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NOTES

SEASONAL EXPONENTIAL SMOOTHING WITH DAMPED TRENDS*

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In this paper we apply the strategy of trend-damping to the popular Winters exponential smoothing systems for seasonal time series. Efficient model formulations are derived for both multiplicative and additive seasonal patterns. An algorithm is given to test the stability of the models in cases where predetermined smoothing parameters are used. Empirical results are presented to show that trend-damping improves ex ante forecast accuracy in seasonal data, especially at long leadtimes.
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1. Introduction

In Gardner and McKenzie (1985), we developed a generalization of Holt's exponential smoothing system for a linear trend. The generalization added a damping parameter to the model to give more control over trend extrapolation. We showed that damping erratic trends improved long-range forecast accuracy with no loss in short-range accuracy. In this paper we apply the strategy of trend-damping to the popular Winters (1960) exponential smoothing systems for seasonal time series. In §§2 and 3, we develop trend-damping systems for multiplicative and additive seasonal series, respectively. In §4, the problems of parameter choice and system stability are examined. Empirical evidence on forecast accuracy is discussed in §5.

2. Multiplicative Seasonality

For a linear trend with multiplicative seasonality, the Winters revision equations are

$$S_t = \alpha(X_t/I_{t-p}) + (1-\alpha)(S_{t-1} + T_{t-1}), \tag{1}$$

$$T_t = \gamma(S_t - S_{t-1}) + (1-\gamma)T_{t-1}, \tag{2}$$

$$I_t = \delta(X_t/S_t) + (1-\delta)I_{t-p}, \tag{3}$$

and the m-step-ahead forecast is

$$\hat{X}_t(m) = (S_t + mT_t)I_{t-p+m}, \qquad m = 1, 2, \ldots, p. \tag{4}$$

We call (1)–(4) the Winters-LM system. X_t is the value of the time series at time t. S_t is the estimated level component of the series and is smoothed using the parameter α. The trend component is T_t, which is smoothed using the parameter γ. In equation (3), the seasonal factors are denoted by I_k, $k = 1, 2, \ldots, p$, where p is the number of periods in one season. The seasonal factors are smoothed separately from the level and trend

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components with δ. All smoothing parameters are usually restricted to the range 0 to 1. Equation (4) computes the forecast for m steps ahead. This equation is valid only for $m = 1, 2, \ldots, p$. For example, to forecast for $m = p + 1, \ldots, 2p$, the seasonal factor in (4) should be I_{t-2p+m}.

To obtain a damped version of Winters-LM, we follow the procedure developed in Gardner and McKenzie (1985) and multiply the trend component by ϕ everywhere it appears. The new forecasting system is called Winters-DM:

$$S_t = \alpha(X_t / I_{t-p}) + (1 - \alpha)(S_{t-1} + \phi T_{t-1}), \tag{5}$$

$$T_t = \gamma(S_t - S_{t-1}) + (1 - \gamma)\phi T_{t-1}, \tag{6}$$

$$I_t = \delta(X_t / S_t) + (1 - \delta) I_{t-p}, \tag{7}$$

and the m-step-ahead forecast is

$$\hat{X}_t(m) = (S_t + \sum_{i=1}^{m} \phi^i T_t) I_{t-p+m}, \qquad m = 1, 2, \ldots, p. \tag{8}$$

Reformulating (5)–(7) in terms of the one-step-ahead error e_t yields a simpler system:

$$S_t = S_{t-1} + \phi T_{t-1} + h_1 e_t / I_{t-p}, \tag{9}$$

$$T_t = \phi T_{t-1} + h_2 e_t / I_{t-p}, \tag{10}$$

$$I_t = I_{t-p} + h_3 e_t / S_t, \tag{11}$$

where $h_1 = \alpha$, $h_2 = \alpha\gamma$, $h_3 = (1 - \alpha)\delta$.

Winters-DM can also be reformulated to provide a direct estimate of the asymptotic level of the trend component of the forecasts. This is done by defining $A_t = S_t + T_t\phi / (1 - \phi)$ and $B_t = -\phi T_t / (1 - \phi)$. Note that the asymptotic level is A_t. The components A_t and B_t are revised using

$$A_t = A_{t-1} + g_1 e_t / I_{t-p}, \tag{12}$$

$$B_t = \phi B_{t-1} + g_2 e_t / I_{t-p}, \tag{13}$$

$$I_t = I_{t-p} + g_3 e_t / (A_t + B_t), \tag{14}$$

and the forecast equation becomes

$$\hat{X}_t(m) = (A_t + \phi^m B_t) I_{t-p+m}, \qquad m = 1, 2, \ldots, p. \tag{15}$$

Equations (12)–(15) are identical to (5)–(8) when $0 < \phi < 1$, $g_1 = h_1 + \phi h_2 / (1 - \phi)$ and $g_2 = -\phi h_2 / (1 - \phi)$. The system defined by (12)–(15) is much simpler, both in revision of the forecasts and in avoidance of a summation in the forecast equation. The disadvantage of (12)–(15) is that separate provision must be made for the case when $\phi = 1$.

3. Additive Seasonality

The development of Winters-DA, to model a damped trend with additive seasonality, is similar to the multiplicative case. The Winters-DA system is:

$$S_t = \alpha(X_t - I_{t-p}) + (1 - \alpha)(S_{t-1} + \phi T_{t-1}), \tag{16}$$

$$T_t = \gamma(S_t - S_{t-1}) + (1 - \gamma)\phi T_{t-1}, \tag{17}$$

$$I_t = \delta(X_t - S_t) + (1 - \delta) I_{t-p}, \tag{18}$$

$$\hat{X}_t(m) = S_t + \sum_{i=1}^{m} \phi^i T_t + I_{t-p+m}, \qquad m = 1, 2, \ldots, p. \tag{19}$$

Using the one-step-ahead error e_t, an equivalent system is defined as

$$S_t = S_{t-1} + \phi T_{t-1} + h_1 e_t, \tag{20}$$

$$T_t = \phi T_{t-1} + h_2 e_t, \tag{21}$$

$$I_t = I_{t-p} + h_3 e_t, \tag{22}$$

where h_1, h_2, h_3 are as before.

Again we reformulate to get a direct estimate of the asymptotic level of the trend component:

$$A_t = A_{t-1} + g_1 e_t, \tag{23}$$

$$B_t = \phi B_{t-1} + g_2 e_t, \tag{24}$$

where g_1 and g_2 are as before. Equation (22) is used to revise the seasonal factors. The forecast equation becomes

$$\hat{X}_t(m) = A_t + \phi^m B_t + I_{t-p+m}, \qquad m = 1, 2, \ldots, p. \tag{25}$$

4. Parameter Choice and System Stability

The damped Winters systems have four parameters: h_1, h_2, h_3, and ϕ. We can reduce the number of parameters to three by using discounted-least-squares (DLS) to smooth the level and trend components. DLS parameters were developed for the nonseasonal Holt-D system in Gardner and McKenzie (1985) and can be applied to the Winters systems. With β as the discount factor, the DLS solution for level and trend parameters in both Winters-DM and DA is:

$$h_1 = 1 - (\beta/\phi)^2; \qquad h_2 = (1 - \beta/\phi)(1 - \beta/\phi^2). \tag{26}$$

Regardless of the number of parameters in the forecasting system, we need some method of determining whether the parameters chosen yield a stable system. If the system is unstable, the forecasts become more dependent on the remote past as additional data become available. This is unreasonable in general and surely intolerable for any forecasting system based on exponential smoothing principles, which stress the importance of recent observations. Although we do not expect that model-fitting will lead to an unstable system, practitioners often choose parameters in advance, without any model-fitting. For the nonseasonal Holt-D system, we were able to derive an equivalent ARIMA process that determined an explicit region of stability (Gardner and McKenzie 1985). However, the Winters-DM system is nonlinear and has no equivalent ARIMA process. This is also true of Winters-LM. Winters-DA and LA are linear systems and thus have equivalent ARIMA processes but the processes are so complex as to be of little practical use.

Although we cannot derive explicit regions of stability for Winters-DM and DA, numerical methods are available to test system stability. In the appendix, a numerical algorithm based on Wilson (1979) is given to test the Winters-DA system. Either system can be accepted as stable if its parameters are stable in Winters-DA. This conclusion is based on the work of Sweet (1985), who showed that Winters-LM and LA have approximately the same stability regions. This result holds for Winters-DM and DA as well.

5. Empirical Results

This section gives empirical forecasting results using the collection of 60 seasonal time series from Makridakis et al. (1982). Forecasting was done in the same manner as Makridakis. For quarterly series, models were fitted to the first N − 8 observations; post-sample forecasts were made at origin N − 8 for horizons 1 to 8. For monthly series, forecasts were made at origin N − 18 for horizons 1 to 18. Other computational details

TABLE 1
MAPE Comparisons for 60 Seasonal Time Series

Forecast Horizon	Winters-DM	Winters-DA	Winters-LM	Box-Jenkins	Parzen	Lewandowski
1	6.2*	7.2	8.9	10.5	11.0	10.8
2	7.7	7.0*	9.8	10.1	10.0	12.9
3	7.6*	8.6	10.5	10.0	10.1	12.6
4	9.9*	10.4	13.6	13.5	12.8	14.3
5	9.8*	10.9	11.5	13.7	12.2	12.9
6	11.7*	12.6	14.7	15.9	13.7	17.3
8	15.5*	15.5*	19.9	20.6	16.1	19.8
12	12.5*	13.3	14.9	15.1	13.9	16.6
15	25.9	23.2	34.0	23.3	19.2*	33.7
18	23.7*	24.3	32.7	30.6	24.8	23.8
All	13.8*	13.9	18.1	17.2	14.9	17.6

* Indicates row minima.

necessary to replicate this research are available in Makridakis et al. (1982) and Gardner (1988).

Mean-absolute-percentage-error (MAPE) comparisons for the series are shown in Table 1. Results for the four-parameter versions of Winters-DM and DA are given along with selected results from Makridakis et al. (1982) for Winters-LM, Box-Jenkins, Parzen, and Lewandowski. The strategy of damping trends improves forecast accuracy compared to Winters-LM, especially at long horizons. Winters-DM and DA also compare favorably to the more complex forecasting systems.

There is little difference in overall MAPE between Winters-DM and DA. The reason appears to be the nature of the seasonal patterns in these time series. Few patterns are purely additive or multiplicative and many series switch between seasonal and nonseasonal behavior.

Another way to judge Winters-DM and DA is to test for autocorrelation in the residuals from model-fitting. For both models, significant first-order autocorrelation was found in the residuals in only 7% of the seasonal series. In contrast, for Winters-LM, autocorrelation was found more than 30% of the time.

In the Makridakis et al. (1982) study, it was found that deseasonalizing the data and then using a nonseasonal predictor provided good forecasts. It is reasonable to ask whether smoothing the seasonal pattern improves accuracy compared to the use of deseasonalized data. The answer appears to be "yes," although the differences are small. If Holt-D is used with deseasonalized series, the overall MAPE is 14.1% versus 13.8% for Winters-DM.

The effects of constraining the parameters with DLS via equation (26) were also tested. As expected, the DLS constraint resulted in shorter computation times but with some loss of accuracy. The overall MAPEs using DLS were 14.1% for Winters-DM and 14.3% for Winters-DA.

6. Conclusions

This paper applies the strategy of trend-damping to exponential smoothing of seasonal time series. Although trend-damping adds some complexity, Winters-DM and DA are still reasonably simple and efficient forecasting systems. They should be suitable for automatic forecasting in large applications such as in inventory control. The main advantages of Winters-DM and DA are that they avoid overshooting the data and excessive amplification of the seasonal pattern, two problems common to seasonal exponential smoothing with linear trends.

Appendix: Stability Algorithm

The stability algorithm tests parameters in the range $0 < h_1, h_2, h_3, \phi < 1$. The algorithm is based on an array W defined recursively. Initially, the elements are:

$$W(p+1, 1) = \phi - h_1 - \phi h_2,$$

$$W(p+1, k) = \phi(h_1 - h_2) - h_1, \qquad k = 2, 3, \ldots, p-1,$$

$$W(p+1, p) = 1 - h_1 - h_3 - \phi(h_2 - h_1),$$

$$W(p+1, p+1) = \phi(h_1 + h_3 - 1),$$

The algorithm is:

1. Set $k = p + 1$.
2. Compute $D = 1 - W(k, k)^2$.
3. If $D \leq 0$, the system is UNSTABLE.
4. Compute $W(k-1, i) = [W(k, i) + W(k, k)W(k, k-i)]/D$ for $i = 1, 2, \ldots, k-1$.
5. Set $k = k - 1$.
6. If $k \geq 2$, go to step 2.
7. If $W(1, 1)^2 < 1$, the system is STABLE. Otherwise the system is UNSTABLE.

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DEVELOPING A GLOBAL DIVERSIFICATION MEASURE*

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Previous measures, focusing on either the international market or product dimension of corporate diversification, have been unsatisfactory for analyzing global diversification since both dimensions

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