THE USE OF COLLATERAL TO ENFORCE DEBT: PROFIT MAXIMIZATION

JAMES HESS*

Collateral lowers the probability of default. This paper modifies the work of Benjamin (1978) by characterizing the competitive lender as a price-taking profit maximizer rather than a profit eliminator. Most of Benjamin's results disappear when profit maximization is assumed; the results are corrected and extended. It is also suggested that the value of the collateral required for a standard loan is a variable that the loan market will adjust until long-run expected profits from secured loans is driven to zero.

In an article in this journal Benjamin (1978) developed a model of a competitive creditor who requires collateral on loans as a means of minimizing the cost of default. Unfortunately the lender does not maximize profit, so the lender's behavior will not correspond to Benjamin's supply curve. In the following a complete model of the profit maximizing lender in a competitive loan market will be developed and contrasted to Benjamin's.

Loans are arranged in a *competitive* market characterized by price-taking and an absence of barriers to entry. For each identical loan customer the lender must determine how much credit to extend, depending on the loan interest rates, the cost of funds and the value of the collateral. The one period loan in dollars, L, and the interest factor, $R^* = 1 + r^*$, combine to give the debt to be repaid at the end of the period, LR^*. The loan is secured by collateral of net market value P_n, and when debt payment LR^* exceeds the value of the collateral, some debtors may default on the loans, forfeiting the collateral. Let $\pi(LR^*/P_n)$ be the probability that debts will be paid, a monotonic decreasing function of LR^*/P_n. If the interest cost of funds is denoted $R = 1 + r$ and the cost of processing a loan is denoted K, the expected profit per customer equals

$$C = LR^*\pi + P_n(1 - \pi) - LR - K, \tag{1}$$

where the dependence of π on LR^*/P_n is suppressed for notational simplicity.

At this juncture Benjamin defined the supply curve of the creditor as the locus of points (L, R^*) such that expected profits are zero. This is premature. The traditional microeconomic model of behavior assumes that the creditor chooses L to maximize profit per customer given R, R^* and P_n. Expected profit maximization will be characterized here by the first order condition

*North Carolina State University.

(2) $$\frac{\partial C}{\partial L} = R^*\pi + R^*\pi' (LR^*/P_n - 1) - R = 0$$

Second order conditions will be satisfied if

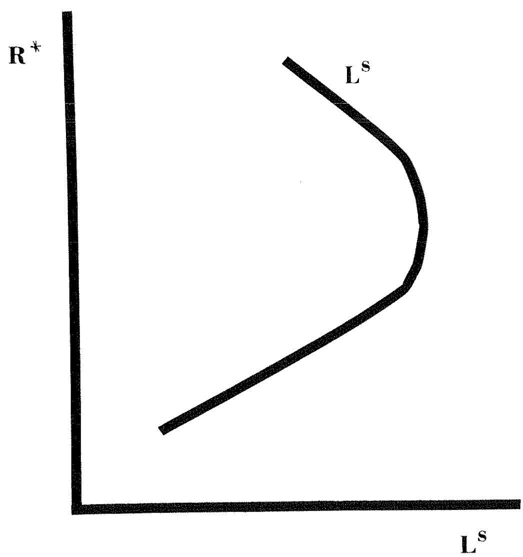
(3) $$\frac{\partial^2 C}{\partial L^2} = 2R^{*2}\pi'/P_n + R^{*2}(LR^*/P_n - 1)\pi''/P_n < 0 .$$

In addition "shutdown" will occur and the profit maximizing loan is zero when C is negative. The profit maximizing loan, denoted

(4) $$L^s(R^*, R, P_n) ,$$

is illustrated in figure 1.

FIGURE 1



The definition of competitive equilibrium requires that the demand for loans and the supply of funds be discussed. The loan customer's behavior is suggested but not developed completely by Benjamin. Consider the following two period consumption-loan model of the borrower.

The borrower has income I_1 and I_2 in periods 1 and 2. A loan L implies that consumption in period 1 equals

(5) $$X_1 = I_1 + L .$$

Some of period 2's income has been committed to supplying the collateral good, so only $I_2 - P_n$ is free and clear. In period 2 the borrower identifies the best price that the collateral will fetch. It is always at least as much as P_n, the open market price. For example, P_n might be the price of a used car when sold to a used car dealer while the best price might be found by selling through the classified ads. Let a be a random variable taking on values greater than or equal to 1 which measures the proportional increase over the open market price that the borrower can achieve. That is, the borrower will be able to sell the collateral at a best price aP_n.

The debtor will default on the loan if the the cost of repaying the debt exceeds the best price of the collateral, so consumption in period 2 is

$$X_2 = \begin{cases} I_2 - P_n + aP_n - LR^* & \text{if } aP_n \geq LR^* \\ I_2 - P_n & \text{if } aP_n \geq LR^* \end{cases} \tag{6}$$

Prior to period 2 the best price is unknown and a has a probability density function $f(a)$, implying the probability of default equals the cumulative distribution evaluated at LR^*/P_n,

$$1 - \pi = \int_1^{LR^*/Pn} f(a)\, da = F(LR^*/P_n)\ . \tag{7}$$

The consumer chooses the amount of money to borrow to maximize the two period expected utility

$$\begin{aligned} EU = E\{U(X1, X2)\} &= \int_1^{LR^*/Pn} U(I_1 + L, I_2 - P_n) f(a)\, da \\ &+ \int_{LR^*/Pn}^{\infty} U(I_1 + L, I_2 - P_n + aP_n - LR^*) f(a)\, da\ . \end{aligned} \tag{8}$$

The first order condition for the optimal loan request is

$$\begin{aligned} \frac{\partial EU}{\partial L} = 0 = &\int_1^{LR^*/Pn} U_1(I_1 + L, I_2 - P_n) f(a)\, da \\ &+ \int_{LR^*/Pn}^{\infty} \{U_1 (I_1 + L, I_2 - P_n + aP_n - LR^*) \\ &- R^* U_2 (I_1 + L, I_2 - P_n + aP_n - LR^*)\} f(a)\, da. \end{aligned} \tag{9}$$

Denote the optimal loan demand as it depends on the loan interest factor and the open market value of the collateral by

$$L^d (R^*, P_n)\ . \tag{10}$$

The total supply of funds to the financial intermediaries may depend on the interest factor, R, although it will continue to be assumed that the lender's are price takers. Denote the total supply of funds by

$$S(R)\ . \tag{11}$$

Short run equilibrium requires that the loan supply per customer equal the loan demand per customer and that the total demand for funds equal the total supply of funds. If there are M borrowers the equilibrium conditions are

$$L^s(R^*, R, P_n) = L^d(R^*, P_n)\,, \tag{12}$$

$$S(R) = M \cdot L^s(R^*, R, P_n)\,. \tag{13}$$

These are not long-run equilibrium conditions because creditors may be earning positive economic profit. The added condition that maximum expected profit per customer is zero,

$$C(R^*, R, P_n) = 0\,, \tag{14}$$

would overdetermine the model if the value of collateral P_n is not considered a variable along with R^* and R. Possibly think of R^* as adjusting to clear the loan market, R adjusting to clear the funds market and P_n adjusting to eliminate excess profits.

Benjamin carefully analyzes the characteristics of the locus of loans and interest rates on loans which produce zero expected profit. This locus is not the supply curve of the expected profit maximizing creditor since creditors will not voluntarily choose to earn zero profit. As a result many of the claims made about the "supply curve" are incorrect. The actual supply behavior of the profit maximizing creditor will be studied below focussing on how the loan supply varies as the interest rate on loans, interest rate on funds, value of collateral and probability of default change.

The slope of the traditional supply curve per customer is found by applying the implicit function rule to the first order condition:

$$\begin{aligned} \frac{\partial L^s}{\partial R^*} &= -\frac{\partial^2 C}{\partial L \partial R^*} \Big/ \frac{\partial^2 C}{\partial L^2} \\ &= -\{\pi + \pi'(3LR^*/P_n - 1) + \pi'' LR^* \\ &\quad (LR^*/P_n - 1)/P_n\}/(\partial^2 C/\partial L^2)\,. \end{aligned} \tag{15}$$

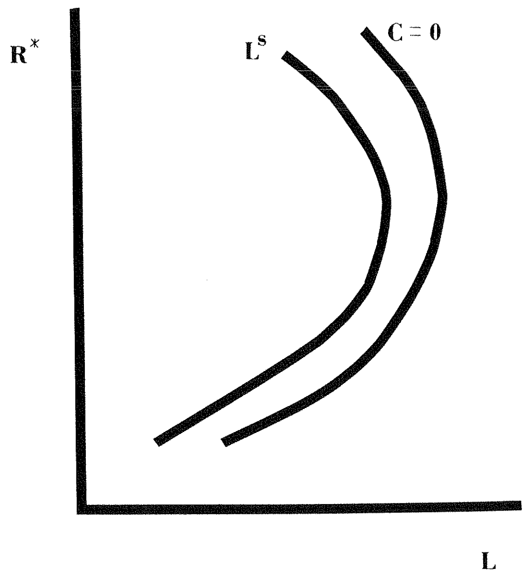
The term in curly brackets may be of either sign, implying the supply curve may have a backward bending section as seen in figure 2. The backward bending section is considered by Benjamin but rejected. His reasoning, based on his assumption that the supply curve is defined by zero profit, is that if a loan L would be supplied at two different interest rates the customers would prefer the lower rate and the lender would not care (since profits are zero in any case). This argument does not hold for the traditionally defined supply curve because a lower interest rate on loans will lead to lower expected profits. Applying the envelope theorem to the lender's problem gives

$$
\begin{aligned}
\partial C/\partial R^* &= L\pi + L^2R^*\pi'/P_n - L\pi' \\
&= (L/R^*)\,(R^*\pi + LR^{*2}\pi'/P_n - R^*\pi') \\
&= (L/R^*)\,R > 0\,,
\end{aligned} \tag{16}
$$

where the last equality follows from the first order condition (2). The lender's backward bending supply curve is thus economically relevant in describing individual behavior, at least in the short run. The response of loan supply to an increase in the cost of funds, R, can be established easily:

$$
\frac{\partial L^s}{\partial R} = \frac{-\dfrac{\partial^2 C}{\partial L \partial R}}{\dfrac{\partial^2 C}{\partial L^2}} = \frac{1}{\dfrac{\partial^2 C}{\partial L^2}} < 0\,. \tag{17}
$$

FIGURE 2



The inequality follows from the second order condition. Unambiguously an increase in the cost of funds will cause the creditor to offer a smaller loan given R^* and P_n.

To find the impact of P_n on L^s, notice that the first order condition depends on P_n through term L/P_n. From the implicit function rule it is found that

$$(18)\qquad \frac{\partial L^s}{\partial P_n} = -\frac{\dfrac{\partial^2 C}{\partial L\,\partial (L/P_n)}\left(\dfrac{-L}{P_n^2}\right)}{\dfrac{\partial^2 C}{\partial L\,\partial (L/P_n)}\left(\dfrac{1}{P_n}\right)} = \frac{L}{P_n} > 0 .$$

When the open market collateral increases in value, the creditor makes larger loan offers at any given interest rate.

Benjamin states that "any change in the density function $f(a)$ that implies an increase in probability of default (a reduction in π) will cause the loan supply curve to shift up." When the supply curve of the lender is defined by the profit maximizing behavior, this unconditional statement no longer holds. Rearrange the first order condition (2) so that it becomes

$$(19)\qquad L^s = P_n R/(R^{*2}\pi') + P_n(1 - \pi/\pi')/R^* .$$

While both π and π' are functions, not parameters, it can still be shown that the response of L^s to a change is signed as follows.

$$(20)\qquad \partial L^s/\partial \pi = -P_n/(R^*\pi') > 0 ,$$

$$(21)\qquad \partial L^s/\partial \pi' = -P_n(LR^*/P_n - 1)/(R^*\pi') > 0 .$$

There may be changes in the density function $f(a)$ which increase the probability of repayment, π, but which simultaneously decrease π' enough so that L^s diminishes. A specific example is found in the Appendix.

In conclusion, Benjamin chooses to study the shape of the average revenue product of loans but the appropriate curve is the *marginal* revenue product of loans. These are two entirely different curves, many of the properties claimed by Benjamin do not carry over to the traditional supply curve. Moreover, Benjamin does not consider the value of collateral to be an endogenous variable and it would seem to be an important extension of his model.

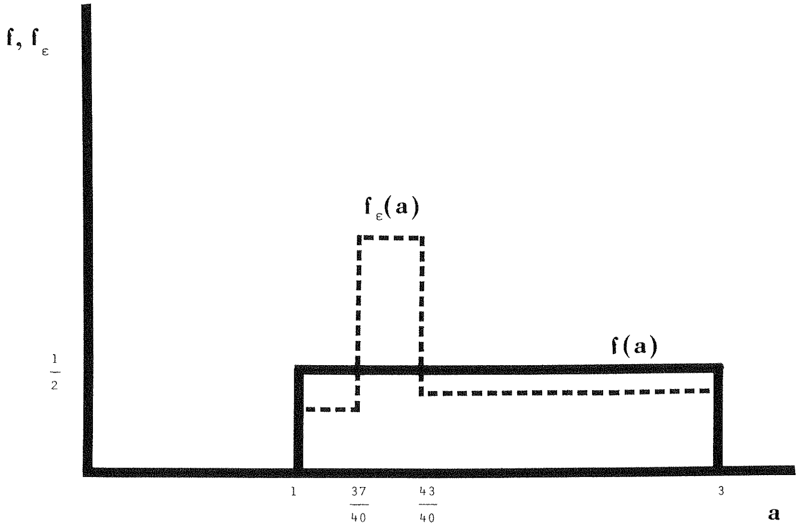
APPENDIX

Suppose the interest rates are $R^* = 5/3$ and $R = 10/9$ and the value of the collateral is $P_n = 6/5$. The probability density $f(a)$ will begin as a uniform density on the interval from 1 to 3 and be modified slightly by adding a constant times another function to it to produce the following density:

$$(22)\qquad f\epsilon(a) = \begin{cases} 1/2 - 7\epsilon & \text{for } 1 \le a \le 37/30 , \\ 1/2 + 16\epsilon & \text{for } 37/30 \le a \le 43/30 , \\ 1/2 - \epsilon & \text{for } 43/30 \le a \le 3 , \\ 0 & \text{elsewhere.} \end{cases}$$

See figure 3.

FIGURE 3



The probability of repayment is found by integrating the density $f\epsilon(a)$ as indicated in Equation (7).

$$(23)\qquad \pi(a) = \begin{cases} 1 - (1/2 - 7\epsilon)(a-1) & \text{for } 1 \le a \le 37/30\,, \\ 3/2 + 641\epsilon/30 - (1/2 + 16\epsilon)\,a & \text{for } 37/30 \le a \le 43/30\,, \\ 3/2 - 3\epsilon - (1/2 - \epsilon)\,a & \text{for } 43/30 \le a \le 3\,. \end{cases}$$

The first order condition for the optimal loan is found to be the following for ϵ very small:

$$(24)\qquad 0 = \frac{\partial C}{\partial L} = \frac{5}{3}\left(\frac{3}{2} + \frac{641}{30}\epsilon - \left(\frac{1}{2} + 16\epsilon\right)\frac{5}{3}\frac{5}{6}L\right) - \frac{5}{6}\left(\frac{5}{3}\right)^2\left(\frac{1}{2} + 16\epsilon\right)L + \frac{5}{3}\left(\frac{1}{2} + 16\epsilon\right) - \frac{10}{9}\,.$$

Solving for L when $\epsilon = 0.0$ and $\epsilon = 0.01$ gives

$$(25)\qquad L = 0.96 \text{ for } \epsilon = 0.0\,,$$

$$(26)\qquad L = 0.93 \text{ for } \epsilon = 0.01\,.$$

Notice that when ϵ increases, π increases in the vicinity of $LR^*/P_n = 4/3$ and at the same time π' decreases.

$$(27)\qquad \left.\frac{\partial \pi}{\partial \epsilon}\right|_{4/3} = 1/30\,,$$

$$(28)\qquad \left.\frac{\partial \pi'}{\partial \epsilon}\right|_{4/3} = -16$$

As a result the lender finds that the optimal loan size decreases when the probability of payment increases.

REFERENCE

Benjamin, Daniel K., "The Use of Collateral to Enforce Debt Contracts," *Economic Inquiry*, July 1978, *16*, 333–359.