Approximate Decision Rules for Continuous Review Inventory Systems

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Constrained multi-item inventory models have long presented significant computational problems. This article presents a general algorithm to obtain simultaneous solutions for order quantities and safety stocks for each line item in an inventory, while satisfying constraints on average inventory investment and reordering workload. Computational experience is presented that demonstrates the algorithm's efficiency in handling large-scale applications. Decision rules for several customer service objectives are developed, with a discussion of the characteristics of the inventory systems in which each objective would be most appropriate. The decision rules are approximations, based on the assumptions commonly used in practice.

1. INTRODUCTION

Several measures of customer service are commonly used by management to evaluate performance in multi-item inventory systems: the dollar value of sales that must be backordered for later delivery to customers, the number of times that a shortage condition occurs, and the number of customer requisitions that must be backordered. The decision-maker is often concerned with minimization of one or more of these measures given constraints on inventory investment and order-processing capacity or workload. This article presents an efficient solution procedure to obtain approximate decision rules for such problems.

The solution of constrained inventory models of this nature has long presented significant computational problems (see the discussion in Hadley and Whitin [15], for example). Several authors have proposed trial-and-error solution strategies [2,5,6,13,17–21], which can be inefficient and tedious in large-scale applications. Others have developed algorithms which guarantee convergence but are severely limited in problem size [23].

In [12], we described a solution algorithm which overcame these difficulties for the objective of minimizing an approximation for the number of customer requisitions backordered. In this article, we show that the algorithm can be modified to minimize approximations for the other commonly used performance measures for customer service. We provide computational experience that demonstrates the algorithm's efficiency in handling large-scale applications. We also comment on the characteristics of the inventory systems in which each customer service objective would be most appropriate. Finally, we illustrate the tradeoffs among customer service measures that should be considered by the decision-maker before selecting an objective.

Since the marginal cost parameters often assumed in inventory models are difficult, if not impossible, to measure accurately [11], the models presented here do not require

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marginal cost estimates. Instead, as surrogates for ordering and holding costs, the models include constraints for both workload and total inventory investment. The efficiency of the general-solution algorithm allows experimentation by the decision-maker with various levels of these constraints and with different customer service objectives. The constraints impute marginal cost information to assist in the decision process.

2. ASSUMPTIONS

Throughout this article we consider a continuous review inventory system, managed with order quantity, order point (Q,r) policies. Aggregate inventory investment is approximated by

$$\sum_i \left(\frac{Q_i}{2} + S_i\right) = I, \tag{1}$$

where Q_i is the order quantity in dollars for the ith item, and S_i is the safety stock.

The probability of a shortage on one order cycle is

$$P_i = \int_{r_i}^{\infty} f(x)dx. \tag{2}$$

r_i is the reorder point in dollars, composed of safety stock plus leadtime demand stock. $f(x)$ is the probability density function for the forecast errors over leadtime, usually assumed to be normal.

The expected dollars short per order cycle is given by the partial expectation function tabled by Brown [1]:

$$E_i = \int_{r_i}^{\infty} (x - r_i)f(x)dx. \tag{3}$$

A number of assumptions are necessary if these expressions are to give a reasonable projection of the steady-state behavior of the inventory. These assumptions generally hold independent of the method of measuring customer service. First, we assume that $f(x)$ is normal, although the general-solution algorithm can be modified for any of the standard probability distributions. Next, we assume that leadtimes are constant, or that any variability is negligible. Since continuous review policies apply, we assume that a replenishment order of fixed size is placed when the inventory position is exactly at the reorder point. If Equation (2) is to hold exactly, it can be proven that demands must arrive according to a Poisson process [15].

Implicit in Equations (1)–(3) is the assumption that there is never more than a single order outstanding. This means that when the reorder point is reached, there will be no previous orders outstanding. If we further assume that the reorder point is positive, which is almost always true in practice, there will be no backorders unfilled at the reorder point. The result is that inventory position (the amount on hand plus reorders minus backorders) is equal to on-hand inventory at the reorder point.

Finally, we assume that management policy will always result in the average level of backorders being negligibly small when compared with the level of on-hand stock. This is almost always true in practice and allows us to neglect the expected value of backorders in Equation (1).

These are strong assumptions, so the models presented in this article must be regarded as heuristics or approximations. A more detailed discussion of these assumptions is given in Hadley and Whitin's heuristic treatment of (Q,r) models [15, Sec. 4-2], and in Peterson and Silver [21]. In the opinion of Hadley and Whitin, these assumptions are frequently unavoidable in developing the decision rules needed for practical applications. This is especially true in large-inventory systems, where many thousands of line items are managed. Hadley and Whitin also offer the opinion that the special cases for which exact models are available do not represent the real world much more accurately than approximate models based on the assumptions above.

Numerous examples of wholesale service parts inventories in which these assumptions are reasonable are given in Brown [1–3]. Peterson and Silver [21] develop a variety of practical decision rules for distribution inventories using these assumptions. The military distribution system described in Gardner and Dannenbring [12] is managed under these assumptions. Other precedents for these assumptions in applied work include Eagle [7], Fetter and Dalleck [8], Gerson and Brown [13], Groff and Muth [14], Flansmann [16], Parker [20], Prichard and Eagle [22], Schrady and Choe [23], and Starr and Miller [24].

3. THE BACKORDERED SALES MODEL

To minimize the approximate expected backordered sales in an inventory, the objective function is

$$\min Z = \sum_i D_i E_i / Q_i, \tag{4}$$

where D_i is the annual demand or sales in dollars. The constraints on aggregate investment and replenishment workload are

$$\sum_i \left(\frac{Q_i}{2} + S_i\right) = I, \tag{5}$$

and

$$\sum_i D_i/Q_i = W. \tag{6}$$

To solve this model for any combination of constraints, we form the Lagrangian function,

$$L = \sum_i \frac{D_i E_i}{Q_i} + \lambda_I \left[\sum_i \left(\frac{Q_i}{2} + S_i\right) - I\right] + \lambda_W \left[\sum_i \frac{D_i}{Q_i} - W\right], \tag{7}$$

where λ_I and λ_W are the Lagrangian multipliers associated with constraints (5) and (6).

Differentiating with respect to Q_i, S_i, λ_I, and λ_W, we obtain the first-order conditions:

$$\frac{\partial L}{\partial Q_i} = \frac{-D_i E_i}{Q_i^2} + \frac{\lambda_I}{2} - \frac{\lambda_W D_i}{Q_i^2} = 0, \tag{8}$$

$$\frac{\partial L}{\partial S_i} = \frac{-D_i P_i}{Q_i} + \lambda_I = 0, \tag{9}$$

$$\frac{\partial L}{\partial \lambda_I} = \sum_i \left(\frac{Q_i}{2} + S_i\right) - I = 0, \tag{10}$$

and

$$\frac{\partial L}{\partial \lambda_W} = \sum_i \frac{D_i}{Q_i} - W = 0. \tag{11}$$

Solutions to the first-order conditions yield the following decision rules for order quantities and safety stocks:

$$Q_i = \left(\frac{2D_i(E_i + \lambda_W)}{\lambda_I}\right)^{1/2}, \tag{12}$$

and

$$P_i = \lambda_I Q_i / D_i. \tag{13}$$

The Lagrangian multipliers can be written as

$$\lambda_I = \sum_i D_i P_i / 2\left(I - \sum_i S_i\right), \tag{14}$$

and

$$\lambda_W = \frac{1}{W}\left(\frac{\lambda_I \sum_i Q_i}{2} - \sum_i \frac{D_i E_i}{Q_i}\right). \tag{15}$$

4. SOLUTION ALGORITHM

Since all model functions are convex (see Gerson and Brown [13] for proofs), any solution to the first-order conditions is optimal. Unfortunately, there is no direct solution to any of the variables in the problem. The approach followed here uses the method of successive approximations developed by Wagner [25], and applied to multi-item, stochastic inventory models by Gardner and Dannenbring [12]. The steps in the solution algorithm are:

(1) Compute initial λ_I, assuming zero safety stock in (14).
(2) Solve (13) for Q_i and compute an initial value for each order quantity, again assuming zero safety stock.
(3) Compute λ_W assuming zero safety stock in (15).
(4) Compute Q_i with (12).
(5) Compute S_i with (13).
(6) If both constraints are fulfilled, stop. Otherwise, recompute λ_I with (14) and λ_W with (15).
(7) Go to step 4.

The first three steps in the algorithm are used to initialize each variable. In step 1, $P_i = 0.5$ for each item, since there is no safety stock. The denominator of (14) is simply $2I$. In step 2, we use (13) for Q_i, rather than (12). Note that (12) includes λ_W, which has not yet been computed. Next, step 3 determines a beginning λ_W with the initial values of λ_I, Q_i, and the E_i values corresponding to zero safety stock ($0.398942\sigma_i$ assuming that forecast errors are normally distributed). From this starting position, we update Q_i, S_i, λ_I, and λ_W iteratively in steps 4–7. The objective function is monotonically decreasing with each iteration, since aggregate safety stocks are built up from a zero starting point.

Convergence of this algorithm cannot be proven to occur in any finite number of iterations. However, extensive computational experience shows that convergence is rapid. CPU time on the UNIVAC 1100/40 is less than 0.1 s per iteration per 1000 line items. In more than 150 runs on data sets ranging from 500 to 40,000 line items, the model has converged to within 1% of the investment constraint in 12 iterations or less. Convergence to within 1% of the workload constraint has occurred in 35 iterations or less.

An example of the performance of the solution algorithm on a sample of 500 line items is given in Table 1. The sample was drawn from a distribution inventory of 12,000 line items. Sales in the sample are about $4.8 million, representing 70% of sales for the inventory. Constraints of $1,115,000 in investment and 4000 annual replenishment orders were imposed. The last ten iterations of the algorithm reduce backordered sales by only 0.02%, indicating that the response surface is rather flat near the optimum solution. This is characteristic of the solutions obtained with the model to date. In most cases, there is very little change in backordered sales after about ten iterations. Thus, it is reasonable to use a stopping rule for the algorithm based on some minimum change in backordered sales rather than on exact fulfillment of the constraints.

5. INTERPRETATION OF THE LAGRANGIAN MULTIPLIERS

The discussion in Gardner [9–11] indicates that it is virtually impossible to measure marginal inventory costs. However, most decision-makers can at least specify reasonable levels of constraints for aggregate inventory investment and replenishment workload. The Lagrangian multipliers can then be interpreted as imputed cost information to assist in finding a good balance among customer service, investment, and workload.

To see this equivalence, suppose that C_s is the shortage or penalty cost per dollar backordered. Let C_h be the annual inventory carrying cost expressed as a percentage

Table 1. Convergence of the backordered sales model.

Iteration	Total investment ($000's)	Total workload	λ_I	λ_W	% Backordered sales
1	950.4	1431	1.093	173.935	9.53
2	981.3	1731	0.909	70.283	7.40
3	1036.0	2004	0.763	44.566	5.75
4	1073.6	2221	0.678	37.323	4.81
5	1093.4	2405	0.632	32.288	4.28
⋮					
10	1113.5	3139	0.572	16.059	3.60
⋮					
20	1115.0	3783	0.558	8.658	3.34
⋮					
30	1115.0	3990	0.555	7.356	3.32

of dollar value and C_o be the marginal ordering cost. Then the total costs for any single item are

$$TC_i = \frac{C_o D_i}{Q_i} + C_h\left(\frac{Q_i}{2} + S_i\right) + \frac{C_s D_i E_i}{Q_i}. \tag{16}$$

Differentiating with respect to Q_i and S_i, we obtain

$$Q_i = \left(\frac{2D_i(C_s E_i + C_o)}{C_h}\right)^{1/2}, \tag{17}$$

and

$$D_i P_i / Q_i = C_h / C_s. \tag{18}$$

Thus, the decision rules for the cost-based, single-item model are equivalent to those for the constrained model, provided that

$$C_o / C_s = \lambda_W, \tag{19}$$

and

$$C_h / C_s = \lambda_I. \tag{20}$$

For example, in Table 1 the investment constraint of \$1,115,000 imputes $C_h/C_s = 0.6$. The workload constraint of 4000 orders imputes $C_o/C_s = 7.4$. That is, the cost to hold one dollar in stock for one year is 60% of the cost of incurring one dollar in backorders. The cost to place one order is 7.4 times the cost to incur one dollar in backorders.

6. THE SHORTAGE OCCURRENCES MODEL

Another popular measure of customer service in inventory control is the number of times each year that a shortage condition or an out-of-stock situation occurs. This measure is appropriate in inventory systems where the occurrence of a shortage can be mitigated by expediting action until the next routine order arrives to replenish supplies. In these systems, total expediting effort will usually be proportional to the number of shortage occurrences.

The objective function to minimize the approximate number of shortage occurrences in an inventory is

$$\min Z = \sum_i D_i P_i / Q_i. \tag{21}$$

Constraints (5) and (6) also apply to this model.

After forming the Lagrangian, differentiating, and solving the first-order conditions, we obtain:

$$Q_i = \left(\frac{2D_i(P_i + \lambda_W)}{\lambda_I}\right)^{1/2}, \tag{22}$$

$$f(S_i) = \lambda_I Q_i \sigma_i / D_i, \tag{23}$$

$$\lambda_I = \frac{\sum_i D_i f(S_i)/\sigma_i}{2(I - \sum_i S_i)}, \tag{24}$$

and

$$\lambda_W = \frac{1}{W}\left(\sum_i \frac{\lambda_I Q_i}{2} - \frac{D_i P_i}{Q_i}\right). \tag{25}$$

The Lagrangian multipliers can be interpreted in a manner similar to the backordered sales case.

Compared with the backordered sales model, the shortage occurrences model usually results in a radically different set of safety stocks for each item. The risk of a shortage condition tends to be concentrated in the high-dollar-value segment of the inventory, where the tradeoff between shortage occurrences and additional safety stock investment is relatively poor. More shortage protection is usually given to the middle-to-lower-value segments of the inventory, where this tradeoff is improved. The result can be a relatively high dollar value of backordered sales in order to obtain minimum shortage occurrences.

With appropriate substitutions, the general-solution algorithm in Section 4 can be used for the shortage occurrences model. One note of caution is in order, however. As Gerson and Brown [13] point out, the objective function is not convex in S_i. In practical terms, this means that (23) has no solution if the right-hand side turns out to be more than 0.398942, which is the maximum value of the density of the normal distribution (at zero safety stock). This problem can occur if the investment constraint is extremely tight, yielding a relatively large λ_I. The solution algorithm must be modified to restrict $f(x) \leq 0.398942$. This does not interfere with convergence to a stationary point of the Lagrangian.

A great deal of trial-and-error experimentation has been done with test problems to determine whether a stationary point with restricted $f(x)$ values could be improved. In no case could a better solution be found. Graphical analysis of several response surfaces also showed no irregularities which would confound the algorithm.

7. THE REQUISITIONS BACKORDERED MODEL

A third important measure of customer service is the annual number of customer requisitions or demand transactions that cannot be filled immediately from stock. One requisition is equivalent to one line on a customer's order. This criterion is appropriate in many repair parts or maintenance inventories, where repair work cannot be completed until the total number of units of stock demanded on each requisition becomes available. The requisitions short measure may also apply in some multiwarehouse distribution systems where attempts are made to satisfy requisitions for items not in stock by shipping from distant warehouses. Expediting effort and costs should then be proportional to the number of requisitions short. Military inventories, where requisitions are often filled from stock in the order of indexes of criticality of need, are another case in which the number of requisitions short is useful as a measure of customer service [12].

One drawback to this measure is that like the shortage occurrences measure, it may result in a high dollar value of backordered sales. Again, relatively more shortage protection is usually given to the lower-value segments of the inventory, where there is a good tradeoff between requisitions short and additional safety stock investment.

The objective function to minimize the approximate annual number of requisitions backordered in an inventory is defined as

$$Z = \sum_i \frac{D_i}{Q_i}\left(\frac{E_i}{m_i}\right), \tag{26}$$

where m_i is the customer requisition size in dollars. We assume that the requisition size is constant for each line item and that it is independent of the level of demand. This is a reasonable assumption in the inventory system studied by Gardner and Dannenbring [12].

Using the same constraints as above, the first-order conditions for this model are

$$Q_i = \left[2D_i\left(\frac{E_i}{m_i} + \lambda_W\right)\Big/\lambda_I\right]^{1/2}, \tag{27}$$

$$P_i = \lambda_I Q_i m_i / D_i, \tag{28}$$

$$\lambda_I = \sum_i \frac{D_i P_i}{m_i}\Big/ 2\left(I - \sum_i S_i\right), \tag{29}$$

$$\lambda_W = \frac{1}{W}\left(\frac{\lambda_I \sum_i Q_i}{2} - \sum_i \frac{D_i E_i / m_i}{Q_i}\right). \tag{30}$$

The Lagrangian multipliers can be interpreted as above. With appropriate substitutions, the general-solution algorithm also applies to this model.

8. OBJECTIVE FUNCTION COMPARISONS

Table 2 compares the values of all three shortage measures, when each objective function is minimized with identical workload and investment constraints. The inventory is the same as in Table 1. As discussed above, both the shortage occurrences and the requisitions short models lead to relatively high dollar values of backorders. To reduce the dollar value of backorders, a significant penalty must be paid in both increased numbers of shortage occurrences and requisitions short. If an acceptable objective function cannot be selected from the information in Table 2, and if sufficient resources are available, the decision-maker may want to consider increasing the levels of the investment and/or workload constraints to reduce shortages.

Table 2. Shortage values for alternative objective functions.

Objective function minimized	Number of shortage occurrences	Dollar value of backorders	Number of requisitions backordered
Number of shortage occurrences	112	$346,232	297
Dollar value of backorders	281	146,320	641
Number of requisitions backordered	163	308,119	222

9. CONCLUSIONS

Constraints on inventory investment and reordering workload are pervasive in practice. This article presented approximate decision rules and an efficient solution procedure for several common objective functions under constraints. The alternative solution procedure to meet constraints is tedious trial-and-error experimentation with different sets of cost parameters.

Another application for the results in this article is in cases where it is difficult to measure marginal inventory costs. If the decision-maker can specify reasonable levels of aggregate investment and workload, these impute marginal cost information in the form of Lagrangian multipliers. The imputed costs can be used to help select the best tradeoff among investment, workload, and customer service.

The decision rules proposed are based on the same assumptions used in numerous inventory systems in practice, which should help minimize implementation problems. The general-solution algorithm is capable of handling large-scale applications, using demand information already available in most inventory systems.

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REFERENCES

[1] Brown, R. G., *Decision Rules For Inventory Management,* Holt, Rinehart, and Winston, New York, 1967.

[2] Brown, R. G., *Materials Management Systems—A Modular Library,* Wiley, New York, 1977.

[3] Brown, R. G., *Advanced Service Parts Inventory Control,* Materials Management Systems, Inc., Norwich, VT, 1982.

[4] Buchan, J., and Koenigsber, E., *Scientific Inventory Management,* Prentice-Hall, Englewood Cliffs, NJ, 1963.

[5] Daeschner, W. E., "Models for Multi-Item Inventory Systems Under Constraints," unpublished Ph.D. dissertation, Naval Postgraduate School, 1975.

[6] Everett, H., "Generalized Lagrange Multiplier Method for Solving Problems of Optimum Allocation of Resources," *Operations Research,* **1**(3),399–417 (May–June 1963).

[7] Eagle, R. H., "Optimal Inventory Decisions Under Constraints," unpublished Ph.D. dissertation, University of North Carolina, 1968.

[8] Fetter, R. B., and Dalleck, W. C., *Decision Models For Inventory Management,* Richard D. Irwin, Inc., Homewood, IL, 1961.

[9] Gardner, E. S., "Aggregate Inventory Models: Theory and Application," unpublished Ph.D. dissertation, University of North Carolina, 1978.

[10] Gardner, E. S., "Investment Allocation in Distribution Inventories," working paper, 1981.

[11] Gardner, E. S., "Inventory Theory and The Gods of Olympus," *Interfaces,* **10**(4),42–45 (August 1980).

[12] Gardner, E. S., and Dannenbring, D. G., "Using Optimal Policy Surfaces to Analyze Aggregate Inventory Tradeoffs," *Management Science,* **25**(8),709–720 (August 1979).

[13] Gerson, G., and Brown, R. G., "Decision Rules for Equal Shortage Policies," *Naval Research Logistics Quarterly,* **17**(3),351–358 (September 1970).

[14] Groff, G. K., and Muth, J. F., *Operations Management: Analysis for Decision,* Richard D. Irwin, Inc., Homewood, IL, 1972.

[15] Hadley, G., and Whitin, T. M., *Analysis of Inventory Systems,* Prentice-Hall, Englewood Cliffs, NJ, 1963.

[16] Hansmann, F., *Operations Research in Production and Inventory Control,* Wiley, New York, 1962.

[17] Herron, D. P., "Profit-Oriented Distribution Management," Annual Conference Proceedings, National Council of Physical Distribution Management, 1980.

[18] Herron, D. P., and Hawley, R. L., "Establishing the Optimum Inventory Size and Stocking Policy for a Warehouse," *AIIE Transactions,* **1**(1),75–80 (March 1969).

[19] Oral, M., "Improved Implementation of Inventory Control Models Through Equivalent Formulations," *Journal of Operations Management,* **I**(4),173–181 (May 1981).

[20] Parker, L. L., "Economical Reorder Quantities and Reorder Points with Uncertain Demand," *Naval Research Logistics Quarterly,* **11**(4),351–358 (December 1964).

[21] Peterson, R., and Silver, E. A., *Decision Systems for Inventory Management and Production Planning,* Wiley, New York, 1979.

[22] Prichard, J. W., and Eagle, R. H., *Modern Inventory Management,* Wiley, New York, 1965.

[23] Schrady, D. A., and Choe, V. E., "Models for Multi-Item Continuous Review Inventory Policies Subject to Constraints," *Naval Research Logistics Quarterly,* **8**(4),451–463 (December 1971).

[24] Starr, M. K., and Miller, D. W., *Inventory Control: Theory and Practice,* Prentice-Hall, Englewood Cliffs, NJ, 1962.

[25] Wagner, H. M., *Principles of Operations Research,* Prentice-Hall, Englewood Cliffs, NJ, 1976.