A COMPARISON OF INVESTMENT ALLOCATION STRATEGIES FOR DISTRIBUTION INVENTORIES

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ABSTRACT

Management goals in distribution inventories are often expressed in terms of the maximum percentage of aggregate sales that should be back ordered. This paper compares several strategies for allocating total inventory investment to each item stocked in order to meet such goals. Computational results are given from a wholesale distribution inventory. The results show that multi-item strategies (which consider the interactions between items) require substantially less investment to meet management goals than strategies that treat each line item independently. All models in this research are approximations based on the assumptions commonly used in practice.

Subject Areas: Logistics and Distribution, Inventory Management, Production/Operations Management, and Heuristics.

INTRODUCTION

A popular measure of customer service in distribution inventories is the percentage of aggregate sales that must be back ordered for later delivery to customers. To meet back-order objectives set by management, single-item inventory models are commonly used. These models ignore the interactions between items in allocating total inventory investment. The IBM IMPACT system is a good example of the single-item approach to inventory decisions. Order quantities in IMPACT are computed with the simple EOQ. Safety stocks are allocated independently of order quantities to give each item the same percentage of back-ordered sales.

This paper compares two multi-item investment allocation strategies to the IMPACT approach. The first strategy is designed to minimize back-ordered sales for any given safety-stock investment. Order quantities are computed independently of safety stocks with the simple EOQ. The second multi-item strategy is based on a Lagrangian model to minimize back-ordered sales subject to two aggregate constraints: investment (cycle plus safety stock) and replenishment workload. Order quantities and safety stocks are computed simultaneously.

The next section of the paper reviews the decision rules used in the IMPACT strategy. The third section discusses the assumptions required by IMPACT. Using the same assumptions, the fourth and fifth sections develop the decision rules for the two multi-item strategies. The sixth section outlines a solution algorithm for the Lagrangian model. The seventh section contrasts the Lagrangian model and more conventional inventory models. The paper concludes with a comparison of the investment allocation strategies, using a sample of 500 line items drawn from a wholesale distribution inventory of service parts.

THE SINGLE-ITEM (IMPACT) STRATEGY

The single-item decision rules in the IMPACT system [12] [13] were developed by Brown [1]. They have been used since the mid-1960s in IMPACT and a variety of other software packages for inventory control [2] [14].

Order quantities in IMPACT are computed with the simple EOQ. Since the investment and service level comparisons below are in aggregate dollar terms, the EOQ in dollars is repeated for reference:

$$Q_i = [(2C_oD_i)/C_h]^{1/2} \tag{1}$$

where Q_i is the EOQ in dollars for the ith item, D_i is the annual demand (sales) in dollars, C_o is the marginal cost per order, and C_h is the holding or carrying cost percentage.

Safety stocks in IMPACT are computed independently of order quantities to give each item the same expected percentage of back-ordered sales. The following approximation is used for this percentage:

$$B = [(D_i/Q_i)\int_{r_i}^{\infty}(x - r_i)f(x)\mathrm{d}x]/D_i. \tag{2}$$

This expression can be interpreted as follows. The probability density function of the forecast errors during lead time is $f(x)$. The reorder point in dollar terms, r_i, is composed of forecast (mean) demand during lead time plus safety stock:

$$r_i = \bar{x} + k_i\sigma_i, \tag{3}$$

where k_i is called the safety factor and σ_i is the standard deviation of the forecast errors. The integral in Equation (2) is thus the expected dollars back ordered during one order cycle. Since D_i/Q_i is the expected number of annual orders, the numerator is the annual expected dollars back ordered.

When the forecast errors are normally distributed, Equation (2) can be simplified with the aid of the unit normal loss function tabled in Peterson and Silver [15, pp. 779-786]. First, we let

$$u = (x - \bar{x})/\sigma_i. \tag{4}$$

Then the unit loss function during each lead time is

$$L = \int_{k_i}^{\infty}(u - k_i)f(u)\mathrm{d}u. \tag{5}$$

This function is equivalent to Brown's partial expectation function tabled in [1, pp. 95-103].

To get the expected dollars back ordered per order cycle, we multiply L by σ_i. Substituting Equation (5) into (2) and simplifying, we have B for the case of normally distributed errors:

$$B=[\sigma_i \int_{k_i}^{\infty}(u-k_i)f(u)\mathrm{d}u]/Q_i. \qquad (6)$$

To implement Equation (6), we select the safety factor k_i so that B gives the same protection from stockout for each item in the inventory.

In some applications of these decision rules, the interactions between order quantities and safety stocks are taken into account by limiting the size of the order quantity to be no smaller than the size of the standard deviation of forecast errors over lead time. This constraint is recommended on an empirical basis by Brown [2] [3] and was used in making the calculations in the section on model comparisons below. Brown demonstrates that the total investment (cycle plus safety stock) necessary to maintain any given service level tends to increase when order quantities are excessively small compared to standard deviations. Herron [9] also discusses this problem.

Using these decision rules, the expected average inventory investment (I) in IMPACT is approximated by

$$\Sigma_i(Q_i/2+k_i\sigma_i)=I. \qquad (7)$$

ASSUMPTIONS

In order for Equations (6) and (7) to give good projections of the steady-state behavior of the inventory, a number of assumptions are required. Lead times must be constant. Since continuous review policies apply, we assume that a replenishment order is placed when the inventory position is exactly at the reorder point. In order for Equation (6) to hold exactly, it can be proven that demands must arrive according to a Poisson process. In order for Equation (7) to hold exactly, only one reorder can be outstanding at any one time. Finally, we assume that the safety factors k_i for any item will be relatively large. That is, management policy will always result in the average level of back orders being small compared to average on-hand stock. This assumption allows us to neglect the expected value of back orders in (7). For a more detailed discussion of these assumptions, see [8] or [15, pp. 257-262].

These are strong assumptions, but they are frequently unavoidable in applied inventory work. This is especially true in large inventory systems, where many thousands of line items are stocked. The series of inventory texts by Brown [1] [2] [3] gives numerous examples of applications in which these assumptions are reasonable. Gardner and Dannenbring [6] describe a large inventory system managed under these assumptions. The text by Peterson and Silver [15] develops a variety of decision rules based on these assumptions.

A MULTI-ITEM DECISION RULE FOR SAFETY STOCKS

Several authors [2] [3] [7] [15] [16] have proposed a simple alternative to the IMPACT strategy of giving each item the same percentage of back-ordered sales. It can be shown that aggregate back-ordered sales are minimized when each item has safety stock yielding the same number of shortage occurrences (NSO), which is defined here as the number of annual order cycles times the probability of a shortage on one order cycle:

$$\text{NSO} = (D_i/Q_i)\int_{k_i}^{\infty} f(u)\text{d}u. \tag{8}$$

Safety factors in this approach are selected to equalize NSO for each item stocked. NSO is treated as a management policy variable which can be changed to obtain different totals for aggregate back-ordered sales.

The mathematical basis for the equal-shortage-occurrences rule is straightforward. Given that order quantities have been computed independently of safety stocks with the EOQ, the total costs (TC) associated with safety stocks for one item are:

$$\text{TC}_i = C_h k_i \sigma_i + C_s (D_i/Q_i)\sigma_i \int_{k_i}^{\infty}(u - k_i) f(u)\text{d}u \tag{9}$$

where C_s is the annual shortage or penalty cost per dollar back ordered. Differentiating with respect to k_i and equating the first derivative to zero yields the following expression:

$$(D_i/Q_i)\int_{k_i}^{\infty} f(u)\text{d}u = C_h/C_s. \tag{10}$$

The ratio C_h/C_s equals the number of annual shortage occurrences for the *i*th item. If these costs are the same for all items, then all items should have the same number of shortage occurrences to minimize the aggregate value of back-ordered sales.

Equivalent versions of this decision rule for safety stocks can be found in [2], [3], [7], [15], and [16].

MULTI-ITEM DECISION RULES FOR ORDER QUANTITIES AND SAFETY STOCKS

Another multi-item strategy does not require that any marginal cost information be specified by the decision maker. Instead, we simply minimize back-ordered sales subject to aggregate replenishment workload and investment constraints. By varying the constraints, different sets of trade-offs among back orders, workload, and investment can be evaluated.

The model formulation is

$$\text{Min } Z = \Sigma_i (D_i/Q_i)\sigma_i \int_{k_i}^{\infty}(u - k_i) f(u)\text{d}u \tag{11}$$

$$\text{s.t.} \quad \Sigma_i (Q_i/2 + k_i\sigma_i) = I, \tag{12}$$

$$\text{and} \quad \Sigma_i (D_i/Q_i) = W. \tag{13}$$

The objective function in Equation (11) is equivalent to the numerator of the approximation in Equation (2). Equations (12) and (13) are the investment and workload constraints. The assumptions discussed in the second section apply to this formulation.

After forming the Lagrangian function, differentiating, and solving the first-order conditions (see [5] for details), the following decision rule for order quantities is obtained:

$$Q_i = [2D_i(\sigma_i \int_{k_i}^{\infty}(u - k_i)f(u)\mathrm{d}u + \lambda_W)/\lambda_I]^{1/2} \tag{14}$$

where λ_I and λ_W are the Lagrangian multipliers associated with Constraints (12) and (13).

The decision rule for safety stocks is

$$\int_{k_i}^{\infty} f(u)\mathrm{d}u = (\lambda_I Q_i)/D_i. \tag{15}$$

Expressions for the Lagrangian multipliers are:

$$\lambda_I = (\Sigma_i D_i \int_{k_i}^{\infty} f(u)\mathrm{d}u)/2(I - \Sigma_i k_i \sigma_i), \tag{16}$$

and

$$\lambda_W = [(\lambda_I \Sigma_i Q_i)/2 - (\Sigma_i D_i \sigma_i \int_{k_i}^{\infty}(u - k_i)f(u)\mathrm{d}u)/\Sigma_i Q_i]/W. \tag{17}$$

SOLUTION ALGORITHM FOR THE LAGRANGIAN MODEL

Since all model functions are convex, any solution to the first-order conditions is optimal. The solution algorithm is a heuristic, based on the method of successive approximations developed by Wagner [17] and applied to multi-item, stochastic inventory models by Gardner and Dannenbring [6].

The steps in the algorithm, using the normal distribution for safety stock, are:

1. Compute $\lambda_I = (.5\Sigma_i D_i)/2I$.
2. Compute $Q_i = (.5D_i)/\lambda_I$.
3. Compute $\lambda_W = [(\lambda_I \Sigma_i Q_i)/2 - (\Sigma_i .398942\sigma_i D_i)/\Sigma_i Q_i]/W$.
4. Compute Q_i with Equation (14).
5. Compute safety stocks with Equation (15).
6. Compute total investment and workload.
7. If both constraints are fulfilled, stop. Otherwise, compute λ_I with Equation (16).
8. Compute λ_W with Equation (17).
9. Go to Step 4.

To initialize the algorithm, we assume zero safety stocks in Equation (16). This gives a starting point for λ_I to compute beginning order quantities. The results of Steps 1 and 2 give a starting value for λ_W in Step 3. Thereafter, we iteratively update Q_i, k_i, λ_I, and λ_W until the constraints are fulfilled.

One peculiarity of this algorithm is that it can result in negative safety stocks for some items. This can happen when a tight investment constraint is applied to items having a relatively large standard deviation in dollar terms. Items with negative safety stocks should be reviewed to ensure that the assumption of normally distributed forecast errors is justified. If the errors do appear to be normal, a constraint should be added to the algorithm to ensure that safety stocks are nonnegative. In my experience, such a constraint does not interfere with convergence.

Since this algorithm is a heuristic, no guarnatees can be made for convergence. As one of the referees pointed out, there are many alternative search routines that could be used. Computational experience with the algorithm has been excellent. In more than 150 runs on data sets ranging in size from 500 to 40,000 line items, the algorithm has converged to within 1 percent of both constraints in 35 iterations or less. CPU time on the UNIVAC 1100/40 is about .1 seconds per iteration per 1,000 line items. Further discussion of this type of solution algorithm is given in [5] and [6].

INTERPRETATION OF THE LAGRANGIAN MULTIPLIERS

The Lagrangian multipliers can be interpreted as imputed marginal cost information. To see this equivalence, suppose we use an objective function with marginal costs for ordering, holding, and shortages of stock. The total cost expression for any item is

$$\mathrm{TC}_i = (C_o D_i/Q_i) + (C_h Q_i/2) + C_h k_i \sigma_i + C_s (D_i/Q_i)\sigma_i \int_{k_i}^{\infty} (u - k_i) f(u)\mathrm{d}u. \quad (18)$$

The solutions for order quantities and safety stocks are:

$$Q_i = [2D_i(C_s\sigma_i \int_{k_i}^{\infty}(u - k_i)f(u)\mathrm{d}u + \mathrm{C_o})/C_h]^{1/2} \quad (19)$$

$$(D_i/Q_i)\int_{k_i}^{\infty} f(u)\mathrm{d}u = (C_h/C_s). \quad (20)$$

Comparing Equations (19) and (20) to Equations (14) and (15), the total cost model gives order quantities and safety stocks equivalent to the Lagrangian model provided that $\lambda_W = C_o/C_s$ and $\lambda_I = C_h/C_s$.

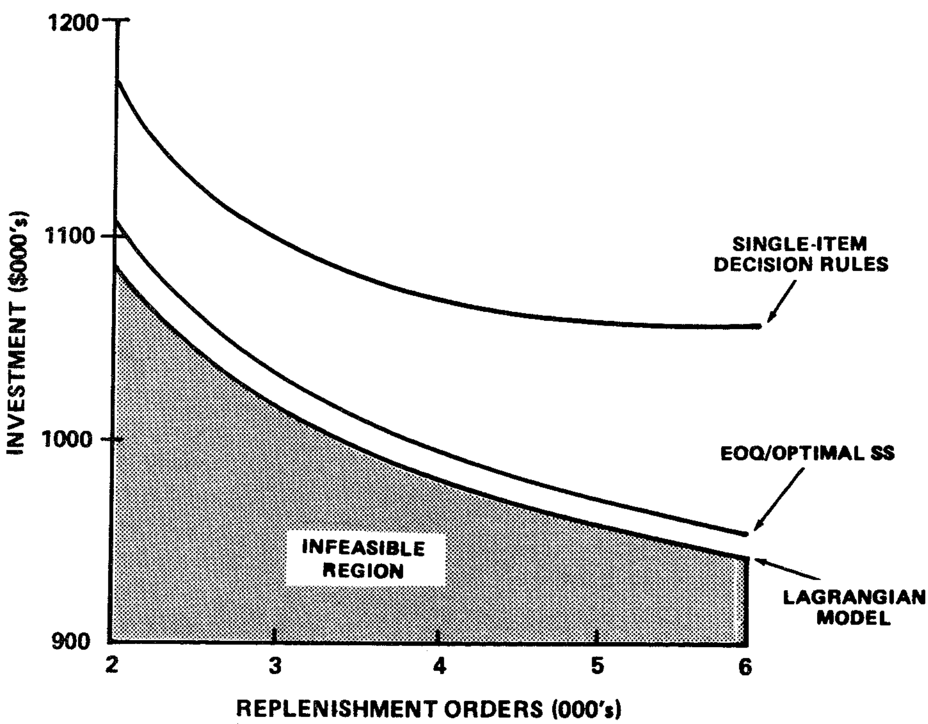
It should be noted that the safety stock rule in Equation (20) is equivalent to the multi-item rule for safety stocks (Equation (8)) in the fourth section. Thus the main difference between the simpler model in the fourth section and the Lagrangian model is the way order quantities are computed. In the fourth section, order quantities are computed independently; in the Lagrangian model, order quantities are computed simultaneously with safety stocks.

MODEL COMPARISONS

The three investment allocation strategies were compared using a sample of 500 line items drawn from a wholesale distribution inventory of service parts. The complete inventory numbers more than 6,000 line items. The sample consists of the "Class A" or the more important items in terms of dollar sales. Annual sales for the sample are about $4.9 million, representing 70 percent of total sales for the inventory. Forecasts for the inventory are generated with standard exponential smoothing models. We assume that lead times are constant and forecast errors are normally distributed (the assumptions currently used for the inventory).

The strategies were compared on the basis of the total investment and workload combinations required to achieve a number of different percentage objectives for back-ordered sales. The relative differences between strategies were about the same for all objectives, so computational results are given to illustrate only one objective, that for back orders equal to 5 percent of annual sales. Figure 1 shows the workload and investment combinations required by each strategy to meet this objective.

FIGURE 1
Isoservice Curves to Back Order 5 Percent of Annual Sales



The curves in Figure 1 are called "isoservice curves" since all points on each curve yield the same 5 percent level of back orders. For the single-item IMPACT strategy, the isoservice curve was generated by varying the costs in the EOQ formula to give a range of order quantity or cycle stock investments. Safety stocks were added to cycle stocks using the decision rule in Equation (6) above.

A similar procedure was followed to generate the curve for the EOQ coupled with the optimal safety stock rule. Costs in the EOQ model were varied to give a range of order quantity investments. A grid search was then made for the values of the NSO policy variable yielding a 5 percent level of back orders.

For the Lagrangian model, a response surface was generated for back-ordered sales as a function of aggregate investment and replenishment workload. The points on the surface yielding a 5 percent level of back orders are plotted in Figure 1.

The results in Figure 1 show that both multi-item strategies require substantially less investment to achieve the back-order objective than the single-item strategy. This is consistent with previous work by Herron [9] [10], Herron and Hawley [11], Brown [2] [3], and Peterson and Silver [15], all of whom have emphasized the potential advantages of multi-item strategies for allocating inventory investment.

Note that the region below the Lagrangian isoservice curve is infeasible. Given the assumptions discussed above, the investment and workload combinations for this model are the best that can be done.

One characteristic of isoservice curves for any back-order objective is that the slope decreases as the workload increases. The reason for this effect is complex, since workload impacts on back orders in two ways. Increasing workload reduces order quantities, so the investment saved can be used to increase safety stocks. More frequent ordering, however, increases the number of exposures to risk of stockout. The net effect allows the decision maker to exchange reduced investment for increased workload for a time, while still maintaining a fixed level of back orders. This trade-off is subject to diminishing marginal returns, so any isoservice curve will eventually become perfectly flat. This is the case with the single-item curve above a workload of 4,000 orders.

CONCLUSIONS

Several generalizations can be made that may be helpful in developing management policy in distribution inventory systems. First, the isoservice curve is a useful way to present inventory trade-offs to management without the need to make any prior assumptions about cost information. There is considerable evidence that marginal inventory costs are virtually impossible to measure. See [4] for a discussion.

Second, users of single-item decision rules like those in the IMPACT system should consider one of the multi-item strategies. Savings in investment and/or workload should be possible. No guarantee can be made that these savings will be significant, but the mathematical basis of the multi-item strategies should always result in some savings.

Finally, this research showed little difference in the performance of the two multi-item strategies. Arguments of simplicity favor the EOQ/optimal–safety-stock strategy, although the Lagrangian model is easier to use when investment and/or workload constraints apply. The Lagrangian model will meet the constraints without the need for trial-and-error experimentation with cost estimates and the NSO policy variable.

The Lagrangian model is also useful when conducting an analysis of inventory performance for top management. The Lagrangian model operates with aggregate inputs and outputs. These are the terms in which senior executives evaluate inventory performance. [Received: April 15, 1983. Accepted: August 4, 1983.]

REFERENCES

[1] Brown, R. G. *Decision rules for inventory management.* New York: Holt, Rinehart and Winston, 1967.

[2] Brown, R. G. *Materials management systems—A modular library.* New York: Wiley, 1977.

[3] Brown, R. G. *Advanced service parts inventory control.* Norwich, Ver.: Material Management Systems, 1982.

[4] Gardner, E. S. Inventory theory and the gods of Olympus. *Interfaces,* 1980, *10*(4), 42-45.

[5] Gardner, E. S. Approximate decision rules for continuous review inventory systems. *Naval Research Logistics Quarterly,* 1983, *20,* 59-68.

[6] Gardner, E. S., & Dannenbring, D. G. Using optimal policy surfaces to analyze aggregate inventory tradeoffs. *Management Science,* 1979, *25,* 709-720.

[7] Gerson, G. K., & Brown, R. G. Decision rules for equal shortage policies. *Naval Research Logistics Quarterly,* 1970, *17,* 351-358.

[8] Hadley, G., & Whitin, T. M. *Analysis of inventory systems.* Englewood Cliffs, N.J.: Prentice-Hall, 1963.

[9] Herron, D. P. A comparison of techniques for multi-item inventory analysis. *Production and Inventory Management,* 1978, *19*(1), 103-115.

[10] Herron, D. P. Profit-oriented distribution management. In *Annual conference proceedings.* Chicago, Ill.: National Council of Physical Distribution Management, 1980.

[11] Herron, D. P., & Hawley, R. L. Establishing the optimum inventory size and stocking policy for a warehouse. *AIIE Transactions,* 1969, *1*(1), 75-80.

[12] International Business Machines Corporation. *Basic Principles of wholesale IMPACT* (Pub. E20-8105-1). White Plains, N.Y.: 1967.

[13] International Business Machines Corporation. *Wholesale IMPACT—Advanced principles and implementation manual* (Pub. E20-0174-1). White Plains, N.Y.: 1971.

[14] Kleijnen, J. P. C., & Rens, P. J. "IMPACT revisited: A critical analysis of IBM's inventory package IMPACT. *Production and Inventory Management,* 1978, *19*(1), 71-90.

[15] Peterson, R., & Silver, E. A. *Decision systems for inventory management and production planning.* New York: Wiley, 1979.

[16] Prichard, J. W., & Eagle, R. H. *Modern inventory management.* New York: Wiley, 1965.

[17] Wagner, H. M. *Principles of operations research.* Englewood Cliffs, N.J.: Prentice-Hall, 1969.

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