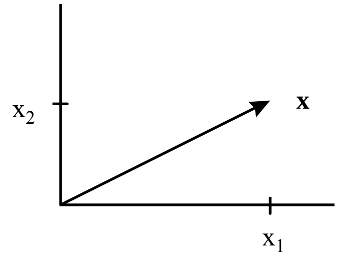
Matrix Manipulation of Data: Matrix Algebra

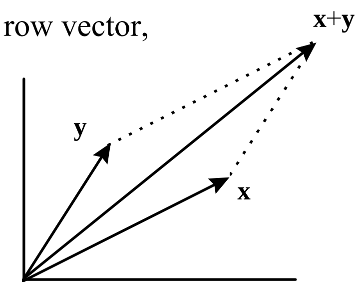
## 1. Vectors

Vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ .. \\ x_n \end{bmatrix}$ column vector

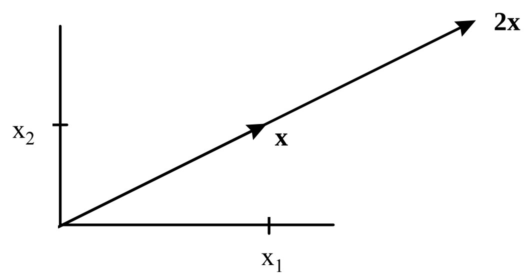



Transpose $\mathbf{x}'=[x_1\ x_2 \ldots x_n]$ row vector,

Addition $\mathbf{x}+\mathbf{y}=\begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ .. \\ x_n+y_n \end{bmatrix}$



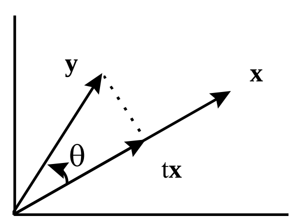

Scalar multiplication $k\mathbf{x} = \begin{bmatrix} kx_1 \\ kx_2 \\ .. \\ kx_n \end{bmatrix}$,




Definition: $\mathbf{x}$ and $\mathbf{y}$ point in the same direction if for some $k\neq 0$, $\mathbf{y}=k\mathbf{x}$.

Vectors have a direction and length. Euclidian Length $\|x\| = \sqrt{\sum x_i^2}$

Inner product $\mathbf{x}'\mathbf{y}=\Sigma x_i y_i$
Note: we think of this as a row times a column
Suppose $t\mathbf{x}$ is a projection of $\mathbf{y}$ into $\mathbf{x}$:
$\mathbf{y}'\mathbf{y}=t^2\mathbf{x}'\mathbf{x} +(\mathbf{y}-t\mathbf{x})'(\mathbf{y}-t\mathbf{x})= t^2\mathbf{x}'\mathbf{x} +\mathbf{y}'\mathbf{y}-2t\mathbf{x}'\mathbf{y}+ t^2\mathbf{x}'\mathbf{x} \Rightarrow t=\mathbf{x}'\mathbf{y}/\mathbf{x}'\mathbf{x}$.
Note: $\cos\theta \equiv t\|\mathbf{x}\|/\|\mathbf{y}\|$, so




$$\mathbf{x}'\mathbf{y} = \|\mathbf{x}\|\ \|\mathbf{y}\| \cos(\theta).$$

The inner product is the product of lengths correcting for not pointing in the same direction. If two vectors $\mathbf{x}$ and $\mathbf{y}$ point in the precisely the same direction, then we say that there are **dependent**. More generally, they are dependent if there exists $k_1, k_2\neq 0$ such that $k_1\mathbf{x}+k_2\mathbf{y}=\mathbf{0}$. That is, a linear combination of $\mathbf{x}$ and $\mathbf{y}$ with weights $\mathbf{k}=(k_1,k_2)'$ equals the zero vector

If $\mathbf{x}$ and $\mathbf{y}$ are not dependent, then they are **independent**: they point in different directions. Maximal independence is **orthogonal** (perpendicular): $\theta=90^o \Rightarrow \cos\theta = 0 \Rightarrow \mathbf{x}'\mathbf{y}=0$.
**Orthonormal** vectors are both orthogonal of unit length: $\mathbf{x}'\mathbf{y}=0$ and $\|\mathbf{x}\|=1=\|\mathbf{y}\|$. They lie on the unit circle.

## 2. Matrices

n×p matrix: $$X = \begin{array}{cc} & \begin{array}{ccc} 1 & j & p \end{array} \\ \begin{array}{c} 1 \\ 2 \\ i \\ n \end{array} & \begin{bmatrix} x_{11} & .. & x_{1p} \\ x_{21} & .. & x_{2p} \\ .. & x_{ij} & .. \\ x_{n1} & .. & x_{np} \end{bmatrix} \end{array}$$ n and p are the dimensions of the matrix. In a data matrix, n is the number of observations and p is the number of variables. $X^{(j)}$ denotes the jth column which holds all the observations of the jth variable, while $X_i$ denotes the ith row which holds the ith observation of all the variables.

<u>Matrix algebra</u>

Equality $X=Y \Leftrightarrow x_{ij}=y_{ij}$ if dimensions conform
Matrix addition $X+Y=Z$ $z_{ij}=x_{ij}+y_{ij}$ if dimensions conform
Scalar multiplication $kX=[kx_{ij}]$
Transpose X' denotes a p×n matrix $=[x_{ji}]$ where rows and columns are interchanged
Symmetric square matrix: X=X'.

Multiplying a matrix X by a vector **b**, X**b**, can be thought of in two different ways depending on whether we focus on the rows or columns of X. Focusing on rows, think of X as rows stacked one upon another. Then the product X**b** is a sequence of inner products $X_i\mathbf{b}=\Sigma x_{ij}b_j$ stacked one upon another. Alternatively, if we think of X as a collections of column vectors, then X**b** is a linear combination of the vectors with weights given in **b**: $\Sigma X^{(j)}b_j$.

Rank(X)=maximum number of independent rows in matrix: rank(X)≤n. If the rank is below n, then some observations are essentially duplicates of others.
Matrix Multiplication: Z=XY is defined if X n×p and Y p×m then Z is n×m where $z_{ij}=X_iY^{(j)}$ = inner product of ith row of X and jth column Y.

**Theorem**: A+B=B+A, A(B+C)=AB+AC, (A')'=A, (A+B)'=A'+B', (AB)'=B'A'
Note: AB=AC $\not\Rightarrow$ B=C
Note: AB≠BA except in special cases.

<u>Special matrices</u>

A is n×n

Square Matrix

$$I_n = I = \begin{bmatrix} 1 & 0 & .. & 0 \\ 0 & 1 & .. & 0 \\ . & . & . & . \\ 0 & 0 & .. & 1 \end{bmatrix}$$

Identity Matrix
IX=X

$$O = \begin{bmatrix} 0 & .. & 0 \\ 0 & .. & 0 \\ 0 & .. & 0 \end{bmatrix}$$

Zero Matrix
OX=O

$$\mathbf{u}_i = \begin{bmatrix} 0 \\ ... \\ 1 \\ 0 \\ ... \\ 0 \end{bmatrix} \begin{array}{l} \\ \\ \leftarrow i^{th} \text{ row} \\ \\ \\ \\ \end{array}$$

$i^{th}$ Unit Vector

$$\text{Diag}(A) = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ ... & ... & ... & ... \\ 0 & 0 & 0 & a_{nn} \end{bmatrix}$$

Diagonal Matrix

$$\mathbf{1} = \begin{bmatrix} 1 \\ 1 \\ .. \\ 1 \end{bmatrix}$$

Unit Vector

$$J = \mathbf{11'} = \begin{bmatrix} 1 & .. & 1 \\ 1 & .. & 1 \\ 1 & .. & 1 \end{bmatrix}$$

Unit Matrix

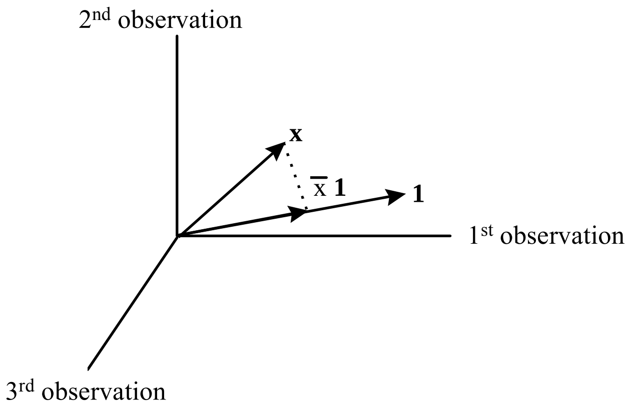
$$H = I - J/n = \begin{bmatrix} \frac{n-1}{n} & \frac{-1}{n} & \cdots & \frac{-1}{n} \\ \frac{-1}{n} & \frac{n-1}{n} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \frac{-1}{n} & \frac{-1}{n} & \cdots & \frac{n-1}{n} \end{bmatrix}$$

Centering Matrix

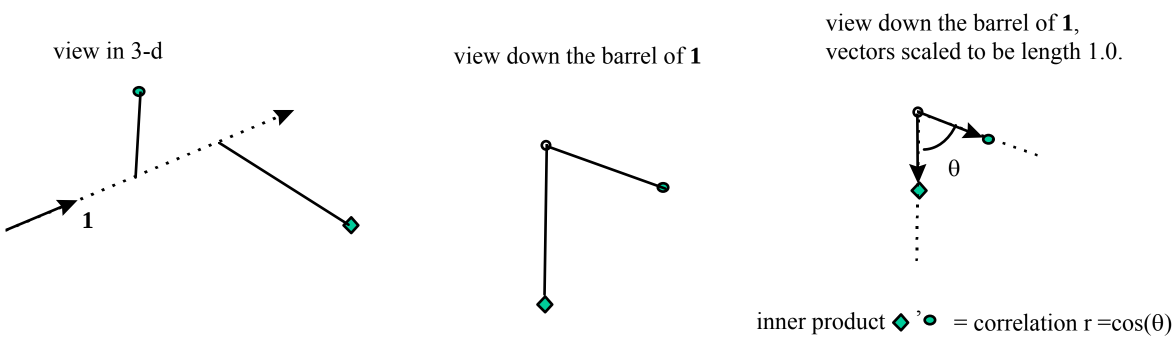
**Theorem**: The centering matrix H is idempotent, HH=H.

**3. Descriptive Statistics Using Matrix Notation**

Mean vector: $\bar{\mathbf{x}} = X'\mathbf{1}/n$ is a p-vector of means. Note that the interpretation of inner product as a projection implies that the mean is a projection into the unit vector in an n-dimensional space. See below for 3 observations of a single variable (this is not a scatterplot).




The mean p-vector $\bar{\mathbf{x}}$ tells us where the center of the scatterplot is located. We are also interested in how the observations differ from the mean. Data as deviations from mean $X_d=HX=X-\mathbf{1}\bar{\mathbf{x}}'$.



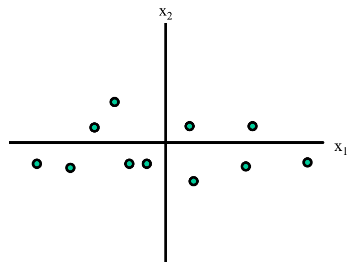

Covariance matrix $S_n=X_d'X_d/n=(HX)'HX/n=X'HX/n$ is a p×p matrix with variance along diagonal and covariances off-diagonal. If we calculate covariance matrix dividing by n-1 rather than n, we write it as S. M=X'X is the matrix of sums of squares and products.

Standard deviation matrix $D^{1/2}=\begin{bmatrix} \sqrt{s_{11}} & 0 & 0 & 0 \\ 0 & \sqrt{s_{22}} & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \sqrt{s_{pp}} \end{bmatrix}$=sqrt(diag(S)) SPSS, N.B. text uses D

Correlation matrix $R = D^{-1/2}SD^{-1/2} = \begin{bmatrix} 1 & \frac{s_{12}}{\sqrt{s_{11}}\sqrt{s_{22}}} & \cdots & \frac{s_{1p}}{\sqrt{s_{11}}\sqrt{s_{pp}}} \\ \frac{s_{21}}{\sqrt{s_{22}}\sqrt{s_{11}}} & 1 & \cdots & \frac{s_{2p}}{\sqrt{s_{22}}\sqrt{s_{pp}}} \\ \vdots & \ldots & \ddots & \vdots \\ \frac{s_{p1}}{\sqrt{s_{pp}}\sqrt{s_{11}}} & \cdots & \cdots & 1 \end{bmatrix}$

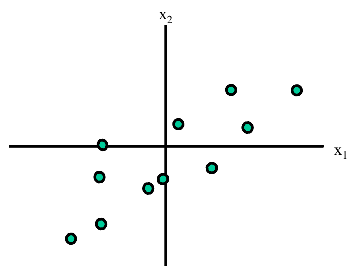
A linear combination of variables is defined by a vector **a**: X**a**. Sample mean of linear combination is $\bar{x}'\mathbf{a}$. Sample variance of a linear combination is **a**'S**a**.

In a p-dimensional scatter plot, we have the n observations of the p variables.




For each observation (row $X_i$ of data matrix X), we could calculate the Euclidian distance $\sqrt{X_i X_i'} = \sqrt{\sum_{j=1}^{p} x_{ij}^2}$. However, it is possible that one variable fluctuates much more than another and yet all variables are treated equally in the summation. To adjust for this, we often take a <u>scaling</u> transformation: $\mathbf{z}_i = D^{-1/2}(\mathbf{x}_i - \bar{\mathbf{x}})$; this subtracts the mean and divides by the standard deviation. The resulting distance does not put extra emphasis on the variables with greater variance.

The variables can be correlated with one another so that the scatter diagram is tilted.




To account for this, we take the <u>Mahalanobis</u> transformation $\mathbf{zm}_i = S^{-1/2}(\mathbf{x}_i - \bar{\mathbf{x}})$, where $S^{1/2}$ is the square root matrix which has the property it is symmetric and $S^{1/2} S^{1/2} = S$ (more on this later). The square root matrix is <u>not</u> a diagonal matrix like the standard deviation matrix, and takes into account the covariance between variables. The statistical length of an observation is the length of the Mahalanobis vector.

The p×p covariance matrix S summarizes all that we know about the shape of the scatter of points around the mean. When there are many variables (p is large), S has so many numbers in it that it too needs to be summarized. If we think of our scatterplot as a football  in a p-dimensional space, we would like to know how long the football compared to how round and in which direction it points. This leads to eigenvalues and eigenvectors of S. See below.

## 4. Data Decompositions

| Y | $X_1$ | $X_2$ |
|---|---|---|
| 6 | 1 | 2 |
| 4 | 2 | 5 |
| 2 | 6 | -3 |

Can we express Y in terms of $X_1$ and $X_2$?

Equations with "row focus": $$\begin{aligned} 1\beta_1 + 2\beta_2 &= 6 \\ 2\beta_1 + 5\beta_2 &= 4 \\ 6\beta_1 - 3\beta_2 &= 2 \end{aligned}$$

Equations with "column focus": $$\begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}\beta_1 + \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix}\beta_2 = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$$

Numbers can be decomposed into products in multiple ways: a positive number can be decomposed into the product of its square roots: $X = \sqrt{X}\,\sqrt{X}$, but it can also be decomposed into the product of a number and one plus that number: $Y(Y+1)=X \Rightarrow Y^2+2\,\tfrac{1}{2}\,Y + \tfrac{1}{2}^{2} - \tfrac{1}{2}^{2}X$ $\Rightarrow (Y+\tfrac{1}{2})^2 = X + \tfrac{1}{4} \Rightarrow Y=\sqrt{X+1/4} - \tfrac{1}{2}$. For example, $0.1 = .316\times.316 = .09\times1.09$. Similarly, a matrix can be decomposed into products of matrices.

<u>Elimination</u>: subtract a multiple of row i from row k with the objective of putting 0's everywhere below a pivot element.

By the process of elimination, we can decompose a data matrix X into the product of a lower triangular matrix and an upper triangular matrix.

We will work on the data matrix $X = \begin{bmatrix} 1 & 2 \\ 2 & 5 \\ 6 & -3 \end{bmatrix}$ using elimination. First, subtract 2 times the first row from the second. Record both a matrix that represents what we have done and the consequences of doing it: $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 6 & -3 \end{bmatrix}$.

The second step of elimination is to change 6 to 0: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 6 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & -15 \end{bmatrix}$.

At this point we have zeroes below the pivot in row 1, so we switch to row 2 and begin with the pivot 1.

The third step of elimination is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -15 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. At this point the adjusted data matrix is an upper triangular matrix - elements on or above the diagonal are non-zero while all the elements below the diagonal are zero.

If we string together the matrices representing the elimination steps we get

$$\begin{bmatrix}1&0&0\\2&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\6&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&-15&1\end{bmatrix}=\begin{bmatrix}1&0&0\\2&1&0\\6&-15&1\end{bmatrix}.$$

The product of $L=\begin{bmatrix}1&0&0\\2&1&0\\6&-15&1\end{bmatrix}$ and $U=\begin{bmatrix}1&2\\0&1\\0&0\end{bmatrix}$ is $X=\begin{bmatrix}1&2\\2&5\\6&-3\end{bmatrix}$. That is, X=LU where L is a lower triangular matrix and U is an upper triangular matrix.

Theorem: Every matrix X can be factored into the product of lower and upper triangular matrices: X=LU.

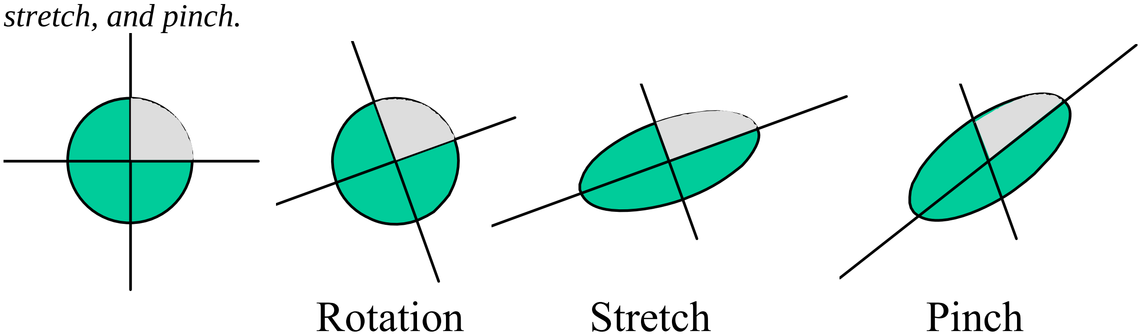
This is most basic of all decompositions, but there are others that are more important in statistics:
Spectral Decomposition: If S is a symmetric matrix, then it can be decomposed into PΛP', where Λ is a diagonal matrix and P is an orthogonal matrix. If S is positive definite, then Λ is positive.
Singular Value Decomposition: every X n×p matrix can be decomposed into QDV' where Q is a n×n orthogonal matrix, D is a n×p positive diagonal matrix, and V is a p×p orthogonal matrix.
Cholesky Decomposition: If S is a symmetric, positive define matrix, then it can be decomposed into LL' where L is a lower triangular matrix with positive elements along the diagonal.

## 5. Algebra of Square Matrices

*From this point forward assume that A is square. You could think of A as a covariance matrix.*

Trace $\mathrm{tr}(A)=\sum_i a_{ii}$ sum along diagonal of square matrix
**Theorem**: tr(A+B)=tr(A)+tr(B), tr(kA)=ktr(A), tr(XY)=tr(YX) if X is n×p and Y is p×n, tr(x'y)=tr(yx').

*A causes a transformation of a vector **x** into **y**: A**x**=**y**. This can be thought of as a rotation, stretch, and pinch.*




Is there are reverse rotation, stretch and pinch that returns you to where you began?
Inverse: B is the inverse matrix of a square matrix A if and only if AB=I=BA. Notationally, we write the inverse $A^{-1}$.

Example: $A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$, then $A^{-1}=\begin{bmatrix}d&-b\\-c&a\end{bmatrix}/(ad-bc)$.

**Theorem**: If $A^{-1}$ exists, then it is unique.
**Theorem**: If A,B are square, then $(AB)^{-1}=B^{-1}A^{-1}$.

Quadratic form in **x**: $\mathbf{x}'A\mathbf{x}$ where A is a square symmetric matrix.
A is positive definite if all non-trivial quadratic forms are positive: $\mathbf{x}'A\mathbf{x}>0$ for all $x\neq 0$.
Example 2×2: Let $A=\begin{bmatrix} a & b \\ b & d \end{bmatrix}$, then $x'Ax=ax_1^2+2bx_1x_2+cx_2^2$. Complete the square to write this
$\mathbf{x}'A\mathbf{x}=a(x_1^2+2b/ax_1x_2)+dx_2^2= a(x_1^2+2b/ax_1x_2+(b/ax_2)^2)-b^2/ax_2^2+dx_2^2=a(x_1+b/ax_2)^2+(ad-b^2)/a\ x_2^2$.
Hence if A is positive definite $a>0$ and $ad-b^2>0$. By completing the square with $x_2$, it is also true that $d>0$.

**Theorem:** $\mathbf{x}'S\mathbf{x}=\mathbf{x}'(HX)'(HX)\mathbf{x}/(n-1)=\mathbf{y}'\mathbf{y}/(n-1)\geq 0$ where $y=HX\mathbf{x}$. Hence the covariance matrix is positive-semidefinite.
**Theorem**: $\mathrm{tr}\left(\sum_i \mathbf{x}_i'A\mathbf{x}_i\right)=\mathrm{tr}\left(A\sum_i \mathbf{x}_i\mathbf{x}_i'\right)$.

Determinants $|x_{11}|=x_{11}$, $\begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix}=x_{11}x_{22}-x_{12}x_{21}$

Minor $|M_{ij}|$ = determinant of X with ith row and jth column deleted.
Cofactor $|C_{ij}|=(-1)^{i+j}|M_{ij}|$
Determinant via Laplace expansion along ith row: $|X|=x_{i1}|C_{i1}|+x_{i2}|C_{i2}|+\ldots+x_{im}|C_{im}|$
**Theorem**: $|AB|=|A|\,|B|$ if A and B are square.

A square p×p matrix X is nonsingular if its rows are independent vectors: $a'X=0$ implies that $a=0$, or equivalently that $\mathrm{rank}(X)=p$.

Three Elementary Row Transformations:
1) $T_{ij}$ Interchange rows i and j, e.g. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}X$
2) $T_i(k)$ Multiply row i by scalar k, e.g. $\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}X$
3) $T_{ij}(k)$ Add row j times k to row i e.g. $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}X$
X is equivalent to Y, denoted X~Y, if X can be transformed into Y by elementary row transformations. Elementary row transformations do not change the rank of a matrix. By elementary row transformations X can be transformed into one of the following four matrices:
$I_m$, $[I_m\ 0]$, $\begin{bmatrix} I_m \\ 0 \end{bmatrix}$, or $\begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix}$. If $X\sim\begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix}$ then the $\mathrm{rank}(X)=m$.
**Theorem**: $T_{ij}$ changes sign of determinant; $T_i(k)$ multiples determinant by k, and $T_{ij}(k)$ does not change the determinant of a square matrix.
If X~I then there exists a sequence of elementary row transformations: $T_qT_{q-1}\ldots T_2T_1X=I$. Since I has rank p, X has rank p and is nonsingular. Multiply both sides by $X^{-1}$ to get $X^{-1}=T_qT_{q-1}\ldots T_2T_1I$. Since $|I|=+1$, $|X|$ cannot be zero.

**Theorem**: The following are equivalent: $|X|\neq 0$, X is nonsingular, and $X^{-1}$ exists.

Eigenvalues and Eigenvectors

A p×p matrix S transforms **x** into a stretched version of **x** (with no rotation) when $S\mathbf{x}=\lambda\mathbf{x}$, for some scalar λ. A nontrivial solution exists if $(S-\lambda I)\mathbf{x}=0$ for $\mathbf{x}\neq 0$. This says the columns of $S-\lambda I$ are dependent but then $|S-\lambda I|=0$. This is a polynomial equation that defines the eigenvalue λ.

Example: eigenvalues of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$: $\left|\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right| = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix}$ $=(a-\lambda)(d-\lambda)-bc=$ $\lambda^2-(a+d)\lambda+ad-bc=0=\lambda^2-tr(S)\lambda+|S|=0$. There will be two eigenvalues (in general, p). The sum of the eigenvalues equals the trace of the matrix tr(S) and the product of the eigenvalues equals the determinant of the matrix |S|. This is true in general.

For each eigenvalue, $\lambda_i$, there is a direction, eigenvector $\mathbf{e_i}$, such that $S\mathbf{e_i}=\lambda_i\mathbf{e_i}$ and we standardize these to have unit length, $\mathbf{e_i}'\mathbf{e_i}=1$. Put these eigenvectors side-by-side into a matrix P.

**Theorem**: if S=S' then P'P=I. That is, the eigenvectors are orthonormal.

Proof: $S\mathbf{e_i}=\lambda_i\mathbf{e_i}$ implies $\mathbf{e_j}'S\mathbf{e_i}=\lambda_i\mathbf{e_j}'\mathbf{e_i}$. By similar reasoning and using symmetry, $\lambda_i\mathbf{e_j}'\mathbf{e_i}=\lambda_j\mathbf{e_j}'\mathbf{e_i}$. Since $\lambda_i\neq\lambda_j$, this implies that $\mathbf{e_j}'\mathbf{e_i}=0$. The length of the eigenvectors equals 1 so that takes care of the diagonal of I. QED

**Spectral Decomposition** of symmetric square matrix S

$Se_i=\lambda_i e_i$ is equivalent to $SP=P\Lambda$ where Λ is a diagonal matrix with eigenvalues along the diagonal. Multiply on the right by P' to get $SPP'=P\Lambda P'$ or (using the above theorem), $S=P\Lambda P'=\sum_i \lambda_i e_i e_i'$. That is, a symmetric square matrix S can be expressed as a combination of elements built from the orthonormal eigenvectors. If we order the eigenvalues from largest ($\lambda_1$) to smallest ($\lambda_m$), then if we used just the first few elements in the sum $\sum_i \lambda_i e_i e_i'$ we get the best approximation of S possible.

Recall that the p×p covariance matrix S can be pretty big in a multivariate study; a survey with 20 questions has a covariance matrix with 210 unique numbers. We might try to capture the essence of what is going on by looking at the 4 or 5 directions suggested by the football-shaped scatterplot that contain the most information. The 4 or 5 largest eigenvalues tells us the ones with the most information and their eigenvectors tell us the direction the football points.

**Theorem**: S is positive definite if and only if the eigenvalues of S are all positive.
Proof: $x'Sx=x'P\Lambda P'x$ by spectral decomposition. Let $y=P'x$, then $x'Sx=\Sigma\lambda_i y_i^2>0$ if all eigenvalues are positive. QED

Singular-Value Decomposition

Spectral decomposition applies only to symmetric square matrices. Suppose that from the data matrix X we compute the p×p square matrix X'X and the n×n square matrix XX'. Use the spectral decomposition to write $X'X=V\Lambda V'$ where V is p×p and write $XX'=U\Lambda_0 U'$ where U is n×n. We assume that n>p. Define the matrix n×p $\Lambda^{1/2}$ to be the diagonal matrix with elements equal to the square root of the eigenvalues in $\Lambda$; the last n-p rows are all zeroes. Compute $Y=U\Lambda^{1/2}V'$. $Y'Y=(U\Lambda^{1/2}V')'U\Lambda^{1/2}V'=V\Lambda^{1/2'}U'U\Lambda^{1/2}V'=V\Lambda^{1/2'}I\Lambda^{1/2}V'=V(\Lambda+0)V'=V\Lambda V'=X'X$ hence $X=U\Lambda^{1/2}V'$. Thus, any data matrix is really composed of a collection of n uncorrelated variables with unit variances (U), a selection of p of these stretched or shrunk ($\Lambda^{1/2}$) and these values rotated (V'). See below. This is called the Singular-Value Decomposition.