{q^2} = \bar{q}^2 + \sigma_q^2.$$

The constant coefficients in Eqs. (1)–(3) are

$$\begin{aligned} &C_1 = 0.1, && C_5 = 0.005, && C_9 = 0.1,\\ &C_2 = 1.0, && C_6 = 0.1, && \sigma_p = 0.5,\\ &C_3 = 0.1, && C_7 = 0.01, && \sigma_q = 0.5,\\ &C_4 = 1.0, && C_8 = 0.1, && \lambda = 1.0. \end{aligned}$$

The optimal air and ground decisions $\alpha(p)$, $\beta(q)$ for $K = 21$, 11, and 1 stages to go are as follows:

$$\begin{aligned} K &= 21, && \alpha(p) = 0.4544 - 9.225 \times 10^{-3}s + 4.792 \times 10^{-4}p,\\ K &= 21, && \beta(q) = 2.002 - 4.659 \times 10^{-2}s + 4.575 \times 10^{-2}q;\\ K &= 11, && \alpha(p) = 0.6285 - 1.758 \times 10^{-2}s + 9.557 \times 10^{-4}p,\\ K &= 11, && \beta(q) = 2.841 - 8.876 \times 10^{-2}s + 8.731 \times 10^{-2}q;\\ K &= 1, && \alpha(p) = 3.967 - 0.1853s + 0.1667p,\\ K &= 1, && \beta(q) = 19.67 - 0.9356s + 0.9524q. \end{aligned}$$

Consider the situation with $K = 11$ days remaining and the front line at $s = 10$ miles. The blue air commander receives the intelligence that the red air strength for the coming day will be $p = 1$. Then, using the above table, the optimal blue air strength is

$$\alpha(p) = 0.6285 - 1.758 \times 10^{-2}(10) + 9.557 \times 10^{-4}(1) = 0.45.$$

Assume that the blue ground commander receives the intelligence that the red ground strength for the coming day will be $q = 0.7$. Then, the optimal blue ground strength is

$$\beta(q) = 2.841 - 8.876 \times 10^{-2}(10) + 8.731 \times 10^{-2}(0.7) = 2.01.$$

Note that the blue air and ground decisions are not dependent on each other, because of the assumed lack of communication. Each commander bases his decision on his own intelligence.

Considering the campaign as a whole with 21 days and 21 miles to go, then, starting with the front line at $s = 0$, the optimal decisions are such that the front line moves forward approximately one mile per day. Furthermore, if the red air and ground strengths are constant throughout the campaign and equal to their average values of one [i.e., $p = \bar{p} = 1$ and $q = \bar{q} = 1$), then the optimal blue air and ground strengths are also constant with

$$\alpha(p) = 0.4549, \qquad \beta(q) = 2.047.$$

The daily blue loss, Eq. (2), is constant in this case and equal to $C = 0.4176$. The minimum expected cost, Eq. (8), is equal to the sum of the daily costs plus the terminal cost when

$$\sigma_p = \sigma_q = 0.$$

Increasing σ_p, σ_q does not change the optimal decisions, but increases the expected cost.

5. Discussion

The above sample campaign is one of many for which results have been obtained. In this example, $\alpha(p)$, $\beta(q)$ are positive. In general, however, for arbitrary choices of the C_i coefficients, $\alpha(p)$, $\beta(q)$ may be negative. An important interpretation of this situation involves the consideration of this model as a perturbation model, so that $\alpha(p)$, $\beta(q)$ represent increments above and below previously assigned values.

In an actual campaign, a coefficient such as C_5 in the cost equation is proportional to the red anti-air strength and may vary during the course of the campaign as the red anti-air fortifications are destroyed. The analysis of such a case involves time-varying coefficients and is easily handled by an extension of the foregoing analysis. The time-varying case will be discussed in a future paper, followed by consideration of different combinations of intelligence and communication capabilities.

References

1. BELLMAN, R., and KALABA, R., *Dynamic Programming and Modern Control Theory*, Academic Press, New York, New York, 1966.
2. KAGIWADA, H., and KALABA, R., *Integral Equations via Imbedding Methods*, Addison-Wesley Publishing Company, Reading, Massachusetts, 1974.
3. KAGIWADA, H., and KALABA, R., *C^3 Modeling*, U.S. Naval Postgraduate School Workshop, Monterey, California, 1978.
4. DRUZHININ, V. V., and KONTOROV, D. S., *Concept, Algorithm, Decision*, USAF Translation, US Government Printing Office, Washington, DC, 1972.
5. THRALL, R., TSOKOS, C., and TURNER, J., *Decision Information for Tactical Command and Control*, R. M. Thrall and Associates, Houston, Texas, 1977.

Stochastic Optimal Control of Internal Hierarchical Labor Markets

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Abstract. This paper develops an optimal control model for a graded manpower system where the demand for manpower is uncertain. The organization's objective is to minimize the discounted costs of operating the manpower system, including excess demand costs. The stock of workers in various grades can be adjusted in two ways. The first method is outside hiring flows, which is the usual control variable used in previous research. The second method is to control the transition rates between grades of the hierarchy, an instrument not previously studied. Incorporating the transition rates into the control variables creates time lags in the control process. The resulting problem is solved numerically using an approximation for the time-lagged control variables. The numerical example is based on the Air Force officer hierarchy. The model is used to examine such issues as the desirability of granting tenure to workers who are not promoted to the highest grade and the effects of length-of-service and demand uncertainty on manpower policy.

Key Words. Stochastic optimal control, labor hierarchy, time-lagged control, internal labor markets, manpower planning.

1. Introduction

A distinguishing feature of many organizations is the hierarchical form of their labor force. This hierarchy is an arrangement of workers according to grade; the stock of workers in each grade is changed by a flow of workers between grades within the organization and also by a flow of people entering and leaving it. The problem posed in this paper is to find optimal worker flow

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