The Equivalence of Team Theory's Integral Equations and a Cauchy System: Sensitivity Analysis of a Variational Problem*†

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ABSTRACT

Team decision theory studies the problem of how a group of decision makers should use information to coordinate their actions. Mathematically, the task is to find functions that maximize an objective functional. The Euler equations take the form of a system of integral equations. In this paper, it will be shown that a class of such integral equations has solutions that are identical to the solutions of a system of initial-valued integrodifferential equations. This Cauchy system describes the sensitivity of the solutions to underlying parameters and provides an efficient technique for solving difficult team decision problems. An analysis of a profit maximizing firm demonstrates the usefulness of the Cauchy system.

1. INTRODUCTION

Team decision theory extends Bayesian statistical decision theory to a group of interdependent decision makers [1,2]. A team is an organization

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whose members share a single, well-defined objective function. Such a harmonious group has but one problem: how are individual activities coordinated in an optimal fashion? Team decision theory explores such problems when the organization is uncertain about its environment and when information about the environment differs among team members. The decision problem reduces to the selection of rules of action that coordinate the interdependent activities of the teammates to maximize the expected payoff of the team. The elements of team decision theory will be developed in Sec. 2.

The optimal decision rules must necessarily satisfy a system of integral equations. In general these equations may be nonlinear, with multiple integrals and infinite limits of integration. The characteristics of their solutions are difficult to define. However, in Sec. 3 it is shown that the optimal decision rules must also satisfy a system of initial-valued integrodifferential equations, which are called the sensitivity equations. These equations describe how the solution and its resolvent kernel (which plays the role of a matrix inverse) depend upon an important parameter, the degree of interdependence between teammates. In Sec. 4, a model of the firm is used to illustrate how the sensitivity equations might be used.

2. TEAM DECISION THEORY

The team consists of n decision makers or teammates, indexed by $i=1,2,\ldots,n$. The basic elements of the team decision problem are as follows:

$\theta\in\Theta\subseteq\mathcal{R}^l$: the unknown state of nature;
$A=(a_1,\ldots,a_n)\in\mathcal{R}^n$: the actions of the teammates;
$P(A,\theta)$: the team's payoff function;
$Y=(y_1,\ldots,y_n)\in\mathcal{Y}\subseteq\mathcal{R}^n$: the information of the teammates;[1]
$f(\theta)$: the team's prior probability density of θ;
$g(Y|\theta)$: the team's conditional prior probability density of Y given θ;
$\alpha(Y)=(\alpha_1(y_1),\ldots,\alpha_n(y_n))\in\Delta$: the team decision function.[2]

Several remarks should be made here. First, there is only one payoff function, agreed upon by all members. Second, the payoff function is not

[1]The information that the ith teammate uses may come from two sources, a personal observation of the environment or a message from another teammate that summarizes his knowledge about the environment. Hence, it may seem more natural to make each component y_i a vector itself; but this will significantly complicate the results that follow. One might imagine that the vector of information has been reduced to a single "statistic".
[2]The function space Δ is presumed to be some complete normed linear vector space. The only important distinction we want to make is that the ith component function, $\alpha_i(\cdot)$, depends only on y_i.

necessarily separable; that is, in general $P_{a_i a_j}\neq 0$. Third, there is only one pair of probability densities, $f(\theta)$ and $g(Y|\theta)$, agreed upon by all members. Fourth, the ith teammate's information, y_i, is different from the jth teammate's information, y_j. Fifth, since the ith teammate knows only y_i, his action depends only on y_i; i.e.,

$$a_i=\alpha_i(y_i).$$

Each teammate wants to select decision rules that are coordinated to maximize the team's expected utility

$$W[\alpha]=\int_\Theta\int_{\mathcal{Y}}P(\alpha(Y),\theta)g(Y|\theta)f(\theta)\,dY\,d\theta. \tag{1}$$

How can the optimal decision rules $\alpha^*(Y)$ be characterized? It has been shown [3] that the optimal decision rules must satisfy a system of integral equations.

Theorem 2.1 (Person-by-person optimality). *If $\alpha^*(Y)$ is the optimal team decision rule, then it must satisfy the following equations:*

$$0=\int_{y_1}\cdots)i(\cdots\int_{y_n}\int_\Theta P_{a_i}(\alpha^*(Y),\theta)h(Y_{)i(},\theta\,|\,y_i))\,dy_{)i(}\,d\theta \tag{2}$$

for all $y_i\in\mathcal{Y}_i$, $i=1,2,\ldots,n$.

Here $h(Y_{)i(},\theta\,|\,y_i)=g(Y|\theta)f(\theta)/g_i(y_i)$ is the posterior probability of θ and

$$Y_{)i(}=(y_1,\ldots,y_{i-1},y_{i+1},\ldots,y_n)$$

given y_i. The integral equations (2) can be written succinctly as

$$E\{P_{a_i}(\alpha^*(Y),\theta)\,|\,y_i\}=0 \qquad \text{for all} \quad y_i\in\mathcal{Y}_i. \tag{3}$$

This is referred to as "person-by-person optimality" because each teammate, assuming his colleagues are using their best decision rules, picks a decision rule such that his posterior expected marginal payoff equals zero no matter what information he might receive.

In the remainder of the paper it will be assumed that the payoff is a linear-quadratic function of the actions; that is,

$$P(A,\theta)=\mu(\theta)'A-1/2A'QA, \tag{4}$$

where $\mu(\theta)$ is a vector of random variables and where Q is a known positive definite matrix. With the payoff function of Eq. (4), the person-by-person optimality conditions are linear Fredholm integral equations. The optimal team decision rules must satisfy the following equations:

$$\begin{aligned}0&=E\{\mu_i(\theta)|y_i\}-q_{ii}\alpha_i^*(y_i)\\&\quad-\sum_{j\neq i}q_{ij}E\{\alpha_j^*(y_j)|y_i\}\\&=E\{\mu_i(\theta)|y_i\}-q_{ii}\alpha_i^*(y_i)\\&\quad-\sum_{j\neq i}q_{ij}\int_{y_i}\alpha_j^*(y_j)h(y_j|y_i)\,dy_j,\end{aligned} \tag{5}$$

where $h(y_j|y_i)$ is the posterior probability of y_j given y_i, $i=1,2,\ldots,n$. These linear Fredholm integral equations may be reexpressed in a standard form as follows:

$$u(t)=b(t)+\int_0^1 k(t,s)u(s)\,ds,\qquad 0\leqslant t\leqslant 1, \tag{6}$$

where $u(t)$ is an unknown vector of functions, $b(t)$ is a vector of forcing functions, and $k(t,s)$ is a matrix of kernels. The transformation of Eq. (5) into Eq. (6) is accomplished by dividing Eq. (5) by q_{ii}, solving for $\alpha_i^*(y_i)$, and relabeling variables.

3. AN EQUIVALENCE THEOREM

A slightly more general class of linear Fredholm integral equations is

$$u(t,\lambda)=b(t)+\lambda\int_0^1 k(t,s)u(s,\lambda)\,ds,\qquad 0\leqslant t\leqslant 1, \tag{7}$$

where λ is a scalar parameter. Equation (6) is a special case of Eq. (7) where $\lambda=1$. The solution of (7) will depend not only on t but on the value of the parameter λ. It will now be shown that the solution, $u(t,\lambda)$, of Eq. (7) is

equivalent to the solution of a particular system of initial-valued integro-differential equations.

THEOREM 3.1. *The vector function $u(t,\lambda)$ which satisfies the system of linear Fredholm integral equations (7) and the matrix of resolvent kernel functions, $K(t,s,\lambda)$, which satisfies the system of linear Fredholm integral equations*

$$K(t,s,\lambda)=k(t,s)+\lambda\int_0^1 k(t,s')K(s',s,\lambda)\,ds',$$

$$0\leqslant t,s\leqslant 1, \tag{8}$$

are equivalent to the solutions of the following initial-valued integrodifferential equations:

$$u_\lambda(t,\lambda)=\int_0^1 K(t,s,\lambda)u(s,\lambda)\,ds, \tag{9}$$

$$K_\lambda(t,s,\lambda)=\int_0^1 K(t,s',\lambda)K(s',s,\lambda)\,ds', \tag{10}$$

$$u(t,0)=b(t), \tag{11}$$

$$K(t,s,0)=k(t,s), \tag{12}$$

$$0\leqslant t\leqslant 1,\qquad 0\leqslant s\leqslant 1.$$

The theorem is proved in two parts. First it will be shown that the solutions of the integral equations are solutions to the Cauchy system (9)–(12). It is well known [4] that the solution of a linear Fredholm system, Eq. (7), may be expressed using a resolvent kernel matrix, $K(t,s,\lambda)$, as follows:

$$u(t,\lambda)=b(t)+\lambda\int_0^1 K(t,s,\lambda)b(s)\,ds. \tag{13}$$

The resolvent kernel must satisfy a related system of linear Fredholm integral equations, given above by Eq. (8). Differentiate the system of equations (7) with respect to λ, to get

$$\begin{aligned}u_\lambda(t,\lambda)&=\int_0^1 k(t,s)u(s,\lambda)\,ds\\&\quad+\lambda\int_0^1 k(t,s)u_\lambda(s,\lambda)\,ds.\end{aligned} \tag{14}$$

This is a new Fredholm system with an unknown function $u_\lambda(t,\lambda)$ but with the same kernel as the original system (7). Therefore its solution may be expressed using the same resolvent kernel,

$$u_\lambda(t,\lambda)=\lambda^{-1}[u(t,\lambda)-b(t)] \\ +\lambda\int_0^1 K(t,s,\lambda)\lambda^{-1}[u(s,\lambda)-b(s)]\,ds, \tag{15}$$

where the forcing term of Eq. (14) has been replaced by an equivalent term using Eq. (7). Equation (15) may be expressed as follows:

$$u_\lambda(t,\lambda)=\lambda^{-1}\left\{u(t,\lambda)-b(t)-\lambda\int_0^1 K(t,s,\lambda)b(s)\,ds\right\} \\ +\int_0^1 K(t,s,\lambda)u(s,\lambda)\,ds. \tag{16}$$

The term in braces in Eq. (16) is zero due to Eq. (13), and thus Eq. (16) reduces to the desired integrodifferential equations (9). To get the integrodifferential equation (10), differentiate Eq. (8) with respect to λ, to get

$$K_\lambda(t,s,\lambda)=\int_0^1 k(t,s')K(s',s,\lambda)\,ds' \\ +\lambda\int_0^1 k(t,s')K_\lambda(s',s,\lambda)\,ds'. \tag{17}$$

Since this equation has the same kernel as Eq. (7), its solution may be expressed using the resolvent kernel, K, as follows:

$$K_\lambda(t,s,\lambda)=\lambda^{-1}[K(t,s,\lambda)-k(t,s)] \\ +\lambda\int_0^1 K(t,s',\lambda)\lambda^{-1}[K(s',s,\lambda)-k(s',s)]\,ds'. \tag{18}$$

Using Eq. (8), it can be shown that Eq. (18) reduces to the desired integrodifferential equations (10). The initial conditions, (11) and (12), are just Eq. (7) and (8) with $\lambda=0$. The second part of the proof is to show that a solution of the Cauchy system is a solution of the integral equations. Define $A(t,s,\lambda)$ by

$$A(t,s,\lambda)=k(t,s)+\lambda\int_0^1 k(t,s')K(s',s,\lambda)\,ds', \tag{19}$$

where $K(t,s,\lambda)$ is the solution of the Cauchy system. If it can be shown that $A(t,s,\lambda)=K(t,s,\lambda)$ for all t and s, then the solution of the Cauchy system satisfies the integral equation (8). Differentiate Eq. (19) with respect to λ, to get

$$A_\lambda(t,s,\lambda)=\int_0^1 k(t,s')K(s',s,\lambda)\,ds' \\ +\lambda\int_0^1 k(t,s')K_\lambda(s',s,\lambda)\,ds'. \tag{20}$$

Substitute Eq. (10) into Eq. (20) to get

$$A_\lambda(t,s,\lambda)=\int_0^1 k(t,s')K(s',s,\lambda)\,ds' \\ +\lambda\int_0^1 k(t,s')\int_0^1 K(s',s'',\lambda) \\ \times K(s'',s,\lambda)\,ds''\,ds'. \tag{21}$$

In the last term of Eq. (21), relabel s' as s'' and s'' as s', reorder the integration, and pass ds'' through all terms independent of s'' to get

$$A_\lambda(t,s,\lambda)=\int_0^1\left\{k(t,s')+\int_0^1 k(t,s'')K(s'',s',\lambda)\,ds''\right\} \\ \times K(s',s,\lambda)\,ds' \tag{22}$$

The term in braces is exactly $A(t,s',\lambda)$, so Eq. (22) is equivalent to

$$A_\lambda(t,s,\lambda)=\int_0^1 A(t,s',\lambda)K(s',s,\lambda)\,ds'. \tag{23}$$

When λ is set equal to zero in Eq. (19), the value of $A(t,s,0)$ is determined to be

$$A(t,s,0)=k(t,s). \tag{24}$$

If the solution of the Cauchy system, (23)–(24), is unique, then since $K(t,s,\lambda)$ satisfies the Cauchy system (23) and (24), it must be true that $A(t,s,\lambda)\equiv K(t,s,\lambda)$, which was to be shown. Define $B(t,\lambda)$ by

$$B(t,\lambda)=b(t)+\int_0^1 k(t,s)u(s,\lambda)\,ds, \tag{25}$$

where $u(s,\lambda)$ is the solution of the Cauchy system. As above, to show that $u(t,\lambda)$ satisfies the integral equation, it suffices to show that $u(t,\lambda)=B(t,\lambda)$ for all t. Differentiate (25) with respect to λ, to get

$$B_\lambda(t,\lambda)=\int_0^1 k(t,s)u(s,\lambda)\,ds + \lambda\int_0^1 k(t,s)u_\lambda(s,\lambda)\,ds. \tag{26}$$

Substitute Eq. (9) into Eq. (26) to get

$$B_\lambda(t,\lambda)=\int_0^1 k(t,s)u(s,\lambda)\,ds + \lambda\int_0^1 k(t,s)\int_0^1 K(s,s',\lambda)u(s',\lambda)\,ds'\,ds. \tag{27}$$

Relabel s as s' and s' as s, reorder the integration, and pass ds' through all terms independent of s to get

$$B_\lambda(t,\lambda)=\int_0^1\left\{k(t,s)+\lambda\int_0^1 k(t,s')K(s',s,\lambda)\,ds'\right\} \times u(s,\lambda)\,ds. \tag{28}$$

From the above we know that $K(t,s,\lambda)$ satisfies the integral equation (9), so the bracketed term of Eq. (28) equals $K(t,s,\lambda)$; that is,

$$B_\lambda(t,\lambda)=\int_0^1 K(t,s,\lambda)u(s,\lambda)\,ds. \tag{29}$$

When $\lambda=0$ in the definition (25), the initial value of $B(t,\lambda)$ is determined to be

$$B(t,0)=b(t). \tag{30}$$

If the solution of the Cauchy system (29)–(30) is unique, this implies that $B(t,\lambda)\equiv u(t,\lambda)$, which completes the proof.

This theorem provides an equivalent way of describing the person-by-person optimality conditions for a linear-quadratic team decision problem. The optimal decision rules must satisfy a system of linear Fredholm integral equations, but in addition, as function of the parameter λ, the decision rules must satisfy the Cauchy system (9)–(12), which we call the *sensitivity* equations. In the next section it will be shown how the sensitivity equations may be used to analyze a team problem drawn from the theory of the firm.

4. MULTIDIVISIONAL FIRM

Suppose a firm consists of two autonomous divisions that produce different commodities in the amounts a_1 and a_2, respectively. The commodities are sold in competitive markets at prices P_1 and P_2. Because of random variations in supply and demand, the prices are not known precisely until the instant the commodities are sold. Each division separately gathers information about the market it sells in and uses this information to help select its quantity of output. Let y_i be the "price forecast" which the ith division uses in its decision making.

The firm's total revenue is $P_1a_1+P_2a_2$. Suppose that the total cost to the firm of producing quantities a_1 and a_2 is

$$c(a_1,a_2)=\tfrac{1}{2}c_{11}a_1^2+c_{12}a_1a_2+\tfrac{1}{2}c_{22}a_2^2. \tag{31}$$

Notice that since c_{12} is nonzero, there is an interdependence between the amounts of production in the two divisions.

Assume that the firm believes that the relative prices of its two commodities are fixed but is uncertain about the price level. That is, the price vector that will be observed is $(\bar{P}_1\theta,\bar{P}_2\theta)$, where θ is a random variable and $\bar{P}_1$ and $\bar{P}_2$ are fixed numbers. The expected profit of the firm is

$$E\left\{\theta\left[\bar{P}_1\alpha_1(y_1)+\bar{P}_2\alpha_2(y_2)\right]-\tfrac{1}{2}c_{11}\alpha_1(y_1)^2 - c_{12}\alpha_1(y_1)\alpha_2(y_2)-\tfrac{1}{2}c_{22}\alpha_2(y_2)^2\right\}. \tag{32}$$

Suppose that the price level, θ, and the price forecasts of the individual divisions, y_1 and y_2, are uniformly distributed across a pyramid, with the joint probability density

$$f(\theta,y_1,y_2)=\begin{cases}3, & 0\leqslant y_1\leqslant\theta\leqslant 1,\quad 0\leqslant y_2\leqslant\theta\leqslant 1,\\ 0 & \text{elsewhere.}\end{cases} \tag{33}$$

The price forecasts give lower bounds on the value of the price "level" θ. One can calculate the needed posterior probability densities from Eq. (33).

They are

$$f(\theta \mid y_i) = \begin{cases} \dfrac{2\theta}{1-y_i^2}, & y_i \leqslant \theta \leqslant 1, \\ 0 & \text{elsewhere,} \end{cases} \tag{34}$$

$$f(y_j \mid y_i) = \begin{cases} \dfrac{2(1-\text{Max}[y_1, y_2])}{1-y_i^2}, & 0 \leqslant y_j \leqslant 1, \\ 0 & \text{elsewhere.} \end{cases} \tag{35}$$

The person-by-person optimal decision rules for the two divisions, $\alpha_1^*(y_1)$ and $\alpha_2^*(y_2)$, must satisfy the following system of linear Fredholm integral equations:

$$\begin{aligned} \alpha_1^*(y_1) &= \frac{\bar{P}_1}{c_{11}} \int_{y_1}^{1} \theta \frac{2\theta}{1-y_1^2}\, d\theta \\ &\quad - \frac{c_{12}}{c_{11}} \int_0^1 \alpha_2^*(y_2) \frac{2(1-\text{Max}[y_1, y_2])}{1-y_1^2}\, dy_2 \\ &= \frac{\bar{P}_1}{c_{11}} \frac{2}{3} \frac{1-y_1^3}{1-y_1^2} \\ &\quad - \frac{c_{12}}{c_{11}} \int_0^1 \alpha_2^*(y_2) \frac{2(1-\text{Max}[y_1, y_2])}{1-y_1^2}\, dy_2, \end{aligned} \tag{36}$$

$$\begin{aligned} \alpha_2^*(y_2) &= \frac{\bar{P}_2}{c_{22}} \int_{y_2}^{1} \theta \frac{2\theta}{1-y_2^2}\, d\theta \\ &\quad - \frac{c_{12}}{c_{22}} \int_0^1 \alpha_1^*(y_1) \frac{2(1-\text{Max}[y_1, y_2])}{1-y_2^2}\, dy_1 \\ &= \frac{\bar{P}_2}{c_{22}} \frac{2}{3} \frac{1-y_2^3}{1-y_1^2} \\ &\quad - \frac{c_{12}}{c_{22}} \int_0^1 \alpha_1^*(y_1) \frac{2(1-\text{Max}[y_1, y_2])}{1-y_2^2}\, dy_1, \end{aligned} \tag{37}$$

for $0 \leqslant y_1 \leqslant 1$, $0 \leqslant y_2 \leqslant 1$. These equations represent the posterior equalization of expected marginal revenue and expected marginal cost.

The integral equations (36) and (37) may be written in the following standard form:

$$\begin{aligned} \alpha_1^*(t) &= b_1(t) + \lambda \int_0^1 k_{11}(t,s)\alpha_1^*(s)\, ds \\ &\quad + \lambda \int_0^1 k_{12}(t,s)\alpha_2^*(s)\, ds, \end{aligned} \tag{38}$$

$$\begin{aligned} \alpha_2^*(t) &= b_2(t) + \lambda \int_0^1 k_{21}(t,s)\alpha_1^*(s)\, ds \\ &\quad + \lambda \int_0^1 k_{22}(t,s)\alpha_2^*(s)\, ds, \end{aligned} \tag{39}$$

where the following definitions hold:

$$b_1(t) = \frac{\bar{P}_1}{c_{11}} \frac{2}{3} \frac{1-t^3}{1-t^2}, \tag{40}$$

$$b_2(t) = \frac{\bar{P}_2}{c_{22}} \frac{2}{3} \frac{1-t^3}{1-t^2}, \tag{41}$$

$$k_{11}(t,s) = 0 = k_{22}(t,s), \tag{42}$$

$$\lambda = c_{12}, \tag{43}$$

$$k_{12}(t,s) = \frac{-2}{c_{11}} \frac{1-\text{Max}(t,s)}{1-t^2}, \tag{44}$$

$$k_{21}(t,s) = \frac{-2}{c_{22}} \frac{1-\text{Max}(t,s)}{1-t^2}. \tag{45}$$

It should be noted that the cost coefficient c_{12} has been selected as the parameter λ. The solution of the two-division team problem will depend, among other things, on the value this *coefficient of interdependence*. The adjustment of the optimal output decision rules, $\alpha_1^*(t,\lambda)$ and $\alpha_2^*(t,\lambda)$, to changes in $\lambda = c_{12}$ is described completely by the Cauchy system of Theorem 3.1.

The Cauchy system (9)–(12) may be used to solve the team's integral equations. Make the following numerical assumptions. Let $c_{11} = 1 = c_{22}$ and $\bar{P}_1 = 1 = \bar{P}_2$. When $\lambda = c_{12} = 0$, it is clear from Eqs. (38) and (39) that the optimal decision rules are just $\alpha_i^*(t,0)=b_i(t)$; the lack of interdependence between divisions greatly simplifies the solution. When $\lambda=c_{12}$ increases, the decision rules and resolvent kernels must follow the integrodifferential equations (9)–(10). Let us compute the decision rules and resolvent kernels at points t,s on a fixed grid of the unit square. Assuming 11 points in the subdivision, a Runge-Kutta start, Adams-Moulton continuation technique for solving differential equations was used, where integrals were approximated by the trapezoid rule [5].

The value of the parameter $\lambda=c_{12}$ was taken from zero up to 1 (beyond this the payoff function is no longer concave) with a step size 0.1. The numerical results are given in Table 1 and Fig. 1.

Several regularities are apparent from this numerical exercise. First, the decision rules are monotonically increasing functions of the price forecast: the higher the forecasted price is, the more output should be produced. Second, the decision rules have a noticeable convex shape: the decisions are more sensitive to price forecasts for larger forecasts. Third, the outputs are uniformly lower as the coefficient of interdependence increases: the larger the cost interdependence, the lower are the individual division's output levels, since they must account for the other division's impact on marginal cost. Fourth, the decision rules as a function of the coefficient of interdependence have a distinct convex shape: the decision rules are less sensitive to λ as λ becomes larger. Fifth, the decision rules for the two divisions are identical; this follows from the symmetry of the numerical assumptions.

The last regularity, identical decision rules, depends on the assumption that $\bar{P}_1=1=\bar{P}_2$. It should be noted that $\bar{P}_1,\bar{P}_2$ only influence the forcing terms $b_1(t)$ and $b_2(t)$. The resolvent kernel depends only on the kernel, not the forcing function [see Eq. (8)]. Once the resolvent kernel has been computed for a particular value of $\lambda=c_{12}$, the optimal decision rules for various values of $b_1(t)$ and $b_2(t)$ may be computed by evaluating the right-hand side of Eq. (13). This permits the easy computation of the supply functions, i.e., the output rules as a function of relative prices $\bar{P}_1$ and $\bar{P}_2$. These supply functions are given in Table 2 for $\lambda=c_{12}=\frac{1}{2}$. It is clear from Eq. (13) that the supply is a linear function of relative prices; i.e.,

$$\alpha_i^*(y_i,\lambda,\bar{P}_1,\bar{P}_2)=\beta_i(y_i,\lambda)\bar{P}_1+\gamma_i(y_i,\lambda)\bar{P}_2, \tag{46}$$

where

$$\beta_1(y_1,\lambda)=b_1(y_1)+\lambda\int_0^1 K_{11}(y_1,s,\lambda)b_1(s)\,ds, \tag{47}$$

$$\gamma_1(y_1,\lambda)=\lambda\int_0^1 K_{12}(y_1,s,\lambda)b_2(s)\,ds \tag{48}$$

TABLE 1

DECISION RULES $\alpha_1^*(y_1,\lambda),\alpha_2^*(y_2,\lambda)$

λ	0.0		0.2		0.4		0.6		0.8		1.0	
y_i	α_1^*	α_2^*	α_1^*	α_2^*	α_1^*	α_2^*	α_1^*	α_2^*	α_1^*	α_2^*	α_1^*	α_2^*
0.0	0.667	0.667	0.544	0.544	0.457	0.457	0.393	0.393	0.343	0.343	0.303	0.303
0.1	0.673	0.673	0.550	0.550	0.463	0.463	0.398	0.398	0.348	0.348	0.308	0.308
0.2	0.690	0.690	0.566	0.566	0.478	0.478	0.413	0.413	0.363	0.363	0.323	0.323
0.3	0.713	0.713	0.589	0.589	0.501	0.501	0.435	0.435	0.384	0.384	0.344	0.344
0.4	0.743	0.743	0.618	0.618	0.529	0.529	0.463	0.463	0.411	0.411	0.370	0.370
0.5	0.778	0.778	0.652	0.652	0.562	0.562	0.495	0.495	0.442	0.442	0.400	0.400
0.6	0.817	0.817	0.690	0.690	0.599	0.599	0.530	0.530	0.476	0.476	0.433	0.433
0.7	0.859	0.859	0.730	0.730	0.638	0.638	0.568	0.568	0.513	0.513	0.469	0.469
0.8	0.904	0.904	0.774	0.774	0.680	0.680	0.608	0.608	0.552	0.552	0.506	0.506
0.9	0.951	0.951	0.819	0.819	0.723	0.723	0.650	0.650	0.593	0.593	0.546	0.546
1.0	1.000	1.000	0.866	0.866	0.768	0.768	0.644	0.644	0.634	0.634	0.586	0.586

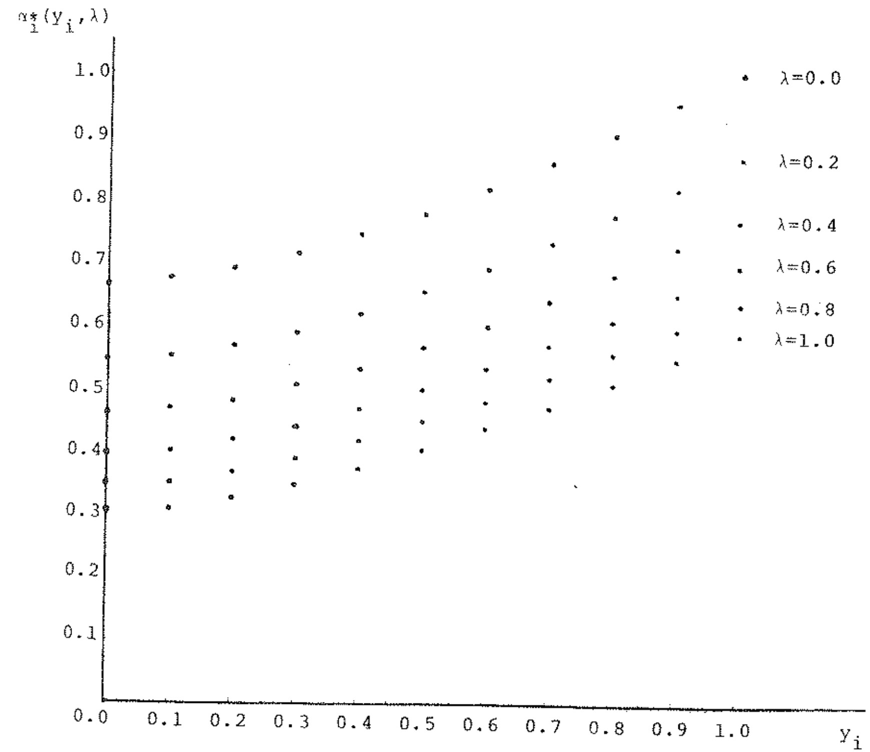

FIG. 1 Decision rules $\alpha_i^*(y_i,\lambda)$.

TABLE 2

DECISION RULES $\alpha_1^*(y_1,\lambda=\frac{1}{2},\bar{P}_1,\bar{P}_2)$, $\alpha_2^*(y_2,\lambda=\frac{1}{2},\bar{P}_1,\bar{P}_2)$

$\bar{P}_1,\bar{P}_2$	$\bar{P}_1=0.92$	$\bar{P}_2=1.0$	$\bar{P}_1=.96$	$\bar{P}_{12}=1.0$	$\bar{P}_1=1.0$	$\bar{P}_2=1.0$	$\bar{P}_1=1.04$	$\bar{P}_2=1.0$	$\bar{P}_1=1.08$	$\bar{P}_2=1.0$
y_i	α_1^*	α_2^*	α_1^*	α_2^*	α_1^*	α_2^*	α_1^*	α_2^*	α_1^*	α_2^*
0.0	0.349	0.462	0.386	0.442	0.423	0.423	0.459	0.403	0.496	0.383
0.1	0.355	0.468	0.392	0.448	0.428	0.428	0.465	0.409	0.502	0.389
0.2	0.369	0.483	0.406	0.464	0.444	0.444	0.481	0.424	0.519	0.404
0.3	0.389	0.506	0.427	0.486	0.466	0.466	0.504	0.446	0.543	0.426
0.4	0.414	0.534	0.454	0.514	0.494	0.494	0.534	0.474	0.573	0.454
0.5	0.444	0.566	0.485	0.546	0.526	0.526	0.567	0.506	0.608	0.486
0.6	0.477	0.602	0.519	0.582	0.562	0.562	0.605	0.542	0.647	0.522
0.7	0.512	0.641	0.556	0.621	0.601	0.601	0.645	0.580	0.689	0.560
0.8	0.549	0.683	0.595	0.662	0.642	0.642	0.688	0.621	0.734	0.600
0.9	0.588	0.726	0.636	0.705	0.685	0.685	0.733	0.664	0.781	0.643
1.0	0.629	0.771	0.679	0.750	0.729	0.729	0.779	0.708	0.829	0.687

TABLE 3

COEFFICIENTS $\beta_i(y_i,\lambda=\frac{1}{2})$, $\gamma_i(y_i,\lambda=\frac{1}{2})$

y	$\beta_1(y,\frac{1}{2})=\gamma_2(y,\frac{1}{2})$	$\gamma_1(y,\frac{1}{2})=\beta_2(y,\frac{1}{2})$
0.0	0.916	−0.493
0.1	0.922	−0.494
0.2	0.938	−0.495
0.3	0.962	−0.496
0.4	0.993	−0.499
0.5	1.028	−0.502
0.6	1.067	−0.505
0.7	1.110	−0.509
0.8	1.155	−0.513
0.9	1.202	−0.518
1.0	1.252	−0.523

$$\beta_2(y_2,\lambda)=\lambda\int_0^1 K_{21}(y_2,s,\lambda)b_1(s)\,ds \tag{49}$$

$$\gamma_2(y_2,\lambda)=b_2(y_2)+\lambda\int_0^1 K_{22}(y_2,s,\lambda)b_2(s)\,ds \tag{50}$$

For $\lambda=c_{12}=\frac{1}{2}$, the values of these coefficient functions are given in Table 3.

5. CONCLUSION

The objective of this paper has been to show that there is an equivalent way of characterizing the optimal decision rules of team decision theory. By techniques of parametric imbedding it has been shown that the optimal decision rules must satisfy a system of initial-valued integrodifferential equations, which are referred to as the sensitivity equations. These sensitivity equations describe the adjustment of the decision rule to changing values of the degree of interdependence within the team. In addition, they permit efficient computation of optimal decision rules. The analysis was restricted to linear-quadratic teams but allowed for general probability distributions. Previous work on team decision theory has been almost exclusively devoted to the linear-quadratic and Gaussian-normal case. The quadratic assumption may be dropped at some loss of simplicity [6].

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A General Population Growth Model with Density Dependence

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ABSTRACT

A model for the growth of a population with $p+q+r$ age groups in which there is competition for limited resources is considered. The steady-state solution is obtained and its stability is discussed. The existence of a time-invariant structure in which the ratios of the populations of the various age groups do not change with time is established under very general conditions, and its relation with the steady-state solution is discussed. The conditions under which we can treat the population as homogeneous with a common birth rate, a common death rate and a common inhibiting constant are also discussed.

1. THE MODEL

We first divide the population into three groups of prereproductive children, of reproductive adults and of postreproductive old persons. We further subdivide the three groups into p,q,r age subgroups respectively. Let $x_{uv}(t)$ be the population of the vth subgroup of the uth group, and let the birth and death rates for this subgroup be b_{uv}, d_{uv} respectively. Let the migration rate from this subgroup to the next be m_{uv}. Also let the decrease in rate of growth of the population of this subgroup due to competition for limited resources be $K_{uv}x_{uv}(x_{11}+\cdots+x_{1p}+x_{21}+\cdots+x_{2q}+x_{31}+\cdots+x_{3r})$. We then get the following system of differential equations:

$$\frac{dx_{11}}{dt}=b_{21}x_{21}+\cdots+b_{2q}x_{2q}-(d_{11}+m_{11})x_{11}$$
$$-K_{11}x_{11}(x_{11}+x_{12}+\cdots+x_{3r}),$$

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